ON THE GEOMETRY OF FREE LOOP SPACES

P. MANOHARAN

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We verify the following three basic results on the free loop space $LM$. (1) We show that the set of all points, where the fundamental form on $LM$ is nondegenerate, is an open subset. (2) The connections of a Fréchet bundle over $LM$ can be extended to $S^1$-central extensions and, in particular, there exist natural connections on the string structures. (3) The notion of Christoffel symbols and the curvature are introduced on $LM$ and they are described in terms of Christoffel symbols of $M$.

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1. Introduction. In this paper, we consider the infinite-dimensional Fréchet manifold $LM$, the free loop space on $M$, which is the space of all smooth maps from the circle $S^1$ to a finite-dimensional manifold $M$. We discuss three different topics as a contribution to the general knowledge on the geometry of loop spaces.

If $M$ is a finite-dimensional Riemannian manifold, Atiyah [1] indicates that $LM$ has a fundamental closed 2-form $\omega$ which, unlike the finite-dimensional case, can be degenerate at certain points. A point in $LM$ is a smooth map $\phi : S^1 \to M$. The Levi-Civita connection on $M$ induces a connection on the pullback bundle $\phi^* TM$, and hence a covariant operator $D_\phi$. The fundamental form $\omega$ is degenerate, precisely at those $\phi$ for which $D_\phi$ has zero eigenvalue. In the first part of this paper, by using the Nash embedding theorem [4], we show that the set of points, where $\omega$ is nondegenerate, is an open set.

Loop spaces are of a particular interest to physicists working on the grand unification theory. String theory involves a theory of spinors on $LM$; a string structure is defined as a lifting of the structural group to an $S^1$-central extension of the loop group [3]. Let $G \to P \to X$ be a principal Fréchet bundle over a Fréchet manifold $X$ that has enough smooth functions to admit a smooth partition of unity. Let $S^1 \to \tilde{G} \to G$ be an $S^1$-central extension of $G$ and let $\tilde{G} \to \tilde{P} \to X$ be a lifting of the principal bundle $G \to P \to X$. Although the existence of connections on a general Fréchet bundle is in general not guaranteed, the second part of this paper verifies that every connection on the principal bundle $G \to P \to X$ together with a $\tilde{G}$-invariant connection on $S^1 \to \tilde{P} \to P$ yields a connection on $\tilde{G} \to \tilde{P} \to X$. In particular, as a corollary of this result, we prove that there exist connections on the string structures of $LM$.

In the last part of this paper, we give a detailed construction of Christoffel symbols on $LM$ by using “Fourier coordinates” and compute the corresponding curvature. Both Christoffel symbols and the curvature on $LM$ are given in terms of the Christoffel symbols on $M$ by a Fourier type series. We hope that these constructions of Christoffel
symbols and the curvature will be useful to obtain local geometrical results on $LM$. The proofs of our results are rather straightforward and do not use any sophisticated method of functional analysis or differential geometry.

2. Fundamental form. The tangent space $T_{\phi}(LM)$ at any $\phi \in LM$ can be identified with $\Gamma(\phi^*TM)$, the sections of the pullback bundle $\phi^*TM \rightarrow S^1$ of the tangent bundle $TM \rightarrow M$. For any fixed nonnegative integer $r$, we have an inner product $\langle\langle,\rangle\rangle_r : \Gamma(\phi^*TM) \times \Gamma(\phi^*TM) \rightarrow R$ defined by

$$\langle\langle s,s' \rangle\rangle_r = \sum_{i=0}^r \int_{S^1} \langle D^i s(t), D^i s'(t) \rangle dt,$$  \hspace{1cm} (2.1)

where $D$ is the covariant derivative of the Riemannian connection on $M$. This can be viewed as a Riemannian structure on $LM$ for each $r \geq 0$. Define $\omega(\phi)(\alpha,\beta) = \langle\langle D\alpha,\beta \rangle\rangle_r$ and we can easily see that the fundamental form $\omega(\phi)$ is bilinear and skew-symmetric. The energy function is defined by $e(\phi) = \langle\langle D\phi, D\phi \rangle\rangle_r$.

