FUZZY NEIGHBORHOOD STRUCTURES ON PARTIALLY ORDERED GROUPS

KAMEL EL-SAADY and M. Y. BAKIER

Received 5 May 2001

Ahsanullah (1988) showed the compatibility between group structures and I -fuzzy neighborhood systems. In this paper, we require not only that the I -fuzzy neighborhood systems be compatible with the group structures, but also compatible with the order relation, in one sense or another.

2000 Mathematics Subject Classification: 54A40, 54H15, 54E15, 06F15, 20F60, 22A05.

1. Introductions. In [8], Katsaras combine the concepts of $[0, 1]$ -topology and order structure to bring out the so-called ordered fuzzy topological spaces. Several authors have continued on the work of Katsaras in the area of $[0, 1]$ -topology and order [3, 4, 10].

In [2] Ahsanullah introduced the notion of I -fuzzy neighborhood groups. In this paper, we aim to introduce and study the concept of I -fuzzy neighborhood structures on ordered groups.

2. Preliminaries. Let X be a nonempty set. A relation $\leq$ on X is said to be preorder if it is reflexive and transitive. An antisymmetric preorder is said to be a partially order. By a preordered (resp., an ordered) set, we mean a set X with a preorder (resp., a partially order) relation on it and we denote it by $(X, \leq)$. Every set can be considered as a partially ordered set equipped with the discrete order ($x \leq y$ if and only if $x = y$).

A function f from a preordered set $(X, \leq)$ to a preordered set $(X', \leq')$ is called isotone or order-preserving (resp., antitone or order-inverting) if $x \leq y$ in X implies $f(x) \leq' f(y)$ (resp., $f(y) \leq' f(x)$) in X' . The function f is said to be order isomorphism if it is bijection and $(\forall x, y \in X) x \leq y \Leftrightarrow f(x) \leq' f(y)$.

Suppose that $(G, *)$ is a semigroup and that G is endowed with an order $\leq$. We say that $(G, *, \leq)$ is an ordered semigroup if the law of composition and the order are related by the property: for all $x, y \in G$

$$x \leq y \Rightarrow (\forall z \in G) x * z \leq y * z, \quad z * x \leq z * y. \quad (2.1)$$

If $(G_1, T_1, \leq_1)$ and $(G_2, T_2, \leq_2)$ are ordered semigroups. A mapping $f: G_1 \rightarrow G_2$ is said to be order-homomorphism if it is both isotone and semigroup homomorphism. By an ordered group we mean an ordered semigroup which is a group.

In this paper, we use the multiplicative ordered group $(G, \cdot, \leq)$ which is sometimes written as $(G, \leq)$.

Combining the notion of order-isomorphism and group isomorphism, we say that an ordered group $(G_1, \leq_1)$ is OG-isomorphic to an ordered group $(G_2, \leq_2)$ if there is a mapping $f: G_1 \rightarrow G_2$ which is both order isomorphism and group isomorphism.

An I -fuzzy set μ , in a preordered set $(X, \leq)$, is called increasing (resp., decreasing) if $x \leq y$ implies $\mu(x) \leq \mu(y)$ (resp., $\mu(y) \leq \mu(x)$) [8].

A Chang-Goguen L -topology (cf. [5, 6, 7]) on a set X is a subset $\tau \subset L^X$, closed under finite infs and arbitrary sups. A pair (X, τ) is called a Chang-Goguen L -topological space; (X, τ) is called stratified L -topological space if τ contains all the constant L -fuzzy sets. The category of Chang-Goguen L -topological spaces (resp., stratified Chang-Goguen L -topological spaces) is denoted by $|L\text{-Top}|$ (resp., $|SL\text{-Top}|$). Both $|L\text{-Top}|$ and $|SL\text{-Top}|$ are topological categories. If $L = I = [0, 1]$, the above categories are denoted by $|I\text{-Top}|$ and $|SI\text{-Top}|$, respectively.

By an I -topological (resp., stratified I -topological) ordered space are we mean a triplet $(X, \leq, \tau)$, consisting of a partially ordered set $(X, \leq)$ and an I -topology (resp., stratified I -topology) τ on X .

