ON A CLASS OF DIOPHANTINE EQUATIONS

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Cohn (1971) has shown that the only solution in positive integers of the equation $Y(Y+1)(Y+2)(Y+3) = 2X(X+1)(X+2)(X+3)$ is $X = 4$, $Y = 5$. Using this result, Jeyaratnam (1975) has shown that the equation $Y(Y+m)(Y+2m)(Y+3m) = 2X(X+m)(X+2m)(X+3m)$ has only four pairs of nontrivial solutions in integers given by $X = 4m$ or $-7m$, $Y = 5m$ or $-8m$ provided that m is of a specified type. In this paper, we show that if $m = (m_1, m_2)$ has a specific form then the nontrivial solutions of the equation $Y(Y+m_1)(Y+m_2)(Y+m_1+m_2) = 2X(X+m_1)(X+m_2)(X+m_1+m_2)$ are m times the primitive solutions of a similar equation with smaller m 's. Then we specifically find all solutions in integers of the equation in the special case $m_2 = 3m_1$.

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We generalize the equations of Cohn [1] and Jeyaratnam [2] by considering the Diophantine equation

$$Y(Y+m_1)(Y+m_2)(Y+m_1+m_2) = 2X(X+m_1)(X+m_2)(X+m_1+m_2). \quad (1)$$

The trivial solutions of (1) are the sixteen pairs obtained by equating both sides of the equation to zero. A nontrivial solution with $(X, Y, m_1, m_2) = 1$ is called a primitive solution.

THEOREM 1. *If every prime p dividing $m = (m_1, m_2)$ is such that*

$$p \equiv 2, 3, 5 \pmod{8} \quad \text{or} \quad p \equiv 1 \pmod{8} \quad \text{with} \quad 2^{(p-1)/4} \equiv -1 \pmod{p}, \quad (2)$$

then every nontrivial solution of (1) is m times a primitive solution of

$$Y\left(Y + \frac{m_1}{m}\right)\left(Y + \frac{m_2}{m}\right)\left(Y + \frac{m_1+m_2}{m}\right) = 2X\left(X + \frac{m_1}{m}\right)\left(X + \frac{m_2}{m}\right)\left(X + \frac{m_1+m_2}{m}\right). \quad (3)$$

THEOREM 2. *If every prime p dividing N is of the form (2), then every nontrivial solution of*

$$Y(Y+N)(Y+cN)(Y+(1+c)N) = 2X(X+N)(X+cN)(X+(1+c)N) \quad (4)$$

is N times a nontrivial solution of

$$Y(Y+1)(Y+c)(Y+1+c) = 2X(X+1)(X+c)(X+1+c), \quad (5)$$

where c is a positive integer.

THEOREM 3. *The equation*

$$Y(Y+1)(Y+3)(Y+4) = 2X(X+1)(X+3)(X+4) \quad (6)$$

has only four pairs of nontrivial solutions in integers given by $X = 14$ or -18 , $Y = 17$ or -21 .

THEOREM 4. *If every prime p dividing N is of the form (2), then the equation*

$$Y(Y+N)(Y+3N)(Y+4N) = 2X(X+N)(X+3N)(X+4N) \quad (7)$$

has only four pairs of nontrivial solutions in integers given by $X = 14N$ or $-18N$, $Y = 17N$ or $-21N$.

Note that Theorem 2 follows immediately by applying Theorem 1 with $m_1 = N$, $m_2 = cN$, and $m = (N, cN) = N$. Also Theorem 4 follows easily by combining Theorem 2, in the case $c = 3$, with Theorem 3.

LEMMA 5. *Every solution of (1) that is not primitive is $K = (X, Y, m_1, m_2)$ times a primitive solution of*

$$Y\left(Y + \frac{m_1}{K}\right)\left(Y + \frac{m_2}{K}\right)\left(Y + \frac{m_1+m_2}{K}\right) = 2X\left(X + \frac{m_1}{K}\right)\left(X + \frac{m_2}{K}\right)\left(X + \frac{m_1+m_2}{K}\right). \quad (8)$$