**Remark 2.1.** Hereafter, we simply use the notations $\omega$ and $e$ for $\omega(r)$ and $e(r)$, respectively.

We verify that (i) $\omega$ is a closed form and (ii) $de + i_A \omega = 0$, where $i_A$ is the contraction along the vector field $A$ associated to the natural $S^1$-action. Since $M$ can be isometrically embedded in $R^N$ for some large $N$ (by the Nash embedding theorem [4]), $LM$ equipped with the Riemannian metric provided by $\langle\langle,\rangle\rangle_r$ is isometrically embedded in $LR^N$ with new metric provided by $\langle\langle,\rangle\rangle_r$ for each $r \geq 0$. Therefore, it is enough to prove (i) and (ii) for $M = R^N$. In this case, the vector field $A$ on an open subset $U$ of $LR^N$ is given by $(A\phi)(t) = \dot{\phi}(t)$. Define the 1-form $\theta : U \times LR^N \rightarrow R$ as the following composition:

$$U \times LR^N \xrightarrow{A \times \text{id}} LR^N \times LR^N \xrightarrow{\langle\langle,\rangle\rangle_r} R$$  \hspace{1cm} (2.2)

so that $\theta(\phi, \alpha) = \langle\langle D\phi, \alpha \rangle\rangle_r$. Clearly $\theta$ is a smooth 1-form and $d\theta = 2\omega$. Therefore, $\omega$ is a smooth closed form. Now

$$de(\phi, \alpha) = \lim_{s \rightarrow 0} \frac{1}{s} \left[ e(\phi + s\alpha) - e(\phi) \right]$$

$$= \lim_{s \rightarrow 0} \frac{1}{s} \left[ \langle\langle D\phi + sD\alpha, D\phi + sD\alpha \rangle\rangle_r - \langle\langle D\phi, D\phi \rangle\rangle_r \right]$$

$$= 2\langle\langle D\phi, D\alpha \rangle\rangle_r = -2\langle\langle D^2 \phi, \alpha \rangle\rangle_r$$

$$= -2\omega(\phi)(\alpha) = -(i_A \omega)(\phi, \alpha).$$  \hspace{1cm} (2.3)

Hence $de + i_A \omega = 0$. The fundamental 2-form $\omega$ described above can be degenerate by the following lemma.

**Lemma 2.2.** The fundamental form $\omega$ is degenerate at $\phi$ if and only if the corresponding covariant derivative $D_\phi$ has zero eigenvalue.
Proof. The fundamental form $\omega$ is degenerate at $\phi$

$\iff \exists \alpha \in T_{\phi}(LM)$ such that $\omega_{\phi}(\alpha, \beta) = 0 \ \forall \beta \in T_{\phi}(LM)$

$\iff \langle \langle D_{\phi}\alpha, \beta \rangle \rangle_r = 0 \ \forall \beta$

$\iff D_{\phi}\alpha = 0$. (2.4)

In other words, $\omega$ has degeneracy at $\phi$ if and only if there exists $\alpha$ in $T_{\phi}(LM)$ which is parallel along $\phi$. For example, $\omega$ is degenerate at any closed geodesic, since $\dot{\phi}$ is parallel along a closed geodesic $\phi$.

Theorem 2.3. The set of points where $\omega$ is nondegenerate is an open subset of $LM$.

Proof. For each $x \in M^n$ and $\tau \in LM$ that passes through $x$, the parallel transport $H_x(\tau)$ of $T_x(M)$ along $\tau$ is an element of the holonomy group $H_x$ of $M$ at $x$. Since the length of a vector and the angle between two vectors are preserved by the parallel transports of the Levi-Civita connection along a curve, the holonomy group at each point $x$ of a connected orientable manifold $M$ is a subgroup of $SO(n)$. The fundamental form $\omega$ is degenerate at some $\phi$ if and only if for some $\theta$ (and hence for any $\theta$) the corresponding element of the holonomy group $H_{\phi(\theta)}$ has eigenvalue 1. Notice that every element of $SO(n)$ has eigenvalue 1 if $n$ is odd and hence in this case $\omega$ is degenerate at every $\phi \in LM$.

