STEADY VORTEX FLOWS OBTAINED FROM A CONSTRAINED VARIATIONAL PROBLEM

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We prove the existence of steady two-dimensional ideal vortex flows occupying the first quadrant and containing a bounded vortex; this is done by solving a constrained variational problem. Kinetic energy is maximized subject to the vorticity, being a rearrangement of a prescribed function and subject to a linear constraint.

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1. Introduction. In this paper, we prove the existence of steady two-dimensional ideal vortex flows occupying the first quadrant, Π_+ , containing a bounded vortex. This is done by solving a constrained variational problem. Such a flow will be described by a stream function $\hat{\psi} : \Pi_+ \rightarrow \mathbb{R}$. At infinity we will have $\hat{\psi} \rightarrow -\lambda x_1 x_2$ which is the stream function for an irrotational flow with velocity field $-\lambda(x_1, x_2)$, where λ is not known a priori. The vorticity is given by $-\Delta\hat{\psi}$, where Δ is the Laplacian, and $-\Delta\hat{\psi}$ vanishes outside a bounded region. It will be shown that $\hat{\psi}$ satisfies the following semilinear partial differential equation:

$$-\Delta\hat{\psi} = \phi \circ \hat{\psi}, \quad (1.1)$$

almost everywhere in Π_+ for ϕ an increasing function, unknown a priori. In our result the vorticity function $\zeta (= -\Delta\hat{\psi})$ is a rearrangement of a prescribed nonnegative, nontrivial function ζ_0 having bounded support, and the *impulse*, $\mathfrak{I}$, given by

$$\mathfrak{I}(\zeta) := \int_{\Pi_+} x_1 x_2 \zeta, \quad (1.2)$$

is a prescribed positive number. We prove that the variational problem, $P(I)$ (see Section 2), is solvable provided that I is sufficiently large. Since the domain of interest Π_+ is unbounded, we first consider the problem over bounded sets, $\Pi_+(\xi, \eta)$, where Burton's theory, related to constrained variational problems, can be applied. We then show that the maximizers are the same for all sufficiently large $\Pi_+(\xi, \eta)$.

Problems of this kind have been investigated by many authors; in particular we cite Badiani [1], Burton [2], Burton and Emamizadeh [3], Elcrat and Miller [7], Emamizadeh [8, 9, 10, 11], Nycander [14] for theoretical results and Elcrat et al. [5, 6] for numerical.

2. Notation, definitions, and statement of the results. Henceforth p denotes a real number in $(2, \infty)$. The first quadrant is denoted Π_+ . Generic points in $\mathbb{R}^2$ are denoted by x, y , and so forth. Thus, for example, $x = (x_1, x_2)$. For $x \in \mathbb{R}^2$, $\bar{x}$, $\underline{x}$, and $\underline{\underline{x}}$ denote

the reflections of x about the x_1 -axis, x_2 -axis, and the origin, respectively. For positive η and ξ we set

$$\begin{aligned}\Pi_+(\eta) &:= \{x \in \Pi_+ \mid x_1 x_2 < \eta\}, \\ \Pi_+(\xi, \eta) &:= \{x \in \Pi_+ \mid x_1 x_2 < \eta, \max\{x_1, x_2\} < \xi\}.\end{aligned}\quad (2.1)$$

For $A \subset \mathbb{R}^2$, $|A|$ denotes the two-dimensional Lebesgue measure of A .

For a measurable function ζ , the strong support of ζ is defined by

$$\text{supp}(\zeta) = \{x \in \text{dom}(\zeta) \mid \zeta(x) > 0\}.\quad (2.2)$$

To define the rearrangement class needed for our variational problem, we fix a non-negative, nontrivial function $\zeta_0 \in L^p(\mathbb{R}^2)$ which vanishes outside a bounded set. In addition, we assume that

$$|\text{supp}(\zeta_0)| = \pi a^2,\quad (2.3)$$

for some $a > 0$. We say that ζ is a rearrangement of ζ_0 if and only if

$$|\{x \mid \zeta(x) \geq \alpha\}| = |\{x \mid \zeta_0(x) \geq \alpha\}|,\quad (2.4)$$

for every positive α . The set of rearrangements of ζ_0 which vanish outside bounded subsets of Π_+ is denoted by $\mathcal{F}$. The set of functions $\zeta \in \mathcal{F}$ that satisfy $\mathfrak{I}(\zeta) = I$, for some $I > 0$, is denoted by $\mathcal{F}(I)$; and the set of functions in $\mathcal{F}(I)$ that vanish outside $\Pi_+(\xi, \eta)$ is denoted by $\mathcal{F}(\xi, \eta, I)$; to ensure that $\mathcal{F}(\xi, \eta, I) \neq \emptyset$, we present the following definition: let $I_1 := \mathfrak{I}(\zeta_0^*)$, where ζ_0^* is the Schwarz-symmetrisation of ζ_0 , and assume that $I > I_1$; we say that $\Pi_+(\xi, \eta)$ satisfies the hypothesis $\mathcal{H}(I)$ if the following two conditions hold:

$$\xi \geq \eta^{1/2},\quad (2.5)$$

$$\eta \geq 4 \max\{a^2, l(I)\},\quad (2.6)$$

where $l(I) := (I - I_1) / \|\zeta_0\|_1$. Now it is immediate that if $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$, for $I > I_1$, then $\mathcal{F}(\xi, \eta, I) \neq \emptyset$. Indeed if we set $t = l(I)^{1/2}$, then $(\zeta_0^*)_t(x) := \zeta_0^*(x_1 - t, x_2 - t)$ belongs to $\mathcal{F}(\xi, \eta, I)$.

The Green's function for $-\Delta$ on Π_+ with homogeneous Dirichlet boundary conditions is denoted by G_+ , hence

$$G_+(x, y) = \frac{1}{2\pi} \log \frac{|x - \overline{y}| |x - \underline{y}|}{|x - y| |x - \overline{\underline{y}}|}.\quad (2.7)$$

Next we define the integral operator K_+

$$K_+ \zeta(x) = \int_{\Pi_+} G_+(x, y) \zeta(y) dy,\quad (2.8)$$

for measurable functions ζ on $\mathbb{R}^2$, whenever the integral exists. The *Kinetic energy* is defined by

$$\Psi(\zeta) = \int_{\Pi_+} \zeta K_+ \zeta,\quad (2.9)$$

whenever the integral exists.

In this paper, we are concerned with constrained variational problems which are defined as follows. For $I > I_1$,

$$P(I) : \sup_{\zeta \in \mathcal{F}(I)} \Psi(\zeta); \quad (2.10)$$

and the corresponding solution set is denoted by $\Sigma(I)$. If $I > I_1$ and $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$, then we define the truncated variational problem

$$P(\xi, \eta, I) : \sup_{\zeta \in \mathcal{F}(\xi, \eta, I)} \Psi(\zeta), \quad (2.11)$$

with the solution set $\Sigma(\xi, \eta, I)$.

We are now in a position to state our main result.

THEOREM 2.1. *There exists $I_0 > 0$ such that if $I > I_0$ then $P(I)$ has a solution, that is, $\Sigma(I) \neq \emptyset$; if ζ is a solution and $\psi := K_+ \zeta$ then the following semilinear elliptic partial differential equation holds*

$$-\Delta \psi = \phi \circ (\psi - \lambda x_1 x_2), \quad \text{a.e. in } \Pi_+, \quad (2.12)$$

where ϕ is an increasing function and $\lambda > 0$, both unknown a priori. Furthermore, I_0 can be chosen to ensure that the vortex core, the strong support of ζ , avoids $\partial \Pi_+$.

