ON AN INFINITE SERIES FOR \((1 + 1/x)^x\) AND ITS APPLICATION

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An infinite series for \((1 + 1/x)^x\) is deduced. As an application, a refinement of Carleman’s inequality is achieved.

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The well-known Carleman’s inequality states that if \(a_n \geq 0, \) \(n = 1, 2, \ldots\), and \(0 < \sum_{n=1}^{\infty} a_n < \infty\), then

\[
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n.
\] (1)

Recently, Yang and Debnath [4] improved (1) to

\[
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2(n+1)}\right) a_n.
\] (2)

In [3], a further refinement of (2) is presented as follows:

\[
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{1}{n+1/5}\right)^{-1/2} a_n.
\] (3)

The key step in the establishment of inequalities (2) and (3) is aimed at estimates of \((1 + 1/x)^x\). In this note, we derive an equality for \((1 + 1/x)^x\) in terms of an infinite series. As an application, we further strengthen inequality (3). The main results of this note are presented as follows.

**Theorem 1.** For any \(x > 0\),

\[
\left(1 + \frac{1}{x}\right)^x = e \left(1 - \sum_{n=1}^{\infty} \frac{b_n}{(1 + x)^n}\right),
\] (4)

where \(b_n > 0\) and satisfies the recurrence relation

\[
b_1 = \frac{1}{2}, \quad b_{n+1} = \frac{1}{(n+1)(n+2)} - \frac{1}{n+1} \sum_{i=1}^{n} \frac{b_i}{n - i + 2}.
\] (5)

Carleman’s inequality (1) is correspondingly refined as follows.
THEOREM 2. If $a_n \geq 0$, $n = 1, 2, \ldots$, and $0 < \sum_{n=1}^{\infty} a_n < \infty$, then
\[
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left( 1 - \sum_{k=1}^{m} \frac{b_k}{(1+n)^k} \right) a_n,
\] (6)

where $m$ is any positive integer and $b_k > 0$ is given by (5).

To prove Theorem 1, we now introduce three lemmas.

LEMMA 3. For $x > 0$, $t = 1/(1+x)$,
\[
\left( 1 + \frac{1}{x} \right)^x = e \exp \left( - \sum_{n=1}^{\infty} \frac{t^n}{n(n+1)} \right).
\] (7)

PROOF. For $x > 0$, $0 < t = 1/(1+x) < 1$, we have
\[
\left( 1 + \frac{1}{x} \right)^x = \left( \frac{1}{1-t} \right)^{(1-t)/t} = \exp \left( - \frac{1-t}{t} \ln(1-t) \right).
\] (8)

Using the power series
\[
\ln(1-t) = - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1},
\] (9)

which converges for $0 < t < 1$, we have
\[
\left( 1 + \frac{1}{x} \right)^x = \exp \left( (1-t) \sum_{n=0}^{\infty} \frac{t^n}{n+1} \right)
= \exp \left( 1 - \sum_{n=1}^{\infty} \frac{t^n}{n(n+1)} \right)
= e \exp \left( - \sum_{n=1}^{\infty} \frac{t^n}{n(n+1)} \right).
\] (10)

This proves (7) as desired. \qed

LEMMA 4. For $0 < t < 1$,
\[
\exp \left( - \sum_{n=1}^{\infty} \frac{t^n}{n(n+1)} \right) = 1 - \sum_{n=1}^{\infty} b_n t^n,
\] (11)

where $b_n$ satisfies the recurrence relation (5).

PROOF. Set
\[
p(t) = - \sum_{n=1}^{\infty} \frac{t^n}{n(n+1)},
\]

\[
f(t) = \exp \left( - \sum_{n=1}^{\infty} \frac{t^n}{n(n+1)} \right) = \exp (p(t)).
\] (12)
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It is clear that the power series of \(p(t)\) converges uniformly for \(0 < t < 1\) and \(f(0) = \exp(p(0)) = 1\). Therefore, we can expand \(f(t)\) as a power series in the form of (11).