So we should restrict our attention to only the even-dimensional manifolds. For every $A \in SO(n)$, let $ev_1(A)$ be the evaluation of the characteristic polynomial of $A$ at 1. Let $\lambda$ be the following composition:

$$LM \overset{i_0}{\to} LM \times S^1 \overset{H}{\to} SO(n) \overset{ev_1}{\to} R,$$

$$\phi \mapsto (\phi, \theta) \mapsto H_{\phi(\theta)}(\phi).$$

We can easily see that $\lambda$ is independent of $\theta$ and $\omega$ is degenerate at $\phi$ if and only if $\lambda(\phi) = 0$. Hence we have proved the theorem. \qed

Remark 2.4. Notice that whether the set of all nonsingular points of $\omega$ is dense or nondense in $LM$ heavily depends on the Levi-Civita connection of $M$. For example, if the curvature of $M$ is zero on a nonempty open subset of $M$, then the set of points, where $\omega$ is degenerate, contains a nonempty open subset of $LM$ and hence its complement, the set of points where $\omega$ is nondegenerate, is not dense in $LM$. This will not be the case if the Riemannian metric on $M$ (supposed to be a real analytic manifold) is real analytic. In this case, the set of degenerate points is a real analytic subset and hence of topological codimension 1. Then the set of nondegenerate points is dense.

3. Lifting of connections. A Fréchet space $F$ is called nice if for every open subset $U$ of $F$ there exists a nonzero real-valued smooth function which vanishes outside $U$. For example, the space of sections of a smooth vector bundle over a compact connected finite-dimensional manifold is a nice Fréchet space and hence $LM$ is locally modeled on nice Fréchet spaces. A manifold locally modeled on nice Fréchet spaces has enough smooth functions to admit smooth partitions of unity. Hereafter, we assume that $X$ is a manifold locally modeled on nice Fréchet spaces.
**Definition 3.1.** A connection on a principal \( G \)-bundle \( G \to P \to X \) is a smooth \( G \)-invariant splitting map \( \mu : \pi^* TX \to TP \) in the exact sequence

\[
0 \to T_v P \to TP \xleftarrow{\mu} \pi^* TX \to 0 \tag{3.1}
\]

of the bundles over \( P \) where \( T_v P \) is the vertical tangent bundle.

Let \( S^1 \to \tilde{G} \to G \) be an \( S^1 \)-central extension of the Fréchet Lie group \( G \), or in other words, \( S^1 \) lies in the center of \( \tilde{G} \) and \( \tilde{G} / S^1 = G \). Let \( \tilde{G} \to \tilde{P} \to X \) be a lifting of \( G \to P \to X \). Then we have the diagram

\[
\begin{array}{ccc}
S^1 & \to & S^1 \\
\downarrow & & \downarrow \\
\tilde{G} & \to & \tilde{P} \\
\downarrow_{\lambda} & & \downarrow_{\tilde{\pi}} \\
G & \to & P \\
\downarrow_{\pi} & & \downarrow \\
& & X \\
\end{array} \tag{3.2}
\]

Of course, a connection on \( G \to P \to X \) together with a connection on \( S^1 \to \tilde{P} \to P \) may not yield a connection on \( \tilde{G} \to \tilde{P} \to X \), because a splitting map \( \lambda^* TP \xleftarrow{\mu} T\tilde{P} \) corresponding to a connection on \( S^1 \to \tilde{P} \to P \) is \( S^1 \)-invariant but not necessarily \( \tilde{G} \)-invariant.

**Proposition 3.2.** The space \( \mathcal{C}_G(S^1 \to \tilde{P} \to P) \) of \( \tilde{G} \)-invariant connections is non-empty.