3. Preliminary results. We present some lemmas that are used in the proof of [Theorem 2.1](#). We begin by stating a lemma from Burton's theory, see for example, Burton and McLeod [4].

LEMMA 3.1. *Let Ω be a nonempty open set in $\mathbb{R}^n$. Let $1 \leq p < \infty$ and p^* denote the conjugate exponent of p . For $\zeta \in L^p(\mu)$ let $\mathcal{F}(\Omega)$ denote the set of rearrangements of ζ on Ω . Let*

$$\mathcal{L} := \sum_{1 \leq |\alpha| \leq m} \mathcal{A}^\alpha(x) \mathcal{D}^\alpha \quad (3.1)$$

be an m th-order linear partial differential operator, whose coefficients $\mathcal{A}^\alpha$ are finite-valued measurable functions on Ω , having no 0th-order term, and suppose that there exists a compact, symmetric, positive linear operator $K : L^p(\Omega) \rightarrow L^{p^}(\Omega)$ such that if $\zeta \in L^p(\Omega)$, then $K\zeta \in L^{p^*}(\Omega) \cap W_{\text{loc}}^{m,1}(\Omega)$ and $\mathcal{L}K\zeta = \zeta$ almost everywhere in Ω . Define*

$$\Psi(\hat{\zeta}) := \int_{\Omega} \zeta K \zeta, \quad \zeta \in L^p(\Omega). \quad (3.2)$$

Let $w \in L^{p^}(\Omega) \cap W_{\text{loc}}^{m,1}(\Omega)$ be such that $\mathcal{L}w$ is essentially constant, and define*

$$\mathcal{T}(\zeta) := \int_{\Omega} w \zeta, \quad \zeta \in L^p(\Omega). \quad (3.3)$$

Let $b \in \mathbb{R}$. Then

(i) *If $b \in \mathcal{T}(\mathcal{F}(\Omega))$ then*

$$\sup \hat{\Psi}(\mathcal{T}^{-1}(b) \cap \mathcal{F}(\Omega)) = \sup \hat{\Psi}(\mathcal{T}^{-1}(b) \cap \overline{\mathcal{F}(\Omega)w}), \quad (3.4)$$

and the supremum is attained by at least one element of $\mathcal{T}^{-1}(b) \cap \mathcal{F}(\Omega)$.

(ii) If b is, relatively, interior to $\mathcal{T}(\mathcal{F}(\Omega))$, and if $\bar{\zeta}$ is a maximizer for Ψ relative to $\mathcal{T}^{-1}(b) \cap \mathcal{F}(\Omega)$, then there exist scalar λ and an increasing function ϕ such that

$$\bar{\zeta} = \phi \circ (K\bar{\zeta} + \lambda w), \quad \text{a.e. in } \Omega. \quad (3.5)$$

REMARK 3.2. It is clear that if $I > I_1$ and $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$ then, by [Lemma 3.1\(i\)](#), $\Sigma(\xi, \eta, I) \neq \emptyset$.

Before stating the next result we give the following definition: for $I > I_1$,

$$\sigma(I) := \inf \{ \Psi(\zeta) \mid \zeta \in \Sigma(\xi, \eta, I), \text{ for some } \Pi_+(\xi, \eta) \text{ satisfying } \mathcal{H}(I) \}. \quad (3.6)$$

We point out that $\sigma(I) = \Psi(\hat{\zeta})$ for some $\hat{\zeta} \in \Sigma(\xi_0, \eta_0, I)$, where $\Pi_+(\xi_0, \eta_0)$ is the minimal region that satisfies $\mathcal{H}(I)$.

LEMMA 3.3. Let σ be as defined in (3.6), then

$$\lim_{I \rightarrow \infty} \sigma(I) = \infty. \quad (3.7)$$

PROOF. Let $I > I_1$ and set $t = l(I)^{1/2}$. If $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$, then $(\zeta_0^*)_t \in \mathcal{F}(\xi, \eta, I)$ and therefore, according to the last remark, we have

$$\sigma(I) \geq \Psi((\zeta_0^*)_t). \quad (3.8)$$

Now applying same method as in Burton [2, Lemma 12], we obtain $\Psi((\zeta_0^*)_t) \geq k \log t$, for all sufficiently large t , hence large I . Thus our claim is done. $\square$

Let $I > I_1$ and $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$. We set

$$\begin{aligned} M(\xi, \eta, I) &:= \{ (\zeta, \phi, \lambda) \mid \zeta \in \Sigma(\xi, \eta, I) \text{ for some } \phi, \lambda \in \mathbb{R} \\ &\text{such that } \zeta = \phi \circ (K_+ \zeta - \lambda x_1 x_2) \text{ a.e. in } \Pi_+(\xi, \eta) \}. \end{aligned} \quad (3.9)$$

Note that under the conditions imposed on ξ, η, I and in view of [Lemma 3.1\(ii\)](#) the set $M(\xi, \eta, I)$ is nonempty. The following two inequalities are standard, see Burton [2]

$$|K_+ \zeta(x)| \leq N \min \{x_1, x_2\}, \quad (3.10)$$

$$|\nabla K_+ \zeta(x)| \leq N, \quad (3.11)$$

for every $x \in \Pi_+$ and every $\zeta \in \mathcal{F}$, where N is a universal constant.

LEMMA 3.4. For $I > I_1$ we define

$$\begin{aligned} \Lambda(I) &:= \sup \{ \lambda \mid (\zeta, \phi, \lambda) \in M(\xi, \eta, I) \text{ for some } \zeta, \phi \\ &\text{and some } \Pi_+(\xi, \eta) \text{ satisfying } \mathcal{H}(I) \}. \end{aligned} \quad (3.12)$$

Then, $\limsup_{I \rightarrow \infty} \Lambda(I) \leq 0$.

PROOF. Assume that the assertion of the lemma is not true and seek a contradiction. Hence, to this end we suppose that there exists $\beta \in (0, \infty]$ such that $\limsup_{I \rightarrow \infty} \Lambda(I) = \beta$. Hence there exists $\Lambda > 0$ such that the set

$$S := \{ I \mid \Lambda(I) > \Lambda \} \quad (3.13)$$

is unbounded. Consider $I \in S$, then from the definition of $\Lambda(I)$, there exists $(\zeta, \phi, \lambda) \in$

$M(\xi, \eta, I)$ such that $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$ and $\Lambda(I) \geq \lambda > \Lambda > 0$. Observe that by taking I sufficiently large we can ensure the existence of ξ_1 such that $\Pi_+(\xi, \eta) \supseteq \Pi_+(\xi_1, a)$ and $|\Pi_+(\xi_1, a)| \geq \pi a^2 = |\text{supp}(\zeta)|$. Now define the set

$$U := \{x \in \Pi_+(\xi, \eta) \mid K_+\zeta(x) - \lambda x_1 x_2 \geq -\lambda a\}. \quad (3.14)$$

Then, $\Pi_+(\xi_1, a) \subseteq U$ and $|U| \geq |\text{supp}(\zeta)|$. Since ζ is essentially an increasing function of $K_+\zeta - \lambda x_1 x_2$ on $\Pi_+(\xi, \eta)$ we deduce that $\text{supp}(\zeta) \subseteq U$.