PROOF. Suppose that X, Y is a solution of (1). By dividing both sides of that equation by K^4 we find

$$\begin{aligned} & \frac{Y}{K}\left(\frac{Y}{K} + \frac{m_1}{K}\right)\left(\frac{Y}{K} + \frac{m_2}{K}\right)\left(\frac{Y}{K} + \frac{m_1+m_2}{K}\right) \\ &= 2 \cdot \frac{X}{K}\left(\frac{X}{K} + \frac{m_1}{K}\right)\left(\frac{X}{K} + \frac{m_2}{K}\right)\left(\frac{X}{K} + \frac{m_1+m_2}{K}\right). \end{aligned} \quad (9)$$

Thus $X/K, Y/K$ is a solution of (8). The lemma follows since $(X/K, Y/K, m_1/K, m_2/K) = 1$. $\square$

LEMMA 6. *Equation (1) cannot have a primitive solution if the greatest common divisor $m = (m_1, m_2)$ is divisible by a prime p of the form (2).*

PROOF. By completing the squares in (1) we find

$$\left[\frac{(2Y + m_1 + m_2)^2 - m_1^2 - m_2^2}{2}\right]^2 - 2\left[\frac{(2X + m_1 + m_2)^2 - m_1^2 - m_2^2}{2}\right]^2 = -m_1^2 m_2^2. \quad (10)$$

Letting

$$y = 2Y + m_1 + m_2, \quad (11)$$

$$x = 2X + m_1 + m_2, \quad (12)$$

$$\begin{aligned} A &= \frac{y^2 - m_1^2 - m_2^2}{2} = 2Y^2 + 2Y(m_1 + m_2) + m_1 m_2, \\ B &= \frac{x^2 - m_1^2 - m_2^2}{2} = 2X^2 + 2X(m_1 + m_2) + m_1 m_2, \end{aligned} \quad (13)$$

we obtain the equations

$$y^2 = 2A + m_1^2 + m_2^2, \quad x^2 = 2B + m_1^2 + m_2^2, \quad (14)$$

$$A^2 - 2B^2 = -m_1^2 m_2^2. \quad (15)$$

If $2 \mid m$, then

$$\begin{aligned} A^2 - 2B^2 = -m_1^2 m_2^2 &\Rightarrow A, B \equiv 0 \pmod{4} \xrightarrow{\text{by (13)}} 2X^2, 2Y^2 \equiv 0 \pmod{4} \\ &\Rightarrow X, Y \equiv 0 \pmod{2} \Rightarrow 2 \mid (X, Y, m_1, m_2) \neq 1. \end{aligned} \quad (16)$$

Let $p \mid m$ such that $p \equiv 3, 5 \pmod{8}$. Assume that $p \nmid A$, then by (15), $p \nmid B$. Also by (15), $1 = (2B^2/p) = (2/p) = -1$, a contradiction. Thus $p \mid A$ and hence $p \mid B$. By (13), $p \mid X$ and Y . Therefore $(X, Y, m_1, m_2) \neq 1$.

Suppose that $p \mid m$ such that $p \equiv 1 \pmod{8}$ and $2^{(p-1)/4} \equiv -1 \pmod{p}$. If $p \nmid A$, then $p \nmid B$. Since $(2/p) = 1$, (13) implies that A and B are quadratic residues mod p . Thus $B^{(p-1)/2} \equiv A^{(p-1)/2} \equiv 1 \pmod{p}$. From (15) we find that

$$2B^2 \equiv A^2 \pmod{p} \Rightarrow 2^{(p-1)/4} B^{(p-1)/2} \equiv A^{(p-1)/2} \Rightarrow 2^{(p-1)/4} \equiv 1 \pmod{p}, \quad (17)$$

a contradiction. Therefore $p \mid A, B$. By (13), $p \mid X, Y$ and hence $(X, Y, m_1, m_2) \neq 1$ and the lemmas follows. $\square$

PROOF OF THEOREM 1. By Lemmas 5 and 6 and the fact that $(m_1/K, m_2/K)$ can only have prime divisors of the form (2), a nontrivial solution of (2) is a multiple of a primitive solution of (3) with $(m_1/K, m_2/K) = 1$. This happens when $K = (m_1, m_2) = m$ and the theorem follows. $\square$

For Theorem 3 we now prove the following lemma.