**Proof.** Let \( T_v^{S^1} \tilde{P} \) be the vertical tangent bundle over \( \tilde{P} \) corresponding to the bundle \( S^1 \to \tilde{P} \to P \). We wish to show that we can produce a \( \tilde{G} \)-invariant splitting map \( \mu \) in

\[
0 \to T_v^{S^1} \tilde{P} \to T\tilde{P} \xleftarrow{\mu} \lambda^* TP \to 0 \tag{3.3}
\]

or equivalently a \( \tilde{G} \)-invariant splitting map \( \eta \) in

\[
0 \to T_v^{S^1} \tilde{P} \xleftarrow{\eta} T\tilde{P} \to \lambda^* TP \to 0. \tag{3.4}
\]

Fortunately, \( T_v^{S^1} \tilde{P} \) is the trivial 1-dimensional bundle over \( \tilde{P} \). Indeed, for each \( x \in \tilde{P} \), consider the map \( \{x\} \times S^1 \to \tilde{P} \) defined by \((x, \theta) \mapsto \theta \cdot x\). Since \( S^1 \) acts locally free, the derivative of this map, defined as

\[
\{x\} \times T_e(S^1) \to (T_v^{S^1} \tilde{P})_x, \quad (x, v) \mapsto v_x, \tag{3.5}
\]

provides a nonzero vector field \( A_v \) and hence \( T_v^{S^1} \tilde{P} \to \tilde{P} \) is trivial.

Since \( S^1 \) lies in the center of \( \tilde{G} \), the action of \( \tilde{G} \) and that of \( S^1 \) commutes. Hence \( A_v \) is \( \tilde{G} \)-invariant and descends to a nonzero vector field on \( T_v^{S^1} \tilde{P} / \tilde{G} \to \tilde{P} / \tilde{G} = X \). Thus
$T^S_{v_1}\tilde{P}/\tilde{G} \to \tilde{P}/\tilde{G}$ is also a trivial line bundle. So it is enough to define a morphism (splitting map) $\gamma$ in

$$
X \times R = T^S_{v_1}\tilde{P}/\tilde{G} \xrightarrow{\gamma} T\tilde{P}/\tilde{G} = E
$$

such that $\gamma \circ i = id$. Since $X$ admits partitions of unity and the convex combination of splittings is a splitting, it is enough to produce a splitting locally on $U \times R \cong \pi^{-1}(U)$

For each $x \in U$, let $v(x) = p_2 i(x, 1) \in F$, where $p_2$ is the projection map. Choose a point $x_0 \in U$. Since Hahn-Banach theorem holds on any Fréchet space $F$, there exists a linear map $\psi : F \to R$ such that $\psi(v(x_0)) = 1$. By restricting to a smaller open subset $U_1 \subseteq U$, we can assume that $\psi(v(x)) \neq 0$ for all $x \in U$. Let $\mathcal{L}(F, R)$ be the space of linear maps from $F$ to $R$. Define $\gamma : U \to \mathcal{L}(F, R)$ by

$$
\gamma(x)(f) = \frac{\psi(f)}{\psi(v(x))} \quad \forall f \in F.
$$

Since $\gamma(x)(v(x)) = 1$ for all $x$, $\gamma$ yields a splitting map. This completes the proof of the lemma and immediately yields the following theorem.

**Theorem 3.3.** Every connection on the principal Fréchet bundle $G \to P \to X$ together with a $\tilde{G}$-invariant connection on $S^1 \to \tilde{P} \to P$ yields a connection on $\tilde{G} \to \tilde{P} \to X$.

**Remark 3.4.** The lifting of a $G$-connection on $P$ to a $\tilde{G}$-connection on $\tilde{P}$ is not unique. The failure of uniqueness is measured by $\tilde{G}$-invariant $S^1$-connections on $S^1 \to \tilde{P} \to P$ and hence by a 1-dimensional form on $\tilde{P}$ which is $\tilde{G}$-invariant.

**Corollary 3.5.** If $F$ is any Fréchet space with a $\tilde{G}$-action, then every connection on $G \to P \to X$ defines at least one connection on $\tilde{P} \times \tilde{G}F \to X$.

**Proof.** The proof is the same as the finite-dimensional case. Every connection on $\tilde{P}$ defines a connection on $\tilde{P} \times F \to X$. Since $\tilde{P} \times F \to \tilde{P} \times \tilde{G}F$ is a submersion and the horizontal subspaces are mapped injectively (or in other words, horizontal subspaces of $\tilde{P} \times F$ intersect trivially with the kernel), there is an induced connection on $\tilde{P} \times \tilde{G}F \to X$.