Next we show that there exists a constant $C > 0$, independent of $I \in S$, such that for $x \in \text{supp}(\zeta)$ we have $x_1 x_2 \leq C$. From (3.10) we observe that for a sufficiently large $k > 0$

$$K_+\zeta(x) \leq \frac{\Lambda}{2} x_1 x_2, \quad (3.15)$$

for all $\zeta \in \mathcal{F}$ and all x for which $\min\{x_1, x_2\} \geq k$. We next define

$$\begin{aligned} S_1 &:= \{x \in \Pi_+ \mid \min\{x_1, x_2\} \geq k\}, \\ S_2 &:= \{x \in \Pi_+ \mid \min\{x_1, x_2\} < k, x_1 < \alpha, x_2 < \alpha\}, \\ S_3 &:= \{x \in \Pi_+ \mid \min\{x_1, x_2\} < k, \max\{x_1, x_2\} \geq \alpha\}, \end{aligned} \quad (3.16)$$

where $\alpha := \max\{2N/\lambda, k\}$. First consider $x \in \text{supp}(\zeta) \cap S_1$; then

$$-\lambda a \leq K_+\zeta(x) - \lambda x_1 x_2 \leq \frac{\Lambda}{2} x_1 x_2 - \lambda x_1 x_2 < -\frac{\lambda}{2} x_1 x_2, \quad (3.17)$$

where the first inequality follows from $\text{supp}(\zeta) \subseteq U$ and the second one from (3.15); whence $x_1 x_2 < 2a$. Next, consider $x \in \text{supp}(\zeta) \cap S_2$; then we have

$$x_1 x_2 < \alpha^2 \leq \left(\max\left\{\frac{2N}{\Lambda}, k\right\} \right)^2, \quad (3.18)$$

since $\lambda > \Lambda$. Finally, consider $x \in \text{supp}(\zeta) \cap S_3$; then an application of (3.10) yields that

$$\begin{aligned} -\lambda a &\leq K_+\zeta(x) - \lambda x_1 x_2 \\ &\leq N \min\{x_1, x_2\} - \lambda x_1 x_2 \\ &= \frac{N}{\alpha} \alpha \min\{x_1, x_2\} - \lambda x_1 x_2 \\ &\leq \frac{N}{\alpha} x_1 x_2 - \lambda x_1 x_2 \\ &\leq N \frac{\lambda}{2N} x_1 x_2 - \lambda x_1 x_2 \\ &= -\frac{\lambda}{2} x_1 x_2, \end{aligned} \quad (3.19)$$

hence $x_1 x_2 \leq 2a$. Therefore, from above argument, it is clear that a constant $C > 0$, as required, exists. This, in turn, implies that

$$I = \mathfrak{I}(\zeta) := \int_{\Pi_+} x_1 x_2 \zeta \leq C \|\zeta_0\|_1. \quad (3.20)$$

Thus S is bounded, which is a contradiction. Hence, the proof of Lemma 3.4. $\square$

LEMMA 3.5. *For $I > I_1$ we define*

$$A(I) := \inf \left\{ \operatorname{ess\,inf}_{x \in \operatorname{supp}(\zeta)} (K_+ \zeta(x) - \lambda x_1 x_2) \mid (\zeta, \phi, \lambda) \in M(\xi, \eta, I) \right. \\ \left. \text{for some } \Pi_+(\xi, \eta) \text{ and some } \phi \right\}, \quad (3.21)$$

where $\Pi_+(\xi, \eta)$ is to satisfy $\mathcal{H}(I)$. Then, $\liminf_{I \rightarrow \infty} A(I) \geq 0$.

PROOF. Fix $\epsilon > 0$. By definition of $A(I)$ there exists $\Pi_+(\xi, \eta)$, satisfying $\mathcal{H}(I)$, and $(\zeta, \phi, \lambda) \in M(\xi, \eta, I)$ such that

$$A(I) + \epsilon \geq \operatorname{ess\,inf}_{x \in \operatorname{supp}(\zeta)} (K_+ \zeta(x) - \lambda x_1 x_2). \quad (3.22)$$

Note that by increasing I , the size of $\Pi_+(\xi, \eta)$ increases as well, hence there is no loss of generality if we assume that $\Pi_+(\xi, \eta)$ contains the square $D := [0, 2a] \times [0, 2a]$, since I will eventually tend to infinity. For $x \in D$ we have

$$K_+ \zeta(x) - \lambda x_1 x_2 \geq -4a^2 \Lambda(I)^+, \quad (3.23)$$

where $\Lambda(I)^+$ denotes the positive part of $\Lambda(I)$, since $K_+ \zeta$ is nonnegative. From this, we infer that

$$D \subseteq \{x \in \Pi_+(\xi, \eta) \mid K_+ \zeta(x) - \lambda x_1 x_2 \geq -4a^2 \Lambda(I)^+\}. \quad (3.24)$$

Hence

$$|\{x \in \Pi_+(\xi, \eta) \mid K_+ \zeta(x) - \lambda x_1 x_2 \geq -4a^2 \Lambda(I)^+\}| > |\operatorname{supp}(\zeta)|, \quad (3.25)$$

since $4a^2 > |\operatorname{supp}(\zeta)|$. Since ζ is essentially an increasing function of $K_+ \zeta - \lambda x_1 x_2$ on $\Pi_+(\xi, \eta)$, we then deduce that

$$\operatorname{supp}(\zeta) \subseteq \{x \in \Pi_+(\xi, \eta) \mid K_+ \zeta(x) - \lambda x_1 x_2 \geq -4a^2 \Lambda(I)^+\}, \quad (3.26)$$

hence, by applying (3.22), we obtain $A(I) + \epsilon \geq -4a^2 \Lambda(I)^+$. Therefore, from Lemma 3.4 we have

$$\liminf_{I \rightarrow \infty} A(I) + \epsilon \geq 0. \quad (3.27)$$

Since ϵ was arbitrary, we derive the desired conclusion. $\square$

The next two results can be proved similarly to Burton [2, Lemmas 8 and 9]; they bear some resemblance to Pohazev-type identities proved in Friedman and Turkington [12] for 3-dimensional vortex rings. We add that, contrary to Burton [2], we can give a direct proof, using the weak divergence theorem (see, e.g., Grisvard [13]) for Lemma 3.6 below without referring to any density theorems.

LEMMA 3.6. *Let $2 < p < \infty$, let $\zeta \in L^p(\Pi_+)$ have bounded support, and let $\psi := K_+ \zeta$. Then*

$$\int_{\Pi_+} (x \cdot \nabla \psi) \zeta = 0. \quad (3.28)$$

LEMMA 3.7. *Let $2 < p < \infty$, let $\zeta \in L^p(\Pi_+)$ be nonnegative, nontrivial and vanish outside the square $D(\xi) := [0, \xi] \times [0, \xi]$, for some $\xi > 0$. Let $\lambda \in \mathbb{R}$, and let $\psi := K_+ \zeta - \lambda x_1 x_2$. Suppose that $\zeta = \phi \circ \psi$ almost everywhere in $D(\xi)$ for some increasing function ϕ , and that ϕ has a nonnegative indefinite integral F . Then*

$$2 \int_{D(\xi)} F \circ \psi - 2\lambda \int_{D(\xi)} x_1 x_2 \zeta = \int_{\partial D(\xi)} (F \circ \psi)(x \cdot \vec{n}), \quad (3.29)$$

where $\vec{n}$ is the outward unit normal, and consequently

$$\int_{D(\xi)} F \circ \psi \geq \lambda \int_{D(\xi)} x_1 x_2 \zeta. \quad (3.30)$$

If additionally $F(s) = 0$ for some $s \leq \beta$, then

$$\int_{D(\xi)} \zeta K_+ \zeta \geq 2\lambda \int_{D(\xi)} x_1 x_2 \zeta + \beta \|\zeta\|_1. \quad (3.31)$$

LEMMA 3.8. *For $I > I_1$ we define*

$$\mu(I) := \inf \left\{ \sup_{x \in \Pi_+(\xi, \eta)} (K_+ \zeta(x) - \lambda x_1 x_2) \mid (\zeta, \phi, \lambda) \in M(\xi, \eta, I) \right. \\ \left. \text{for some } \Pi_+(\xi, \eta) \text{ satisfying } \mathcal{H}(I), \text{ and some } \phi \right\}. \quad (3.32)$$

Then $\lim_{I \rightarrow \infty} \mu(I) = \infty$.