LEMMA 7. *The only solution in positive integers of (6) is $X = 14, Y = 17$.*

PROOF. Note that (6) can be obtained from (1) by letting $m_1 = 1$ and $m_2 = 3$. Then (11), (12), (13), (14), and (15) become

$$y = 2Y + 4, \quad x = 2X + 4, \quad (18)$$

$$A = 2Y^2 + 8Y + 3, \quad B = 2X^2 + 8X + 3, \quad (19)$$

$$y^2 = 2A + 10, \quad x^2 = 2B + 10, \quad (20)$$

$$A^2 - 2B^2 = -9. \quad (21)$$

All solutions in positive integers of (21) are given by

$$A = V_n, \quad B = U_n, \quad (22)$$

where

$$V_n + \sqrt{2}U_n = (3 + 3\sqrt{2})(3 + 2\sqrt{2})^n = 3(1 + \sqrt{2})^{2n+1}, \quad n = 0, 1, 2, \dots \quad (23)$$

Thus

$$\begin{aligned} V_n &= \frac{3(1+\sqrt{2})^{2n+1} + 3(1-\sqrt{2})^{2n+1}}{2}, \\ U_n &= \frac{3(1+\sqrt{2})^{2n+1} - 3(1-\sqrt{2})^{2n+1}}{-2\sqrt{2}}. \end{aligned} \quad (24)$$

Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$, then

$$\begin{aligned} \alpha + \beta &= 2, & \alpha - \beta &= -2\sqrt{2}, & \alpha\beta &= -1, \\ V_n &= 3 \left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{\alpha + \beta} \right), & U_n &= 3 \left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} \right). \end{aligned} \quad (25)$$

From (20) and (22), we must have

$$y^2 = 2V_n + 10, \quad (26)$$

$$x^2 = 2U_n + 10. \quad (27)$$

Using (25), we can easily find that

$$V_{-n} = -V_{n-1}, \quad (28)$$

$$U_{-n} = U_{n-1}, \quad (29)$$

$$U_{n+2} = 6U_{n+1} - U_n, \quad (30)$$

$$V_{n+2} = 6V_{n+1} - V_n. \quad (31)$$

Let

$$\eta_r = \frac{\alpha^r + \beta^r}{\alpha + \beta}, \quad \xi_r = \frac{\alpha^r - \beta^r}{\alpha - \beta}, \quad (32)$$

then we easily find that

$$V_n = 3\eta_{2n+1}, \quad U_n = 3\xi_{2n+1}, \quad (33)$$

$$\xi_{2r} = 2\xi_r \eta_r, \quad (34)$$

$$\eta_{2r} = 2\eta_r^2 + (-1)^{r+1} = 4\xi_r^2 + (-1)^r, \quad (35)$$

$$\eta_{m+n} = \eta_m \eta_n + 2\xi_m \xi_n, \quad (36)$$

$$\xi_{m+n} = \xi_m \eta_n + \xi_n \eta_m. \quad (37)$$

Using relations (33), (34), (35), (36), and (37), we get

$$V_{n+r} \equiv (-1)^{r+1} V_n \pmod{\eta_r}, \quad (38)$$

$$V_{n+2r} \equiv V_n \pmod{\eta_r}, \quad (39)$$

$$U_{n+r} \equiv (-1)^{r+1} U_n \pmod{\eta_r}, \quad (40)$$

$$U_{n+2r} \equiv U_n \pmod{\eta_r}, \quad (41)$$

$$\eta_{3r} = \eta_r \left[4\eta_r^2 + 3(-1)^{r+1} \right], \quad (42)$$

$$\xi_{3r} = \xi_r \left[4\eta_r^2 + (-1)^{r+1} \right]. \quad (43)$$

Let

$$\theta_t = \xi_{2^t}, \quad \phi_t = \eta_{2^t}, \quad (44)$$

then we get

$$\theta_{t+1} = 2\theta_t\phi_t, \quad (45)$$

$$\phi_{t+1} = 2\phi_t^2 - 1 = 4\theta_t^2 + 1 = \phi_t^2 + 2\theta_t^2, \quad (46)$$

$$\phi_t^2 = 2\theta_t^2 + 1. \quad (47)$$

Using (42), (43), and (44), we find that for $k = 2^t$ we have

$$\eta_{6k} = \phi_{t+1}[4\phi_{t+1}^2 - 3], \quad (48)$$

$$\xi_{6k} = \theta_{t+1}[4\phi_{t+1}^2 - 1]. \quad (49)$$

We will need some of the entries in Tables 1 and 2.