Let $M^n$ be an even-dimensional smooth compact connected orientable manifold with the spin structure $\text{Spin}(n) \to Q \to M$. Let $L\text{Spin}(n) \to LQ \to LM$ be the associated $L\text{Spin}(n)$-principal bundle. It is shown in [3] that such a bundle can be lifted to a
new bundle $\widetilde{L\text{Spin}}(n) \rightarrow \widetilde{LQ} \rightarrow LM$ provided that $p_1(M) = 0$, where $p_1(M)$ is the first Pontrjagin class of $M$ and $L\text{Spin}(n)$ is the nontrivial $S^1$-central extension of $L\text{Spin}(n)$. The spinors on $LM$ are defined as the sections of the vector bundles $\widetilde{LQ} \times_{L\text{Spin}(n)} \Lambda^\pm \rightarrow LM$ where $L\text{Spin}(n)$ acts on $\Lambda^\pm$ by two inequivalent irreducible representations. The reader may consult [5] for the construction of such infinite-dimensional irreducible representations.

**Corollary 3.6.** Any connection on $L\text{Spin}(n) \rightarrow LQ \rightarrow LM$ induces connections on the spin bundles $\widetilde{LQ} \times_{L\text{Spin}(n)} \Lambda^\pm \rightarrow LM$.

4. Christoffel symbols on $LM$. If $M^n$ is an orientable Riemannian manifold and $\phi \in LM$, we can choose a frame $\{e_1, \ldots, e_n\}$ of the pullback bundle $\phi^*TM \rightarrow S^1$. Consider the Riemannian structures $\langle\langle,\rangle\rangle_{r,\phi}$ induced by the Riemannian structure of $M$. Let $s_1, s_2 \in T_\phi(LM)$ with $s_1 = \sum s_1^i e_i$ and $s_2 = \sum s_2^j e_j$. We can explicitly calculate $\langle\langle s_1, s_2 \rangle\rangle_{r,\phi}$. For example, if $r = 0$,

$$\langle\langle s_1, s_2 \rangle\rangle_{0,\phi} = \int_{S^1} \langle s_1, s_2 \rangle(\theta) \, d\theta = \sum_{i,j=1}^n \int_{S^1} s_1^i(\theta) s_2^j(\theta) g_{ij}(\phi(\theta)) \, d\theta,$$

where $g_{ij}(\phi(\theta)) = \langle e_i(\phi(\theta)), e_j(\phi(\theta))\rangle$; and if $r = 1$,

$$\langle\langle s_1, s_2 \rangle\rangle_{1,\phi} = \sum \int_{S^1} s_1^i(\theta) s_2^j(\theta) g_{ij}(\phi(\theta)) \, d\theta + \sum \int_{S^1} \phi(s_1^i(\theta)) \phi(s_2^j(\theta)) g_{ij}(\phi(\theta)) \, d\theta + \sum \int_{S^1} \phi(s_1^i(\theta)) \phi^t(\theta) s_2^j(\theta) \Gamma^k_{ij}(\phi(\theta)) g_{kl}(\phi(\theta)) \, d\theta + \sum \int_{S^1} \phi(s_2^j(\theta)) \phi^u(\theta) s_1^i(\theta) \Gamma^l_{ji}(\phi(\theta)) g_{lj}(\phi(\theta)) \, d\theta + \sum \int_{S^1} \phi^t(\theta) \phi^u(\theta) s_1^i(\theta) s_2^j(\theta) \Gamma^k_{li}(\phi(\theta)) \Gamma^l_{mj}(\phi(\theta)) g_{kl}(\phi(\theta)) \, d\theta,$$

where $\{\Gamma^k_{ij}(x)\}$ represent the Christoffel symbols of the Levi-Civita connection on $M$. We use the Einstein convention where the sums run through super- and subscripts. In all of our further calculations, we restrict ourselves to the case $r = 0$, and denote $\langle\langle,\rangle\rangle_{0,\phi}$ simply by $\langle\langle,\rangle\rangle_\phi$ for each $\phi \in LM$.