PROOF. It clearly suffices to show that

$$\liminf_{I \rightarrow \infty} \mu(I) = \infty. \quad (3.33)$$

Let $I > I_1$ and consider $(\zeta, \phi, \lambda) \in M(\xi, \eta, I)$ for some $\Pi_+(\xi, \eta)$ satisfying $\mathcal{H}(I)$. Since $K_+ \zeta(x) - \lambda x_1 x_2 \geq \Lambda(I)$ for almost every $x \in \text{supp}(\zeta)$, we may assume that $\phi(s) = 0$ for $-\infty < s < A(I)$. Now write

$$F(s) = \int_{-\infty}^s \phi, \quad (3.34)$$

for all s in the domain of ϕ . Now, by Lemma 3.7, we have

$$\begin{aligned} \int_{\Pi_+} \zeta (K_+ \zeta - \lambda x_1 x_2) &= 2\Psi(\zeta) - \lambda I \\ &= \frac{1}{2} (2\Psi(\zeta) - 2\lambda I - A(I) \|\zeta\|_1) + \Psi(\zeta) + \frac{1}{2} A(I) \|\zeta\|_1 \\ &\geq \Psi(\zeta) + \frac{1}{2} A(I) \|\zeta\|_1 \\ &\geq \sigma(I) + \frac{1}{2} A(I) \|\zeta\|_1. \end{aligned} \quad (3.35)$$

Hence

$$\sup_{\Pi_+(\xi, \eta)} (K_+ \zeta(x) - \lambda x_1 x_2) \geq \frac{\sigma(I)}{\|\zeta\|_1} + \frac{1}{2} A(I). \quad (3.36)$$

Therefore

$$\mu(I) \geq \frac{\sigma(I)}{\|\zeta\|_1} + \frac{1}{2}A(I). \quad (3.37)$$

Thus by applying Lemmas 3.4 and 3.5 we obtain (3.33). $\square$

LEMMA 3.9. *There exists $I_2 > I_1$ such that*

$$A(I) \geq aN, \quad I \geq I_2. \quad (3.38)$$

PROOF. By Lemma 3.7 there exists $I_2 > I_1$ such that

$$\mu(I) \geq 7aN, \quad I \geq I_2; \quad (3.39)$$

moreover by taking I_2 sufficiently large we can ensure that if $I \geq I_2$, then any $\Pi_+(\xi, \eta)$ satisfying $\mathcal{H}(I)$, also satisfies

$$\left| \Pi_+(\xi, \eta) \setminus \Pi_+\left(\xi, \frac{\eta}{2}\right) \right| \geq \pi a^2. \quad (3.40)$$

To see it, observe that in general we have

$$|\Pi_+(\xi, \eta)| = \eta \left(1 + \log \frac{\xi^2}{\eta} \right), \quad (3.41)$$

for any $\Pi_+(\xi, \eta)$ satisfying (2.5); therefore

$$\left| \Pi_+(\xi, \eta) \setminus \Pi_+\left(\xi, \frac{\eta}{2}\right) \right| \geq \frac{1}{2}(1 - \log 2)\eta. \quad (3.42)$$

Hence, in view of (2.6), for sufficiently large I we derive (3.40). Now, fix $I \geq I_2$ and consider $(\zeta, \phi, \lambda) \in M(\xi, \eta, I)$ for some $\Pi_+(\xi, \eta)$ satisfying $\mathcal{H}(I)$. Since $K_+\zeta - \lambda x_1 x_2 \in C(\overline{\Pi_+(\xi, \eta)})$, it attains its maximum at, say, $z \in \overline{\Pi_+(\xi, \eta)}$. Now from the definition of $\mu(I)$ and (3.10) we infer that

$$\mu(I) \leq K_+\zeta(z) - \lambda z_1 z_2 \leq N \min\{z_1, z_2\} - \lambda z_1 z_2; \quad (3.43)$$

and applying (3.39), we obtain

$$7aN \leq N \min\{z_1, z_2\} - \lambda z_1 z_2. \quad (3.44)$$

Clearly, if $\lambda \geq 0$ we obtain $\min\{z_1, z_2\} \geq 7a$. If $\lambda < 0$, then

$$7aN \leq N \min\{z_1, z_2\} - \lambda \eta, \quad (3.45)$$

or

$$N \min\{z_1, z_2\} \geq 7aN + \lambda \eta. \quad (3.46)$$

Now we consider two cases.

CASE 1. When $\lambda\eta \geq -2aN$, then $N \min\{z_1, z_2\} \geq 5aN$, hence $\min\{z_1, z_2\} \geq 5a$. Therefore, when $\lambda \geq 0$, or when $\lambda < 0$, and $\lambda\eta \geq -2aN$ we find that $\min\{z_1, z_2\} \geq 5a$. Thus $\Pi_+(\xi, \eta)$ must contain at least a quadrant of $B_{4a}(z)$, denoted by Q . For $x \in Q$, by the mean value inequality, we have

$$\begin{aligned}
 K_+\zeta(x) - \lambda x_1 x_2 &\geq K_+\zeta(x) \\
 &\geq K_+\zeta(z) - 4aN \\
 &= K_+\zeta(z) - \lambda z_1 z_2 - 4aN + \lambda z_1 z_2 \\
 &\geq \mu(I) - 4aN + \lambda z_1 z_2 \\
 &\geq \mu(I) - 4aN + \lambda\eta \\
 &\geq 7aN - 4aN - 2aN \\
 &= aN.
 \end{aligned} \tag{3.47}$$

This means that

$$Q \subseteq \{x \in \Pi_+(\xi, \eta) \mid K_+\zeta(x) - \lambda x_1 x_2 \geq aN\}. \tag{3.48}$$

CASE 2. When $\lambda\eta < -2aN$, then for $x \in \Pi_+(\xi, \eta) \setminus \Pi_+(\xi, \eta/2)$ we have

$$\begin{aligned}
 K_+\zeta(x) - \lambda x_1 x_2 &\geq -\lambda x_1 x_2 > -\frac{\lambda\eta}{2}; \\
 \Pi_+(\xi, \eta) \setminus \Pi_+\left(\xi, \frac{\eta}{2}\right) &\subset \{x \in \Pi_+(\xi, \eta) \mid K_+\zeta(x) - \lambda x_1 x_2 \geq aN\}.
 \end{aligned} \tag{3.49}$$