By $|I\text{-TopOS}|$ (resp., $|SI\text{-TopOS}|$), we mean the category of all I -topological (resp., stratified I -topological) ordered spaces as object and all order-preserving continuous mappings between them as morphisms.

The order $\leq$, in an I -topological ordered space $(X, \leq, \tau)$, is said to be closed [8] if and only if the following condition holds: if $x \not\leq y$, then there are neighborhoods μ , ρ of x , y , respectively, such that $i(\mu) \wedge d(\rho) = 0$.

Let $(X, \leq, \tau)$ be an L -topological ordered space. If the order is closed, then X is Hausdorff [8].

An I -fuzzy quasi-uniformity [9] is a subset U of $I^{X \times X}$ which is prefilter and has the following three properties:

- (1) $\alpha(x, x) = 1 \forall \alpha \in U$ and $\forall x \in X$,
- (2) $\forall \alpha \in U, \forall \varepsilon > 0, \exists \alpha_1 \in U$ such that $\alpha_1 \circ \alpha_1 - \varepsilon \leq \alpha$,
- (3) $U = U$, that is, for every family $\{\alpha_\varepsilon \in U, \varepsilon \in I_0\}$ we have $\sup_{\varepsilon \in I_0 (\alpha_\varepsilon - \varepsilon) \in U}$.

The family $U^{-1} = \{\alpha^{-1} : \alpha \in U, \alpha^{-1}(x, y) = \alpha(y, x)\}$ is an I -fuzzy quasi-uniformity on X called the conjugate of U . We denote by U^* the I -fuzzy uniformity which generated by U , that is, $U^* = U \vee U^{-1} = \{\alpha \wedge \alpha^{-1} : \alpha \in U, \alpha^{-1} \in U^{-1}\}$. The I -fuzzy quasi-uniformity U can generate an order, say $\leq_u$, by setting

$$x \leq_u y \iff \begin{cases} \alpha(x, z) \leq \alpha(y, z) & \forall z \geq x, y, \\ \alpha(x, z) \geq \alpha(y, z) & \forall z \leq x, y. \end{cases} \quad (2.2)$$

A triplet $(X, \leq, U^*)$, consisting of an ordered set $(X, \leq)$ and an I -fuzzy uniformity U^* , is called an I -fuzzy uniform ordered space [10] if there exists an I -fuzzy quasi-uniformity U on X such that $U^* = U \vee U^{-1}$ and $G(\leq) = G(\leq_u)$.

DEFINITION 2.1 [10]. Let (X_1, U_1) and (X_2, U_2) be I -fuzzy quasi-uniform spaces. A mapping $f: (X_1, U_1) \rightarrow (X_2, U_2)$ is said to be quasi-uniformly continuous if and only if $\forall \alpha_2 \in U_2, \exists \alpha_1 \in U_1$ such that $\alpha_1 \in (f \times f)^{-1}(\alpha_2)$. Where f is called quasi-uniform equivalence if f is bijective and both f and f^{-1} are quasi-uniformly continuous.

DEFINITION 2.2 [10]. A mapping $f : (X, \leq, \mathbf{U}^*) \rightarrow (X_1, \leq_1, \mathbf{U}_1^*)$ is said to be uniformly order-mapping if there exist I -fuzzy quasi-uniformities u and u_1 on X and X_1 , respectively such that

- (i) $\mathbf{U}^* = \mathbf{U} \vee \mathbf{U}^{-1}$ and $G(\leq) = G(\leq_u)$;
- (ii) $\mathbf{U}_1^* = \mathbf{U}_1 \vee \mathbf{U}_1^{-1}$ and $G(\leq_1) = G(\leq_{u_1})$;
- (iii) $f : (X, \mathbf{U}) \rightarrow (X_1, \mathbf{U}_1)$ is quasi-uniformly continuous.

DEFINITION 2.3 [2]. Let $(G, \cdot)$ be a group and let $\varkappa$ be an I -fuzzy neighborhood system on G . Then, the triplet $(G, \cdot, t(\varkappa))$ is called I -fuzzy neighborhood group if and only if the following conditions are fulfilled:

- (1) the mapping $m : (G \times G, t(\varkappa) \times t(\varkappa)) \rightarrow (G, t(\varkappa)) : (x, y) \rightarrow xy$ is continuous;
- (2) the mapping $r : (G, t(\varkappa)) \rightarrow (G, t(\varkappa)) : x \rightarrow x^{-1}$ is continuous.