A connection on a Fréchet vector bundle $E \overset{\pi}{\rightarrow} X$ over a Fréchet manifold $X$ is a rule that assigns to each point of $E$ a complementary subspace for the vertical tangent space, called the horizontal subspace, such that the local representation of the connection is given by a smooth map $\Gamma$ as follows: if $\pi$ is locally $U \times G \rightarrow U$ where
U ⊆ F is open and F and G are Fréchet spaces (i.e., X is locally modeled by F and G is the fiber), then the horizontal vectors consists of all \((b, c) \in F \times G\) with \(c = \Gamma(u, a, b)\) where \(\Gamma: U \times G \times F \to G\) is bilinear in \(a\) and \(b\) (see [2]).

By a connection on a manifold \(X\), we always understand a connection on its tangent bundle. Unlike Banach manifolds, the existence of a connection on an arbitrary Fréchet manifold is not guaranteed. However, every connection \(\Gamma\) on \(M\) determines a connection on \(LM\), also denoted by \(\Gamma\), which is given locally at \(\phi \in LM\) by the formula

\[
\Gamma(\phi, f, g)(\theta) = \Gamma(\phi(\theta), f(\theta), g(\theta)),
\]

where \(f(\theta), g(\theta) \in T_{\phi(\theta)}M\) and \(\Gamma\) is bilinear in the last two variables.

Every \(f \in LR^n\) has Fourier series

\[
f \sim \sum_{p \in \mathbb{Z}} (f_{1p}, f_{2p}, \ldots, f_{np}) \alpha_p(\theta)
\]

such that \(\sum_p \|\tilde{f}_p\|^2 2^k < \infty\) for every integer \(k\), where

\[
\alpha_p(\theta) = \begin{cases} 
\cos p\theta & \text{if } p \leq 0, \\
\sin p\theta & \text{if } p > 0.
\end{cases}
\]

In other words, \(f \sim \sum_{p \in \mathbb{Z}} \sum_{k=1}^n f_{kp} \alpha_p(\theta) \tilde{e}_k\) where \(\{\tilde{e}_1, \ldots, \tilde{e}_n\}\) is the standard basis of \(R^n\). The numbers \(\{f_{kp}\}\) are called the Fourier coordinates of \(f \in LR^n\). Notice that though every element \(f \in LR^n\) has Fourier coordinates which is a sequence \(\{f_{kp}\}\) of real numbers, only the sequences \(\{f_{kp}\}\) satisfying additional convergence conditions are coordinates for elements of \(LR^n\) (see [6, Proposition 10.2]).

By choosing a frame \(\{e_1, \ldots, e_n\}\) of the pullback bundle \(\phi^*TM \to S^1\), just like above (beginning of Section 4), each \(s \in \Gamma(\phi^*TM)\) can be assigned Fourier coordinates \(\{s_{kp}\}\) where \(s \sim \sum_{p \in \mathbb{Z}} \sum_{k=1}^n s_{kp} \alpha_p(\theta) e_k\). Let \(\Gamma\) be a connection on \(LM\). For a fixed \(\phi \in LM\), let \(\{E_{kp}(\phi)\}_{1 \leq k \leq n, p \in \mathbb{Z}}\) be defined by \(E_{kp}(\phi)(\theta) = \alpha_p(\theta) e_k(\phi(\theta))\). We define the Christoffel symbols \(\{\tilde{\Gamma}_{kp, lq, mr}(\phi)\}\) of \(\Gamma\) with respect to \(\{E_{kp}(\phi)\}\) by

\[
\tilde{\Gamma}_{kp, lq, mr}(\phi) = \langle \Gamma(\phi, E_{kp}, E_{lq}), E_{mr} \rangle \phi,
\]

where \(1 \leq k, l, m \leq n\) and \(p, q, r \in \mathbb{Z}\).