From (3.40) and the fact that $|Q| = 4\pi a^2$, we infer that

$$|\{x \in \Pi_+(\xi, \eta) \mid K_+\zeta(x) - \lambda x_1 x_2 \geq aN\}| \geq |\text{supp}(\zeta)|. \tag{3.50}$$

Since ζ is an increasing function of $K_+\zeta - \lambda x_1 x_2$ on $\Pi_+(\xi, \eta)$, we derive

$$\text{supp}(\zeta) \subseteq \{x \in \Pi_+(\xi, \eta) \mid K_+\zeta(x) - \lambda x_1 x_2 \geq aN\}, \tag{3.51}$$

modulo a set of zero measure, from which we obtain (3.38). $\square$

LEMMA 3.10. *Let $b > 0$, let $2 < p < \infty$ and $0 < \gamma < 1$. Then there exist positive constants M_1, M_2 , and M_3 such that*

$$\begin{aligned}
 K_+\zeta(x) &\leq M_1(x_1 x_2)^{-1} \mathfrak{I}(\zeta) + M_2(x_1 x_2)^{-1} \mathfrak{I}(\zeta) \log \frac{25x_1 x_2}{4|x|} \\
 &\quad + M_3(x_1 x_2)^{-\gamma} \mathfrak{I}(\zeta)^\gamma \|\zeta\|_p^{1-\gamma},
 \end{aligned} \tag{3.52}$$

for every $x \in \Pi_+$ such that $\min\{x_1 x_2\} \geq b/2$ and every nonnegative $\zeta \in L^p(\Pi_+)$ that vanishes outside a set of measure πb^2 .

PROOF. Fix $x \in \Pi_+$ such that $v := \min\{x_1 x_2\} \geq b/2$. For $y \in \Pi_+$ we define

$$\alpha := |x - \overline{y}|, \quad \beta := |x - \underline{y}|, \quad \rho := |x - y|, \quad \delta := |x - \underline{y}|. \tag{3.53}$$

Thus

$$\begin{aligned}
 K_+ \zeta(x) &= \frac{1}{2\pi} \int_{\Pi_+} \log \frac{\alpha\beta}{\rho\delta} \zeta(y) dy \\
 &= \frac{1}{2\pi} \int_{B_{\nu/2}(x)} \log \frac{\alpha\beta}{\rho\delta} \zeta(y) dy \\
 &\quad + \frac{1}{2\pi} \int_{\Pi_+ \setminus B_{\nu/2}(x)} \log \frac{\alpha\beta}{\rho\delta} \zeta(y) dy,
 \end{aligned} \tag{3.54}$$

where $B_{\nu/2}(x)$ denotes the ball centered at x with radius ν . From the identity

$$\alpha^2 \beta^2 = \rho^2 \delta^2 + 16x_1 x_2 y_1 y_2, \tag{3.55}$$

we obtain

$$\begin{aligned}
 \int_{\Pi_+ \setminus B_{\nu/2}(x)} \log \frac{\alpha\beta}{\rho\delta} \zeta(y) dy &= \frac{1}{2} \int_{\Pi_+ \setminus B_{\nu/2}(x)} \log \left(1 + \frac{16x_1 x_2 y_1 y_2}{\rho^2 \delta^2} \right) \zeta(y) dy \\
 &\leq 8x_1 x_2 \int_{\Pi_+ \setminus B_{\nu/2}(x)} \frac{y_1 y_2}{\rho^2 \delta^2} \zeta(y) dy \\
 &\leq \frac{32x_1 x_2}{\nu^2 |x|^2} \int_{\Pi_+ \setminus B_{\nu/2}(x)} y_1 y_2 \zeta(y) dy \\
 &\leq 32(x_1 x_2)^{-1} \mathfrak{I}(\zeta),
 \end{aligned} \tag{3.56}$$

where the first inequality follows from the fact that $\log(1+x) \leq x$, for $x \geq 0$. To estimate $\int_{B_{\nu/2}(x)} \log(\alpha\beta\rho^{-1}\delta^{-1})\zeta(y)dy$, we note that for $y \in B_{\nu/2}(x)$ we have

$$\alpha \leq |x - \bar{x}| + |\bar{x} - \bar{y}| = 2x_2 + \rho < \frac{5}{2}x_2. \tag{3.57}$$

Similarly, $\beta < 5/2x_1$. Therefore

$$\begin{aligned}
 \int_{B_{\nu/2}(x)} \log \frac{\alpha\beta}{\rho\delta} \zeta(y) dy &\leq \int_{B_{\nu/2}(x)} \log \frac{25x_1 x_2}{4\rho|x|} \zeta(y) dy \\
 &= \log \frac{25x_1 x_2}{4|x|} \int_{B_{\nu/2}(x)} \zeta(y) dy \\
 &\quad + \int_{B_{\nu/2}(x)} \log \frac{1}{\rho} \zeta(y) dy.
 \end{aligned} \tag{3.58}$$

Observe that for $y \in B_{\nu/2}(x)$ we have $y_1 y_2 \geq x_1 x_2 / 4$, hence

$$\int_{B_{\nu/2}(x)} \zeta(y) dy \leq 4(x_1 x_2)^{-1} \int_{B_{\nu/2}(x)} y_1 y_2 \zeta(y) dy \leq 4(x_1 x_2)^{-1} \mathfrak{I}(\zeta). \tag{3.59}$$

On the other hand, if we let $\hat{\zeta}$ denote the Schwarz-symmetrisation of $\bar{\zeta} := \zeta \chi_{B_{\nu/2}(x)}$, where $\chi_{B_{\nu/2}(x)}$ is the characteristic function of $B_{\nu/2}(x)$ in Π_+ , about x ; then by a standard inequality (see, e.g., [3]) and Hölder's inequality we obtain

$$\begin{aligned} \int_{B_{\nu/2}(x)} \log \frac{1}{\rho} \zeta(y) dy &\leq \int_{B_{\nu/2}(x)} \log \frac{1}{\rho} \hat{\zeta}(y) dy \\ &\leq \left(\int_{B_{\hat{b}}(x)} \left| \log \frac{1}{\rho} \right|^\tau dy \right)^{1/\tau} \|\hat{\zeta}\|_\epsilon, \end{aligned} \quad (3.60)$$

where $\hat{b} := |\text{supp}(\zeta \chi_{B_{\nu/2}(x)})| \leq b$, $\epsilon := p/(1 + p\gamma - \gamma)$ and τ is the conjugate exponent of ϵ . It is elementary to show that

$$\int_{B_{\hat{b}}(x)} \left| \log \frac{1}{\rho} \right|^\tau dy \leq C, \quad (3.61)$$

where C is a constant independent of x . Next observe that $\epsilon = \epsilon\gamma + (1 - \epsilon\gamma)p$ and $\epsilon\gamma < 1$, hence applying the standard interpolation inequality yields

$$\|\hat{\zeta}\|_\epsilon^\epsilon \leq \|\hat{\zeta}\|_1^{\epsilon\gamma} \|\hat{\zeta}\|_p^{(1-\epsilon\gamma)p}, \quad (3.62)$$

or

$$\|\hat{\zeta}\|_\epsilon \leq \|\hat{\zeta}\|_1^\gamma \|\hat{\zeta}\|_p^{(1-\epsilon\gamma)p/\epsilon} = \|\hat{\zeta}\|_1^\gamma \|\hat{\zeta}\|_p^{1-\gamma}. \quad (3.63)$$

Therefore, we obtain

$$\|\hat{\zeta}\|_\epsilon \leq 4^\gamma (x_1 x_2)^{-\gamma} \mathfrak{I}(\zeta)^\gamma \|\zeta\|_p^{1-\gamma}. \quad (3.64)$$

Finally from (3.56), (3.58), (3.60), (3.61), and (3.64) we derive (3.52). $\square$

By a simple modification of Burton [2, Lemma 1] we get the following lemma.