To show that the recurrence relation (5) holds, by the chain rule, we have

\[
b_1 = -f'(0) = -f(0)p'(0) = \frac{1}{2},
\]

(13)

Next we have, using the Leibniz rule,

\[
f^{(k+1)}(x) = (f(x)p'(x))^k = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(x)p^{(k-i+1)}(x),
\]

(14)

where \(f^{(i)}\) indicates the \(i\)th derivative of \(f(x)\) for \(i \geq 1\) and \(f^{(0)} = f\). By virtue of the facts

\[
b_{k+1} = -\frac{f^{(k+1)}(0)}{(k+1)!}, \quad p^{(i)}(0) = -\frac{i!}{i(i+1)}, \quad \binom{k}{i} = \frac{k!}{i!(k-i)!},
\]

(15)

separating the first term in (14) from the summation, we get

\[
b_{k+1} = \frac{1}{(k+1)(k+2)} - \frac{1}{k+1} \sum_{i=1}^{k} \frac{b_i}{k-i+2},
\]

(16)

from which the recurrence relation (5) follows. This proves Lemma 4.

To find \(b_n\) in (11), starting with \(b_1 = 1/2\), and applying the recurrence relation (5) repeatedly, we obtain

\[
b_2 = \frac{1}{6} - \frac{1}{4} b_1 = \frac{1}{24},
\]

\[
b_3 = \frac{1}{12} - \frac{1}{9} b_1 - \frac{1}{6} b_2 = \frac{1}{48},
\]

\[
b_4 = \frac{1}{20} - \frac{1}{16} b_1 - \frac{1}{12} b_1 - \frac{1}{8} b_3 = \frac{73}{5760}.
\]

(17)

For \(n \geq 5\), the computation of \(b_n\) is considerably longer and complicated. Implementing the recurrence relation (5) with Maple, we easily find the next six coefficients as follows:

\[
b_5 = \frac{11}{1280}, \quad b_6 = \frac{3625}{580608}, \quad b_7 = \frac{5525}{1161216},
\]

\[
b_8 = \frac{5233001}{1393459200}, \quad b_9 = \frac{1212281}{398131200}, \quad b_{10} = \frac{927777937}{367873228800}.
\]

(18)

Those calculations suggest the following lemma.

**Lemma 5.** If \(b_n\) satisfies the recurrence relation (5), then \(b_n > 0\) for all \(n \geq 1\).
**Proof.** In view of the recurrence relation (5), we see that $b_{n+1} > 0$ is equivalent to

$$
\sum_{i=1}^{n} \frac{b_i}{n-i+2} < \frac{1}{n+2}.
$$

We make the inductive hypothesis that (19) is true for all positive integers $n$. This hypothesis is true for $n = 1$ as $b_1 = 1/2$ and

$$
\frac{b_1}{2} = \frac{1}{4} < \frac{1}{3}.
$$

Now, by the recurrence relation (5), we have

$$
\frac{1}{k+3} - \sum_{i=1}^{k} \frac{b_i}{k-i+3} = \frac{1}{k+3} - \sum_{i=1}^{k} \frac{b_i}{k-i+3} - \frac{b_{k+1}}{2}
$$

$$
= \frac{1}{k+3} - \sum_{i=1}^{k} \frac{b_i}{k-i+3} - \frac{1}{2(k+1)} \left( \frac{1}{k+2} - \sum_{i=1}^{k} \frac{b_i}{k-i+2} \right)
$$

$$
= \frac{2(k+1)(k+2) - (k+3)}{2(k+1)(k+2)(k+3)} - \sum_{i=1}^{k} \frac{2(k+1)(k-i+2) - (k-i+3)}{2(k+1)(k-i+3)} \frac{b_i}{k-i+2}
$$

$$
= \frac{2k^2 + 5k + 1}{2(k+1)(k+3)} \left\{ \frac{1}{k+2} - \sum_{i=1}^{k} \frac{b_i}{k-i+2} \right\}
$$

$$
= \frac{2k^2 + 5k + 1}{2(k+1)(k+3)} \left\{ \frac{1}{k+2} - \sum_{i=1}^{k} \frac{b_i}{k-i+2} \right\}
$$

$$
> 0,
$$

from which (19) holds for $n = k + 1$. Here we have used the fact

$$
\frac{2(k+1)(k-i+2) - (k-i+3)}{2(k+1)(k-i+3)} \frac{b_i}{k-i+2} < 1, \quad \text{for } 1 \leq i \leq k
$$

and the inductive hypothesis for $n = k$. Therefore, the lemma now follows by the principle of mathematical induction. \qed

Now, we turn to the proof of Theorem 1.