**Note 1.** Similar definition in the finite-dimensional case yields \(\{\tilde{\Gamma}_{\beta y, \delta}(x)\}\) which are connected to the usual Christoffel symbols \(\{\Gamma^\delta_{\beta y}(x)\}\) by

\[
\Gamma^\delta_{\beta y}(x) = \tilde{\Gamma}_{\beta y, \delta}(x) (g^{-1}(x))^{\delta \mu}.
\]

Now we can describe the Christoffel symbol \(\tilde{\Gamma}\) in terms of the Christoffel symbols of \(M\) as follows.

**Proposition 4.1.** Let \(\{\Gamma^\delta_{\beta y}(x)\}\) be the Christoffel symbols with respect to a connection \(\Gamma\) of \(M\) and \(\{\tilde{\Gamma}_{kp, lq, mr}(\phi)\}\) be the Christoffel symbols of the induced connection \(\Gamma\) of \(LM\). Then

\[
\tilde{\Gamma}_{kp, lq, mr}(\phi) = \sum_{i=1}^{n} \int_{S^1} \alpha_p(\theta) \alpha_q(\theta) \alpha_r(\theta) \Gamma^i_{kp}(\phi(\theta)) g_{im}(\phi(\theta)) d\theta.
\]
**Proof.** The formula for Christoffel symbols can be obtained as follows:

\[
\left\langle \left\langle \Gamma(\phi, E_k, E_l, E_m) \right\rangle, E_n \right\rangle \phi = \int_{S^1} \alpha_p(\theta) \alpha_q(\theta) \alpha_r(\theta) \left\langle \Gamma(\phi(\theta), e_k(\phi(\theta)), e_l(\phi(\theta)), e_m(\phi(\theta))) \right\rangle d\theta \]

\[
= \int_{S^1} \alpha_p(\theta) \alpha_q(\theta) \alpha_r(\theta) \tilde{\Gamma}_{klm}(\phi(\theta)) d\theta \]

\[
= \sum_{i=1}^n \int_{S^1} \alpha_p(\theta) \alpha_q(\theta) \alpha_r(\theta) \tilde{\Gamma}^i_{klm}(\phi(\theta)) g_{im}(\phi(\theta)) d\theta.
\]

This completes the proof of the proposition. \( \square \)

The curvature \( R \) of a connection \( \Gamma \) is given locally by the map

\[
R(\phi, f, g, h) = D_\phi \Gamma(f, g, h) - D_\phi \Gamma(f, h, g) - \Gamma(\phi, \Gamma(\phi, f, g), h) + \Gamma(\phi, \Gamma(\phi, f, h), g)
\]

which is trilinear in the last three variables. Here \( D_\phi \Gamma(f, g, h) \) is the derivative of \( \Gamma \) at \((\phi, f, g)\) in the direction of \( h \).

If \( \Gamma \) (with the curvature \( R \)) is induced by the connection \( \Gamma_M \) (with the curvature \( R_M \)) then

\[
R(\phi, f, g, h)(\theta) = R_M(\phi(\theta), f(\theta), g(\theta), h(\theta))
\]

for all \( \theta \in S^1 \). As before let

\[
\tilde{R}_{kp, lq, mr, ns}(\phi) = \left\langle \left\langle R(\phi, E_k, E_l, E_m) \right\rangle, E_n \right\rangle \phi.
\]

**Proposition 4.2.** By a similar argument as in Proposition 4.1, we have the following formula for the curvature

\[
\tilde{R}_{kp, lq, mr, ns}(\phi) = \sum_{i=1}^n \int_{S^1} \alpha_p(\theta) \alpha_q(\theta) \alpha_r(\theta) \alpha_s(\theta) R^i_{klm}(\phi(\theta)) g_{ir}(\phi(\theta)) d\theta,
\]

where \( R^i_{klm}(x) \) is defined by

\[
R_M(x, e_k(x), e_l(x), e_m(x)) = R^i_{klm}(x) e_i(x)
\]

for \( x = \phi(\theta) \in M \).

**References**


P. Manoharan: Mathematics Department, University of Maryland, University College, 3501 University Boulevard East, Adelphi, MD 20783, USA

E-mail address: pmanoharan@umuc.edu