LEMMA 3.11. *Let ζ be a nonnegative measurable function on Π_+ , let $t > 0$. Let ζ_t be the function, defined on Π_+ , obtained by translating ζ along the diagonal of Π_+ , $\text{diag}(\Pi_+)$, $\sqrt{2}t$ units, that is,*

$$\zeta_t(x_1, x_2) := \begin{cases} \zeta(x_1 - t, x_2 - t), & x_1 \geq t, x_2 \geq t \\ 0, & 0 < x_1 < t, 0 < x_2 < t. \end{cases} \quad (3.65)$$

Then

$$\int_{\Pi_+} \zeta_t K_+ \zeta_t \geq \int_{\Pi_+} \zeta K_+ \zeta. \quad (3.66)$$

LEMMA 3.12. *Let $2 < p < \infty$ and $\zeta \in L^p(\Pi_+)$ be a nonnegative, nontrivial function which vanishes outside $\Pi_+(h)$ for some $h > 0$. Then*

$$K_+ \zeta(x) \leq \frac{4hx_1 x_2}{\pi |x_1^2 - x_2^2|} \|\zeta\|_1 + N \min\{x_1, x_2\}, \quad (3.67)$$

provided that $x \in \Pi_+ \setminus \text{diag}(\Pi_+)$.

PROOF. Fix $x \in \Pi_+ \setminus \text{diag}(\Pi_+)$ and define

$$U(x) := \left\{ y \in \Pi_+ \mid |(\gamma_1^2 - \gamma_2^2) - (x_1^2 - x_2^2)| < |x_1^2 - x_2^2|^{1/2} \right\}. \quad (3.68)$$

Next we decompose ζ as follows: $\zeta := \zeta_1 + \zeta_2$, where

$$\zeta_1(y) := \begin{cases} \zeta(y), & y \in \Pi_+(h) \cap U(x), \\ 0, & \text{otherwise.} \end{cases} \quad (3.69)$$

Again by setting $\alpha := |x - \overline{y}|$, $\beta := |x - \underline{y}|$, $\rho := |x - y|$, $\delta := |x - \overline{y}|$, we obtain

$$\begin{aligned} K_+ \zeta_2(x) &= \frac{1}{4\pi} \int_{\Pi_+} \log \frac{\alpha^2 \beta^2}{\rho^2 \delta^2} \zeta_2(y) dy \\ &= \frac{1}{4\pi} \int_{\Pi_+} \log \left(1 + \frac{16x_1 x_2 y_1 y_2}{\rho^2 \delta^2} \right) \zeta_2(y) dy \\ &\leq \frac{4hx_1 x_2}{\pi} \int_{\Pi_+ \setminus U(x)} \frac{1}{\rho^2 \delta^2} \zeta_2(y) dy. \end{aligned} \quad (3.70)$$

In view of the following identity:

$$\rho^2 \delta^2 = ((\gamma_1^2 - \gamma_2^2) - (x_1^2 - x_2^2))^2 + 4(x_1 x_2 - y_1 y_2)^2, \quad (3.71)$$

we infer that if $y \in \Pi_+ \setminus U(x)$, then $\rho^2 \delta^2 > |x_1^2 - x_2^2|$. This, in conjunction with (3.70), yields

$$K_+ \zeta_2(x) \leq \frac{4hx_1 x_2}{\pi |x_1^2 - x_2^2|} \|\zeta\|_1. \quad (3.72)$$

Finally, recalling (2.5) we obtain

$$K_+ \zeta_1(x) \leq N \min\{x_1, x_2\}. \quad (3.73)$$

Since $K_+ \zeta(x) = K_+ \zeta_1(x) + K_+ \zeta_2(x)$, (3.67) follows from (3.72) and (3.73). $\square$

REMARK 3.13. Under the hypotheses of Lemma 3.12 with b replaced by a and an additional assumption, namely, $\Im(\zeta) \geq 1$ we can show the existence of a positive constant P such that

$$K_+ \zeta(x) \leq P(x_1 x_2)^{-\gamma} \Im(\zeta), \quad (3.74)$$

provided that $\min\{x_1, x_2\} \geq a/2$ and $\zeta \in \mathcal{F}$. Clearly, the truth of (3.74) emerges from the elementary fact that $s^{\gamma-1} \log s$ is bounded on any interval of the form $[d, \infty)$, $d > 0$.

4. Proof of Theorem 2.1. We first show that, for I sufficiently large, there exists a positive constant $R(I)$ such that if $\Pi_+(\xi, \eta)$ is sufficiently large (satisfying $\mathcal{H}(I)$) and $\zeta \in \Sigma(\xi, \eta, I)$, then

$$\text{supp}(\zeta) \subset \Pi_+(R(I)), \quad (4.1)$$

modulo a set of zero measure. From Lemma 3.3, there exists $I_3 > I_1$ such that if $I > I_3$, then

$$\sigma(I) > \frac{5}{2}aN\|\zeta_0\|_1. \quad (4.2)$$

Fix $I > I_3$ and consider $\zeta \in \Sigma(\xi, \eta, I)$ for some $\Pi_+(\xi, \eta)$ satisfying $\mathcal{H}(I)$. From (4.2) and definition of σ , we infer that

$$\frac{5}{2}aN\|\zeta\|_1 \leq \Psi(\zeta) \leq \frac{1}{2}\|\zeta\|_1 \sup_{x \in \text{supp}(\zeta)} K_+\zeta(x), \quad (4.3)$$

thus

$$\sup_{x \in \text{supp}(\zeta)} K_+\zeta(x) \geq 5aN. \quad (4.4)$$

Since $K_+\zeta \in C(\mathbb{R}^2)$, it attains its maximum relative to $\overline{\text{supp}(\zeta)}$ at z , say. Therefore, by applying (4.4), we obtain

$$5aN \leq K_+\zeta(z) \leq N \min\{z_1, z_2\}, \quad (4.5)$$

whence $\min\{z_1, z_2\} \geq 5a$. Without loss of generality, we may assume that $\mathfrak{I}(\zeta) \geq 1$, hence, by (3.74) we obtain

$$5aN \leq K_+\zeta(z) \leq PI(z_1 z_2)^{-\gamma}, \quad (4.6)$$

so

$$z_1 z_2 \leq \left(\frac{PI}{5aN} \right)^\gamma. \quad (4.7)$$

Now we define

$$R(I) := \max \left\{ \left(\frac{PI}{5aN} \right)^\gamma, 25a^2 \right\}. \quad (4.8)$$