**Proof of Theorem 1.** By virtue of (7) and (11), taking $t = 1/(1+x)$, we have

$$
\left(1 + \frac{1}{x}\right)^x = e \left(1 - \sum_{n=1}^{\infty} \frac{b_n}{(1+x)^n}\right).
$$

(23)
By Lemmas 4 and 5, we have that \( b_n > 0 \) and satisfies the recurrence relation (5). This proves Theorem 1.

**Remark 6.** As an added bonus, taking \( x = n \) in (23), we have

\[
(1 + \frac{1}{n})^n = e \left( 1 - \sum_{k=1}^{\infty} \frac{b_k}{(1+n)^k} \right). \tag{24}
\]

Thus, for any positive integer \( m \geq 1 \), we obtain

\[
(1 + \frac{1}{n})^n < e \left( 1 - \sum_{k=1}^{m} \frac{b_k}{(1+n)^k} \right). \tag{25}
\]

On the other hand, noticing that \( b_k \leq 1/k(k+1) \) from (5), we have

\[
(1 + \frac{1}{n})^n > e \left( 1 - \sum_{k=1}^{\infty} \frac{1}{k(k+1)(1+n)^k} \right). \tag{26}
\]

Combining inequalities (24) and (26), we deduce that

\[
e \left( 1 - \sum_{k=1}^{\infty} \frac{1}{k(k+1)(1+n)^k} \right) < (1 + \frac{1}{n})^n < e \left( 1 - \sum_{k=1}^{m} \frac{b_k}{(1+n)^k} \right). \tag{27}
\]

This improves Kloosterman’s inequality [2, pages 324–325] and [4, inequality (2.7)].

Next, we prove Theorem 2 by modifying the approach used to prove Hardy’s inequality [1].

**Proof of Theorem 2.** For any positive sequence \( \{c_n\} \), using the arithmetic-geometric average inequality, we have

\[
\left( \prod_{k=1}^{n} c_k a_k \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} c_k a_k. \tag{28}
\]

So that

\[
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} = \sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} c_k a_k \right)^{1/n} \leq \sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} c_k \right)^{-1/n} \left( \frac{1}{n} \sum_{k=1}^{n} c_k a_k \right). \tag{29}
\]

Exchanging the order of the summation in the last inequality, we have

\[
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq \sum_{k=1}^{\infty} c_k a_k \sum_{n=k}^{\infty} \frac{1}{n} \left( \prod_{k=1}^{n} c_k \right)^{-1/n}. \tag{30}
\]
Set
\[ c_k = \left(1 + \frac{1}{k}\right)^k, \quad k = 1, 2, \ldots, \]  \hfill (31)
we have
\[ \prod_{k=1}^{n} c_k = (1 + n)^n, \]  \hfill (32)
and hence
\[ \sum_{n=1}^{\infty} \frac{1}{n} \left(\prod_{k=1}^{n} c_k\right)^{-1/n} = \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = \frac{1}{n}. \]  \hfill (33)
Thus, by virtue of (30), we deduce that
\[ \sum_{n=1}^{\infty} \left(a_1 a_2 \cdots a_n \right)^{1/n} \leq \sum_{k=1}^{\infty} \frac{1}{k} c_k a_k = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n a_n. \]  \hfill (34)
Taking \( x = n \) in Theorem 1, we have refined Carleman’s inequality (1) as
\[ \sum_{n=1}^{\infty} \left(a_1 a_2 \cdots a_n \right)^{1/n} \leq e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{\infty} \frac{b_k}{(1+n)^k}\right) a_n < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{b_k}{(1+n)^k}\right) a_n, \]  \hfill (35)
where \( m \) is any positive integer. This proves Theorem 2 as required.

**Remark 7.** It is clear that (2) is the special case of (35) at \( m = 1 \). Furthermore, by the binomial series, we have
\[ \left(1 + \frac{1}{n + 1/5}\right)^{-1/2} > 1 - \frac{1}{2(n + 1)} - \frac{1}{24(n + 1)^2}, \quad \text{for} \ n = 1, 2, \ldots. \]  \hfill (36)
Therefore, when \( m = 2 \), (35) strengthens (3).

**References**


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