PROPOSITION 2.4 [2]. Let $(G, \cdot)$ be a group and let $\varkappa$ be an I -fuzzy neighborhood system on G . Then, $(G, \cdot, t(\varkappa))$ is an I -fuzzy neighborhood group if and only if the mapping

$$h : (G \times G, t(\varkappa) \times t(\varkappa)) \rightarrow (G, t(\varkappa)) : (x, y) \rightarrow xy^{-1} \quad (2.3)$$

is continuous

3. Fuzzy neighborhood ordered groups

DEFINITION 3.1. A triplet $(G, \leq, t(\varkappa))$ is called I -fuzzy neighborhood ordered groups if the following statements hold:

- (1) $(G, \leq)$ is a partially ordered group;
- (2) $(G, t(\varkappa))$ is an I -fuzzy neighborhood group;
- (3) the order $\leq$ is closed.

By $|I - \text{FNOGr}|$, we mean the category of all I -fuzzy neighborhood ordered groups as objects and all order-preserving homeomorphisms between them as morphisms.

In agreement with [1], a faithful functor $T : A \rightarrow \text{Set}$ is said to be topological (mono-topological) if and only if, given any index class $((X_j, \xi_j) : j \in J)$ of A -objects indexed by a class J and any source (resp., mono-source) $(f_j : X \rightarrow X_j)$ in Set , there exists a unique A -structure ξ on X which is initial with respect to $(f_j : X \rightarrow (X_j, \xi_j))_{j \in J}$, that is, such that for any A -object (Y, ζ) , a mapping $h : (Y, \zeta) \rightarrow (X, \xi)$ is an A -morphism if and only if for every $j \in J$, the composition $f_j \circ h : (Y, \zeta) \rightarrow (X_j, \xi_j)$ is an A -morphism. Also, we have that the constant function lift to morphism in A and the A -fibre $T^{-1}(S)$ for any set S is small.

PROPOSITION 3.2. The category $|I - \text{FNOGr}|$ is mono-topological.

PROOF. The forgetful functor $T : |I - \text{FNOGr}| \rightarrow |\text{Group}|$ is given by $T(G, \leq, t(\varkappa)) = G$. For some index class J , let $(G_\alpha, \leq_\alpha, t(\varkappa_\alpha)) \in |I - \text{FNOGr}|$ and $(f_\alpha : G \rightarrow G_\alpha)_{\alpha \in J}$ be a monosource in $|\text{Group}|$. Let $\varkappa$ be the I -fuzzy neighborhood system making the monosource

$$(f_\alpha : (G, t(\varkappa)) \rightarrow (G_\alpha, t(\varkappa_\alpha)))_{\alpha \in J} \quad (3.1)$$

initial and let $\leq$ be the order defined by $x \leq y$ if and only if $f_\alpha(x) \leq_\alpha f_\alpha(y)$ for all $\alpha \in J$. Then $(G, \leq, t(\mathfrak{N})) \in |I\text{-FNOGr}|$. Initiality of the mono-source

$$(f_\alpha : (G, \leq, t(\mathfrak{N})) \rightarrow (G_\alpha, \leq_\alpha, t(\mathfrak{N}_\alpha)))_{\alpha \in J} \quad (3.2)$$

can easily be checked; thus T is mono-topological. The other conditions for a mono-topological category are clearly met. $\square$

PROPOSITION 3.3. *Let $(G, \leq, t(\mathfrak{N})) \in |I\text{-FNOGr}|$. Then, for $x, a \in G$,*

- (i) *the mapping $L_a : G \rightarrow G$ (resp., $R_a : G \rightarrow G$) defined by $x \rightarrow ax$ (resp., $x \rightarrow xa$) is an order-preserving homeomorphism;*
- (ii) *the mapping $r : (G, t(\mathfrak{N})) \rightarrow (G, t(\mathfrak{N})) : x \rightarrow x^{-1}$ is an order-inverting homeomorphism.*

PROOF. The proof follows from [Definition 2.3](#). $\square$

LEMMA 3.4. *Let $(G, \leq, t(\mathfrak{N})) \in |I\text{-FNOGr}|$ and μ be an increasing (resp., decreasing) I -fuzzy set in G , then*

- (i) *$R_a^{-1}(\mu)$ is increasing (resp., decreasing);*
- (ii) *$r^{-1}(\mu)$ is decreasing (resp., increasing).*

PROOF. Let μ be an increasing I -fuzzy set.