Then $V := \{x \in \Pi_+ \mid x_1 x_2 \leq R(I), \min\{x_1, x_2\} \geq 5a\}$ is not empty and $z \in V$. Note that at least a quadrant of $B_{4a}(x)$, for every $x \in V$, is contained in $\Pi_+(R(I))$ and, in fact, contained in $\Pi_+(\xi_1, R(I))$ for some $\xi_1^2 > R(I)$. By $\Pi_+(\xi_1, R(I))$ we denote the translation of $\Pi_+(\xi_1, R(I))$ along $\text{diag}(\Pi_+)$, $\sqrt{2}t$ units. Observe that the family of translations $\{\Pi_+(\xi_1, R(I))\}_{0 \leq t \leq t_0}$, where $t_0 := (I/\|\zeta_0\|_1)^{1/2}$, is uniformly contained in $\Pi_+(\xi_2, \eta_2)$, for some ξ_2 and η_2 (in fact we can take $\xi_2 = \xi_1 + t_0$). From now on we assume that $\xi > \xi_2$ and $\eta > \eta_2$. Since a quadrant of $B_{4a}(z)$, designated by Q , is contained in $\Pi_+(R(I))$ we can apply the mean value inequality and (2.5) to deduce that

$$K_+\zeta(x) \geq K_+\zeta(z) - 4aN \geq aN, \quad x \in Q, \quad (4.9)$$

where the last inequality is obtained from (4.4). To seek a contradiction we assume that $E := \text{supp}(\zeta) \setminus \Pi_+(R(I))$ has a positive measure and write $\zeta = \zeta_0 + \zeta_1$, where

$$\zeta_1 := \zeta \chi_E. \quad (4.10)$$

Since $|Q| = 4\pi a^2 > |\text{supp}(\zeta)| = \pi a^2$, there exists a measure preserving bijection, denoted by T , from E onto a subset of $Q \setminus \text{supp}(\zeta)$, say G , see Royden [15]. Now define

$$\zeta_2 := \zeta_1 \circ T^{-1}, \quad (4.11)$$

on the range of T and zero elsewhere, that is,

$$\zeta_2 = (\zeta_1 \circ T^{-1}) \chi_{\text{im}(T)}, \quad (4.12)$$

where $\text{im}(T)$ is the range of T , and let $\zeta' := \zeta_0 + \zeta_2$. Clearly $\zeta' \in \mathcal{F}(\xi, \eta)$. We show that $\mathfrak{I}(\zeta') < \mathfrak{I}(\zeta)$:

$$\begin{aligned} \mathfrak{I}(\zeta') &= \int_{\Pi_+} x_1 x_2 \zeta_0 + \int_{\Pi_+} x_1 x_2 \zeta_2 \\ &= \int_{\Pi_+} x_1 x_2 \zeta_0 + \int_{\Pi_+} x_1 x_2 \zeta_1 \circ T^{-1} \\ &= \int_{\Pi_+} x_1 x_2 \zeta_0 + \int_E (x_1 x_2 \circ T) \zeta_1 \\ &< \int_{\Pi_+} x_1 x_2 \zeta_0 + \int_{\Pi_+} x_1 x_2 \zeta_1 \\ &= \mathfrak{I}(\zeta). \end{aligned} \quad (4.13)$$

On the other hand, we have

$$\Psi(\zeta') - \Psi(\zeta) = \int_{\Pi_+} (\zeta_2 - \zeta_1) K_+ \zeta + \Psi(\zeta_2 - \zeta_1) > \int_{\Pi_+} (\zeta_2 - \zeta_1) K_+ \zeta, \quad (4.14)$$

since K_+ is strictly positive, see Emamizadeh [10]. Hence

$$\begin{aligned} \Psi(\zeta') - \Psi(\zeta) &> \int_{\Pi_+} \zeta_2 K_+ \zeta - \int_{\{x \in \Pi_+ \mid x_1 x_2 > R(I)\}} \zeta_1 K_+ \zeta \\ &\geq aN \int_{\Pi_+} \zeta_2 - \int_{\{x \in \Pi_+ \mid x_1 x_2 > R(I)\}} \zeta_1 K_+ \zeta, \end{aligned} \quad (4.15)$$

by (4.9). Now we proceed to estimate $\int_{\{x \in \Pi_+ \mid x_1 x_2 > R(I)\}} \zeta_1 K_+ \zeta$. For this purpose we set

$$\text{supp}(\zeta) = J_1 \cup J_2, \quad (4.16)$$

where

$$\begin{aligned} J_1 &:= \left\{ x \in \text{supp}(\zeta) \mid x_1 x_2 > R(I), \min \{x_1, x_2\} \geq \frac{a}{2} \right\}, \\ J_2 &:= \left\{ x \in \text{supp}(\zeta) \mid x_1 x_2 > R(I), \min \{x_1, x_2\} < \frac{a}{2} \right\}. \end{aligned} \quad (4.17)$$

If $x \in J_1$, then by (3.74)

$$K_+ \zeta(x) \leq PI(x_1 x_2)^{-\gamma} \leq PIR(I)^{-\gamma}. \quad (4.18)$$

On the other hand, if $x \in J_2$ then by (2.5)

$$K_+ \zeta(x) \leq N \min \{x_1, x_2\} \leq \frac{a}{2}. \quad (4.19)$$

Therefore, if $x \in \text{supp}(\zeta_1)$

$$K_+ \zeta(x) \leq \max \left\{ PIR(I)^{-\gamma}, \frac{aN}{2} \right\}. \quad (4.20)$$

Assume that $R(I)$ is large enough to ensure

$$aN - \max \left\{ PIR(I)^{-\gamma}, \frac{aN}{2} \right\} > 0. \quad (4.21)$$

Therefore, we obtain

$$\Psi(\zeta') - \Psi(\zeta) \geq \left(aN - \max \left\{ PIR(I)^{-\gamma}, \frac{aN}{2} \right\} \right) \|\zeta_1\|_1 > 0. \quad (4.22)$$

This implies that $\Psi(\zeta') > \Psi(\zeta)$. Finally, we define ζ'' to be the function obtained by translating ζ' along $\text{diag}(\Pi_+)$ so that $\mathfrak{I}(\zeta'') = I$. If we denote the amount of translation by t , then it is clear that t is the bigger root of the following algebraic equation:

$$\|\zeta'\|_1 t^2 + 2 \left(\int_{\Pi_+} (x_1 + x_2) \zeta' \right) t + \int_{\Pi_+} x_1 x_2 \zeta' = I. \quad (4.23)$$

Note that t depends on ζ ; but we are able to find a uniform bound, independent of ζ , as follows. Solving (4.23) for t yields

$$\begin{aligned} t &= \frac{- \int_{\Pi_+} (x_1 + x_2) \zeta' + \left(\left(\int_{\Pi_+} (x_1 + x_2) \zeta' \right)^2 - \|\zeta'\|_1 (\mathfrak{I}(\zeta') - I) \right)^{1/2}}{\|\zeta'\|_1} \\ &< (\|\zeta'\|_1 (I - \mathfrak{I}(\zeta')))^{1/2} < \left(\frac{I}{\|\zeta'\|_1} \right)^{1/2}, \end{aligned} \quad (4.24)$$

as desired. Note that the choices of ξ_2 and η_2 ensure that $\zeta'' \in \mathcal{F}(\xi, \eta, I)$. Now, by Lemma 3.11 we have

$$\Psi(\zeta'') \geq \Psi(\zeta') > \Psi(\zeta). \quad (4.25)$$

This is a contradiction to the maximality of ζ . Therefore we have been able to show that if $I > I_3$, then there exists $R(I)$ given by (4.8) such that if $\Pi_+(\xi, \eta)$ is sufficiently large ($\xi \geq \xi_2$ and $\eta \geq \eta_2$) and $\zeta \in \Sigma(\xi, \eta, I)$, then, for almost every $x \in \text{supp}(\zeta)$, (4.1) holds.