TABLE 1

n	U_n	V_n
1	15	21
3	507	717
4	2955	4179
11	675176043	954843117
8	3410067	4822563
23	1037608383669414483	1467399848617311837
24	6047624848242867123	8552633080529593443

TABLE 2

k	η_k
2	3
3	7
4	17
6	$3^2 \cdot 11$
8	577
12	$17 \cdot 1153$
24	$97 \cdot 577 \cdot 13729$
48	$193 \cdot 9188923201 \cdot 665857$

Now we consider the following cases.

(a) Equation (26) is impossible if $n \equiv 1 \pmod{3}$. Let $n = 1 + 3r$ where $r \geq 0$, then using (38) we get

$$\begin{aligned} V_n &\equiv \pm V_1 \pmod{\eta_3}, \\ V_n &\equiv \pm 21 \equiv 0 \pmod{7}. \end{aligned} \quad (50)$$

Hence $2V_n + 10 \equiv 10 \equiv 3 \pmod{7}$. Since $(3/7) = -1$, (26) is impossible.

(b) Equation (27) is impossible if $n \equiv 1, 2 \pmod{4}$. Using (40), we get

$$\begin{aligned} U_n &\equiv \pm U_1, \pm U_2 \pmod{\eta_4}, \\ U_n &\equiv \pm 15, \pm 87 \equiv \pm 2 \pmod{17}. \end{aligned} \quad (51)$$

Hence $2U_n + 10 \equiv \pm 4 + 10 \equiv 6, -3 \pmod{17}$. Since $(6/17) = (-3/17) = -1$, (27) is impossible.

(c) Equation (26) is impossible if $n \equiv 8 \pmod{12}$. Using (39) and (28) we get

$$\begin{aligned} V_n &\equiv V_{-4} = -V_3 \pmod{\eta_6}, \\ V_n &\equiv -717 \equiv -2 \pmod{11} \quad \text{since } 11 \mid \eta_6. \end{aligned} \quad (52)$$

Hence $2V_n + 10 \equiv 6 \pmod{11}$. Since $(6/11) = -1$, (26) is impossible.

(d) Equation (26) is impossible if $n \equiv 11 \pmod{16}$. Using (39) and (28) we get

$$\begin{aligned} V_n &\equiv V_{-5} = -V_4 \pmod{\eta_8}, \\ V_n &\equiv -4179 \equiv -140 \pmod{577}. \end{aligned} \quad (53)$$

Hence $2V_n + 10 \equiv -270 \pmod{577}$. Since $(-270/577) = -1$, (26) is impossible.

(e) Equation (26) is impossible if $n \equiv 11, 12 \pmod{24}$. Using (38) and (28) we get

$$\begin{aligned} V_n &\equiv \pm V_{11}, \pm V_{-12} = \pm V_{11}, \mp V_{11} \pmod{\eta_{24}}, \\ V_n &\equiv \pm 954843117 \equiv \pm 46 \pmod{97} \quad \text{since } 97 \mid \eta_{24}. \end{aligned} \quad (54)$$

Hence $2V_n + 10 \equiv \pm 102 + 10 \equiv 5, 15 \pmod{97}$. Since $(5/97) = (15/97) = -1$, (26) is impossible.

(f) Equation (26) is impossible if $n \equiv 15 \pmod{24}$. Using (38) and (28) we get

$$\begin{aligned} V_n &\equiv \pm V_{-9} = \mp V_8 \pmod{\eta_{24}}, \\ V_n &\equiv \mp 4822563 \equiv \pm 504289 \pmod{1331713} \quad \text{since } 1331713 \mid \eta_{24}. \end{aligned} \quad (55)$$

Hence $2V_n + 10 \equiv 323145, 1008588 \pmod{1331713}$. Since $(323145/1331713) = (1008588/1331713) = -1$, (26) is impossible.

(g) Equation (26) is impossible if $n \equiv 23, 24 \pmod{48}$. Using (38) and (28) we get

$$V_n \equiv \pm V_{23}, \pm V_{-24} = \pm V_{23}, \mp V_{23} \pmod{\eta_{48}}. \quad (56)$$

Since $V_{23} = 1467399848617311837$ and $\tau = 9188923201 \mid \eta_{48}$, we have $2V_n + 10 \equiv 11299978, -11299958 \pmod{\tau}$. Since $(11299978/\tau) = (-11299958/\tau) = -1$, (26) is impossible.