(i) We have

$$\begin{aligned} R_a^{-1}(\mu)(x) &= \mu(R_a(x)) = \mu(xa) \leq \mu(ya) = \mu(R_a(y)), \\ R_a^{-1}(\mu)(x) &= \mu(R_a(x)) \leq \mu(R_a(y)) = R_a^{-1}(\mu)(y), \end{aligned} \quad (3.3)$$

that is, $R_a^{-1}(\mu)(x) \leq R_a^{-1}(\mu)(y)$ whenever $x \leq y$.

(ii) The mapping $r : G \rightarrow G$ is decreasing, then

$$r^{-1}(\mu)(x) = \mu(r(x)) = \mu(x^{-1}) \geq \mu(y^{-1}) = \mu(r(y)) = r^{-1}(\mu)(y), \quad (3.4)$$

that is, $r^{-1}(\mu)$ is decreasing. $\square$

PROPOSITION 3.5. *If $(G, \leq, t(\mathfrak{N})) \in |I\text{-FNOGr}|$ and μ is an increasing (resp., decreasing) open I -fuzzy set in G and $\rho \in I^G$, then the I -fuzzy set $(\mu \cdot \rho)$ is an increasing (resp., decreasing) open I -fuzzy set in G .*

PROOF. By [\[2, Proposition 1.10\]](#), an I -fuzzy set $(\mu \cdot \rho)$ is open. To prove the second part, let $x, y \in G$ with $x \leq y$ and μ be increasing I -fuzzy, then

$$\begin{aligned} \mu \cdot \rho(x) &= \sup_{x=s \cdot t} \mu(s) \wedge \rho(t) = \sup_{t \in G} \mu(xt^{-1}) \wedge \rho(t) \\ &= \sup_{t \in G} \mu(R_t^{-1}(x)) \wedge \rho(t) = \sup_{t \in G} R_t(\mu)(x) \wedge \rho(t). \end{aligned} \quad (3.5)$$

But the mapping $R_t : G \rightarrow G : x \rightarrow xt$ is increasing, then, by fixing $t \in G$, it follows that

$$\mu \cdot \rho(x) = \sup_{t \in G} R_t(\mu)(x) \wedge \rho(t) \leq \sup_{t \in G} R_t(\mu)(y) \wedge \rho(t) = \mu \cdot \rho(y), \quad (3.6)$$

that is, I -fuzzy set $\mu \cdot \rho$ is increasing. $\square$

PROPOSITION 3.6. *Let $(G, \leq, t(\mathfrak{N})) \in |I\text{-FNOGr}|$, then for all increasing (resp., decreasing) I -fuzzy set $\mu \in \mathfrak{N}(e)$ and for all $\varepsilon \in I_0$, there exists $\rho \in \mathfrak{N}(e)$ such that $i(\rho \cdot \rho) - \varepsilon \leq \mu$ (resp., $d(\rho \cdot \rho) - \varepsilon \leq \mu$).*

PROOF. Since $(G, t(\mathfrak{N}))$ is an I -fuzzy neighborhood group, then the continuity of the mapping $m : (G \times G, t(\mathfrak{N}) \times t(\mathfrak{N})) \rightarrow (G, t(\mathfrak{N})) : (x, y) \rightarrow xy$ is equivalent to the fact that $\forall \mu \in \mathfrak{N}(e)$ and $\forall \varepsilon \in I_0$, there exists $\rho \in \mathfrak{N}(e)$ (see [2, Proposition 2.5]) such that $\rho \cdot \rho - \varepsilon \leq \mu$. If we choose μ to be increasing then

$$\rho \cdot \rho \leq i(\rho \cdot \rho) \leq \mu + \varepsilon, \quad (3.7)$$

where $i(\rho \cdot \rho)$ is the smallest increasing I -fuzzy set containing $(\rho \cdot \rho)$ and it follows that $i(\rho \cdot \rho) - \varepsilon \leq \mu$ and this completes the proof. $\square$