However, the possibility that the vortex core runs off to infinity, as $\Pi_+(\xi, \eta)$ exhausts Π_+ , still exists. We now show that this situation is ruled out once I is sufficiently large. For this purpose, fix $I > I_3$ and consider $(\zeta, \phi, \lambda) \in M(\xi, \eta, I)$. We claim that if ξ and

η are large enough then λ can not be too negative. For this purpose let $\xi \geq \xi_2$ and $\eta \geq \max\{h, \eta_2\}$, ξ_2 and η_2 are as above, where

$$h := \left(N|\lambda^*|^{-1} + 1\right)R(I), \quad \lambda^* := -\frac{aN}{3R(I)}, \quad (4.26)$$

such that $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$. We show that

$$\lambda > \lambda^*. \quad (4.27)$$

To seek a contradiction suppose that $\lambda \leq \lambda^*$. Without loss of generality we may assume that $R(I) \geq 1$. Let $x \in W := \{y \in \Pi_+(\xi, \eta) \mid y_1 y_2 > h\}$. Then

$$\begin{aligned} K_+ \zeta(x) - \lambda x_1 x_2 &> -\lambda x_1 x_2 = |\lambda| x_1 x_2 > |\lambda| h \\ &= |\lambda| \left(N|\lambda^*|^{-1} + 1\right)R(I) > (N + |\lambda|)R(I). \end{aligned} \quad (4.28)$$

Now consider $x \in \text{supp}(\zeta)$. If $\max\{x_1, x_2\} \geq 1$, then $\min\{x_1, x_2\} \leq x_1 x_2$, hence $\min\{x_1, x_2\} \leq R(I)$. If, however, $\max\{x_1, x_2\} < 1$ then $\min\{x_1, x_2\} < 1 \leq R(I)$. Therefore in either case we have $\min\{x_1, x_2\} \leq R(I)$. This, in turn, implies that

$$K_+ \zeta(x) - \lambda x_1 x_2 \leq N \min\{x_1, x_2\} - \lambda x_1 x_2 < (N + |\lambda|)R(I), \quad (4.29)$$

whence

$$\sup_{x \in \text{supp}(\zeta)} (K_+ \zeta(x) - \lambda x_1 x_2) \leq (N + |\lambda|)R(I). \quad (4.30)$$

Therefore $K_+ \zeta(x) - \lambda x_1 x_2$ takes greater values on a nonempty subset of $\Pi_+(\xi, \eta)$, namely W , than its supremum on $\text{supp}(\zeta)$. This is impossible, since ζ is essentially an increasing function of $K_+ \zeta(x) - \lambda x_1 x_2$ on $\Pi_+(\xi, \eta)$. Hence we derive (4.27). For the rest of the proof we fix $I > I_0 := \max\{I_1, I_2, I_3\}$. Let $\xi > \xi_2$, $\eta > h$ (as above) be such that $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$. Consider $(\zeta, \phi, \lambda) \in M(\xi, \eta, I)$. Now fix $x \in \text{supp}(\zeta) \setminus \text{diag}(\Pi_+)$ such that $\min\{x_1, x_2\} < a/6$. Then by Lemmas 3.9 and 3.12, in conjunction with (4.27),

$$\begin{aligned} aN &\leq K_+ \zeta(x) - \lambda x_1 x_2 \\ &\leq \frac{4R(I)x_1 x_2}{\pi |x_1^2 - x_2^2|} \|\zeta\|_1 + N \min\{x_1, x_2\} - \lambda^* x_1 x_2 \\ &\leq \frac{4R(I)x_1 x_2}{\pi |x_1^2 - x_2^2|} \|\zeta\|_1 + N \min\{x_1, x_2\} - \lambda^* R(I) \\ &\leq \frac{4R(I)x_1 x_2}{\pi |x_1^2 - x_2^2|} \|\zeta\|_1 + \frac{aN}{6} + \frac{aN}{3}. \end{aligned} \quad (4.31)$$

Hence

$$|x_1^2 - x_2^2| < \frac{8R(I)\|\zeta_0\|_1}{a\pi N}. \quad (4.32)$$

To summarise, we have shown that if $x \in \text{supp}(\zeta)$ is such that $\min\{x_1, x_2\} > a/6$, then $x \in \Pi_+(R(I)) \cap \{y \in \Pi_+ \mid \min\{y_1, y_2\} > a/6\}$; otherwise x satisfies (4.32). This clearly concludes the existence part of the theorem.

Now consider $\zeta \in \Sigma(I)$. Then there exists $\hat{\xi} > 0$ such that $\overline{\text{supp}(\zeta)}$ is a compact subset of $D(\hat{\xi}) := (0, \hat{\xi}) \times (0, \hat{\xi})$ and, according to [Lemma 3.1](#),

$$\zeta = \phi \circ (K_+ \zeta - \lambda x_1 x_2), \quad \text{a.e. in } D(\hat{\xi}), \quad (4.33)$$

for some increasing function ϕ and $\lambda \in \mathbb{R}$. Note that from [Lemma 3.9](#)

$$\kappa := \text{ess sup} \{K_+ \zeta(x) - \lambda x_1 x_2 \mid x \in \text{supp}(\zeta)\} \geq aN > 0. \quad (4.34)$$

Since the level sets of $K_+ \zeta - \lambda x_1 x_2$, on $\text{supp}(\zeta)$, have zero measure, in particular we have

$$|\{x \in \text{supp}(\zeta) \mid K_+ \zeta - \lambda x_1 x_2 = \kappa\}| = 0. \quad (4.35)$$

Therefore

$$K_+ \zeta - \lambda x_1 x_2 > \kappa, \quad \text{a.e. in } \text{supp}(\zeta). \quad (4.36)$$

Thus we may suppose that $\phi(s) = 0$ for $s \leq \kappa$. Now if we define $F(s) := \int_0^s \phi(t) dt$, then [Lemma 3.7](#) yields

$$2 \int_{D(\hat{\xi})} F \circ \psi - 2\lambda I = \int_{\partial D(\hat{\xi})} (F \circ \psi)(x \cdot \vec{n}), \quad (4.37)$$

where $\psi := K_+ \zeta - \lambda x_1 x_2$. We claim that for $x \in \partial D(\hat{\xi})$ we have $\psi \leq \kappa$. Otherwise, by the continuity of ψ we can find $B_\epsilon(x)$ such that $B_\epsilon(x) \cap \text{supp}(\zeta)$ has positive measure, since $\overline{\text{supp}(\zeta)}$ is a compact subset of $D(\hat{\xi})$, and $\psi(s) > \kappa$ for $s \in B_\epsilon(x)$; but this is a contradiction to (4.33). Therefore, if $x \in \partial D(\hat{\xi})$ we have $F \circ \psi(x) = 0$. Hence from (4.37) we deduce that $\lambda > 0$, as required.

Now fix $x \in \text{supp}(\zeta)$. Since $\lambda > 0$, we can employ [Lemma 3.9](#) to obtain

$$aN \leq K_+ \zeta(x) - \lambda x_1 x_2 < K_+ \zeta(x) \leq N \min\{x_1, x_2\}. \quad (4.38)$$

Thus $\min\{x_1, x_2\} \geq a$. This proves the vortex core avoids $\partial\Pi_+$. The validity of (2.12) is established as in Emamizadeh [11]. $\square$

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