4. Fuzzy quasi-uniformity on I -fuzzy neighborhood ordered groups. As given in [2], if $(G, \cdot)$ is a group, then we define

$$\begin{aligned} \mu_L : G \times G &\rightarrow I, \quad \text{where } \mu_L(x, y) = \mu(x^{-1}y), \\ \mu_R : G \times G &\rightarrow I, \quad \text{where } \mu_R(x, y) = \mu(yx^{-1}). \end{aligned} \quad (4.1)$$

If $(G, \cdot, t(\mathfrak{N}))$ is an I -fuzzy neighborhood group and $\mu \in \mathfrak{N}(e)$, then μ_L (resp., μ_R) is called the left (resp., right) I -fuzzy entourages associated with μ . We can easily note that the left (resp., right) I -fuzzy entourages μ_L (resp., μ_R) is not symmetric, if $x \neq y$, then $y^{-1}x \neq e \neq x^{-1}y$ and this implies that $\mu_L(x, y) = \mu(x^{-1}y) \neq \mu(y^{-1}x) = \mu_L(y, x)$. Also, $\mu_R(x, y) \neq \mu_R(y, x)$.

In the sequel, we use $\mathfrak{N}^i(e)$ (resp., $\mathfrak{N}^d(e)$) to denote the system of all increasing (resp., decreasing) I -fuzzy neighborhoods of e . From the above discussion we have the following easily established result.

THEOREM 4.1. *Let $(G, \leq, t(\mathfrak{N})) \in |I\text{-FNOGr}|$ and $\mathfrak{N}^i(e)$ (resp., $\mathfrak{N}^d(e)$) denote the system of all increasing (resp., decreasing) I -fuzzy neighborhoods of e . Then,*

- (i) *the family β_L (resp., β_R) = $\{\mu_L$ (resp., μ_R) : $\mu \in \mathfrak{N}^i(e)\}$ is a basis for the left (resp., right) I -fuzzy quasi-uniformity u_L (resp., u_R) on G ;*
- (ii) *the family β_L^{-1} (resp., β_R^{-1}) = $\{\mu_L^{-1}$ (resp., μ_R^{-1}) : $\mu \in \mathfrak{N}^d(e)\}$ is a basis for the conjugate left (resp., right) I -fuzzy quasi-uniformity U_L^{-1} (resp., U_R^{-1}) on G ;*
- (iii) *the family β_s = $\{\mu_L \wedge \mu_R : \mu \in \mathfrak{N}^i(e)\}$ is a basis for the two-sided I -fuzzy quasi-uniformity $(u_R \vee u_L)$ on G .*

We denote $U_L \vee U_L^{-1}$ (resp., $U_R \vee U_R^{-1}$) by U_L^* (resp., U_R^*). It is clear that U_L^* (resp., U_R^*) is an I -fuzzy uniformity on G called the left (resp., right) I -fuzzy uniformity generated by U_L (resp., U_R). Also, the two-sided I -fuzzy uniformity $U^* = U_R^* \vee U_L^*$ can be generated by the two-sided I -fuzzy quasi-uniformity $(U_R \vee U_L)$.

It is known that the entourages of the above I -fuzzy quasi-uniformities can generate an order on G by setting

$$x \leq^* y \iff (\forall z \in G) \mu_L(y, z) \leq \mu_L(x, z). \quad (4.2)$$

The partial order $\leq^*$ is said to be generated by the left I -fuzzy quasi-uniformity U_L .

DEFINITION 4.2. Let G_1, G_2 be groups and U_1, U_2 be quasi-uniformities on G_1 and G_2 , respectively. A mapping $f : G_1 \rightarrow G_2$ is called a quasi-uniform isomorphism if it is a quasi-uniform equivalence (see [Definition 2.1](#)) and group isomorphism.

PROPOSITION 4.3. Let $(G, \leq, t(\kappa)) \in |I\text{-FNOGr}|$ and let U_L be the associated left I -fuzzy quasi-uniformity on G , then

- (i) L_x (resp., R_x) : $(G, U_L) \rightarrow (G, U_L)$ is a quasi-uniform isomorphism;
- (ii) L_x (resp., R_x) : $(G, \leq, U_L^*) \rightarrow (G, \leq, U_L^*)$ is a uniformly order isomorphism.

PROOF. (i) It follows immediately from the formulas

$$(L_x \times L_x)^{-1}(\mu_L) = \mu_L. \quad (4.3)$$

(ii) The existence of the associated left I -fuzzy quasi-uniformity U_L which generate the I -fuzzy uniformity U_L^* and the order $\leq^*$ with $G(\leq^*) = G(\leq)$ and from (i) the proof becomes clear. $\square$

PROPOSITION 4.4. Let $(G, \leq, t(\kappa)) \in |I\text{-FNOGr}|$ and U_L (resp., U_R) be the associated left (resp., right) I -fuzzy quasi-uniformity on G , then

- (i) the mapping $r : (G, U_L) \rightarrow (G, U_R)$ is a quasi-uniform isomorphism;
- (ii) the mapping $r : (G, \leq, U_L^*) \rightarrow (G, \leq, U_R^*)$ is a uniform order-isomorphism.

PROOF. (i) The mapping $r : G \rightarrow G$ is a group isomorphism. But for $\mu_R \in U_R$, we have that $(r \times r)^{-1}(\mu_R)(x, y) = \mu_R(r(x), r(y)) = \mu_R(X^{-1}, Y^{-1}) = \mu(Y^{-1}X)$, that is, $(r \times r)^{-1}(\mu_R) = \tilde{\mu}_L$. And this means that $r : (G, U_L) \rightarrow (G, U_R)$ is a quasi-uniform equivalence and so it is quasi-uniform isomorphism.

(ii) This can be proven by [Definition 2.1](#) and part (i) and this completes the proof. $\square$

We omit the proof of the following easily established proposition.

PROPOSITION 4.5. Let $(G, \leq, t(\kappa))$ and $(G', \leq', t(\kappa')) \in |I\text{-FNOGr}|$ and let U_L, U'_L be the associated left I -fuzzy quasi-uniformities on G and G' , respectively. Then, the order-preserving homeomorphism $f : G \rightarrow G'$ is uniformly order-mapping.

REFERENCES

- [1] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1990.
- [2] T. M. G. Ahsanullah, *On fuzzy neighborhood groups*, J. Math. Anal. Appl. **130** (1988), no. 1, 237-251.
- [3] A. A. Allam, S. A. Hussein, and K. El-Saady, *Fuzzy syntopogenous structures and order*, Fuzzy Sets and Systems **63** (1994), no. 1, 91-98.
- [4] M. Y. Bakier and K. El-Saady, *Fuzzy topological ordered vector spaces. I*, Fuzzy Sets and Systems **54** (1993), no. 2, 213-220.
- [5] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182-190.
- [6] J. A. Goguen, *The fuzzy Tychonoff theorem*, J. Math. Anal. Appl. **43** (1973), 734-742.
- [7] U. Höhle and A. P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, Mathematics of Fuzzy Sets, Fuzzy Sets Ser., vol. 3, Kluwer, Massachusetts, 1999, pp. 123-272.
- [8] A. K. Katsaras, *Ordered fuzzy topological spaces*, J. Math. Anal. Appl. **84** (1981), no. 1, 44-58.

- [9] ———, *Fuzzy neighborhood structures and fuzzy quasi-uniformities*, Fuzzy Sets and Systems **29** (1989), no. 2, 187–199.
- [10] A. S. Mashhour, A. A. Allam, and K. El-Saady, *Fuzzy uniform structures and order*, J. Fuzzy Math. **2** (1994), no. 1, 57–67.

KAMEL EL-SAADY: MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, SOUTH VALLEY UNIVERSITY, QENA 83523, EGYPT

E-mail address: el-saady@lycos.com

URL: <http://at.yorku.ca/h/a/a/a/43.htm>

M. Y. BAKIER: MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, ASSIUT UNIVERSITY, ASSIUT, EGYPT