(h) Equation (27) is impossible if $n \equiv 3 \pmod{48}$, $n \neq 3$. That is, $n = 3 + 3 \cdot 2^t \cdot r$, where $t \geq 4$ and r is an odd positive integer. Using (40) we get $U_n \equiv -U_3 = -507 \pmod{\eta_{3 \cdot 2^t}}$. Hence

$$2U_n + 10 \equiv -1004 \pmod{\eta_{3 \cdot 2^t}}. \quad (57)$$

From (48) we get $\eta_{3 \cdot 2^t} = \eta_{6 \cdot 2^{t-1}} = \phi_t[4\phi_t^2 - 3]$. Using this in (57) we simultaneously get

$$\begin{aligned} 2U_n + 10 &\equiv -1004 \pmod{\phi_t}, \\ 2U_n + 10 &\equiv -1004 \pmod{4\phi_t^2 - 3}. \end{aligned} \quad (58)$$

Since $\phi_{t+1} = 2\phi_t^2 - 1$ and $\phi_3 = 577$ we can easily show, by induction, the following for $t \geq 3$

$$\phi_t \equiv 1 \pmod{8}, \quad (59)$$

$$\phi_t \equiv 81, 69, -17, 75, -46, -36 \pmod{251}, \quad (60)$$

when

$$t \equiv 0, 1, 2, 3, 4, 5 \pmod{6}, \quad (61)$$

respectively. By (59) we get

$$\left(\frac{-1004}{\phi_t}\right) = \left(\frac{-1}{\phi_t}\right) \left(\frac{4}{\phi_t}\right) \left(\frac{251}{\phi_t}\right) = (1)(1) \left(\frac{\phi_t}{251}\right) = \left(\frac{\phi_t}{251}\right). \quad (62)$$

Similarly $(-1004/(4\phi_t^2 - 3)) = ((4\phi_t^2 - 3)/251)$. Using (60) we find that $(\phi_t/251) = -1$ if $t \equiv 2, 5 \pmod{6}$ and $((4\phi_t^2 - 3)/251) = -1$ if $t \equiv 0, 1, 3, 4 \pmod{6}$. Therefore (27) is always impossible in this case.

Note that for $n = 3$ we have $U_3 = 507$ and $V_3 = 717$. Now (22) and (19) imply that $X = 14$, $Y = 17$, a nontrivial solution of (6).

(i) Equation (27) is impossible if $n \equiv \delta \pmod{48}$ and $n > 0$, where $\delta = 0, -1$. That is $n = \delta + 3k(2r + 1) = \delta + 6kr + 3k$, where $k = 2^t$, $t \geq 4$, and $r \geq 0$. Using (40) and (33) we get

$$U_n \equiv \pm U_{3k+\delta} = \pm 3\xi_{6k+2\delta+1} \pmod{\eta_{6k}}. \quad (63)$$

The upper and the lower signs depend on whether r is even or odd. Using (37), we get

$$\xi_{6k+2\delta+1} = \xi_{6k}\eta_{2\delta+1} + \xi_{2\delta+1}\eta_{6k}, \quad (64)$$

where $\eta_{2\delta+1} = 1, -1$ for $\delta = 0, -1$ and $\xi_{2\delta+1} = 1$ for $\delta = 0, 1$. Now (64) becomes $\xi_{6k+2\delta+1} = \pm \xi_{6k} + \eta_{6k}$, where the upper and lower signs depend on whether $\delta = 0$ or $\delta = 1$, respectively. Using this in (63) we get

$$U_n \equiv \pm 3\xi_{6k} \pmod{\eta_{6k}}. \quad (65)$$

For $\delta = 0$, the upper sign holds if r is even and the lower sign holds if r is odd. For $\delta = -1$, upper sign holds if r is odd and the lower sign holds if r is even. Using (48) and (49) in (65) we get

$$U_n \equiv \pm 3\theta_{t+1} [4\phi_{t+1}^2 - 1] = \pm 3\theta_{t+1} [4\phi_{t+1}^2 - 3 + 2] \pmod{\phi_{t+1} [4\phi_{t+1}^2 - 3]}. \quad (66)$$

Therefore we simultaneously get $U_n \equiv \pm 6\theta_{t+1} \pmod{4\phi_{t+1}^2 - 3}$ and $U_n \equiv \mp 3\theta_{t+1} \pmod{\phi_{t+1}}$. Thus

$$\begin{aligned} 2U_n + 10 &\equiv 10 \pm 12\theta_{t+1} \pmod{4\phi_{t+1}^2 - 3}, \\ 2U_n + 10 &\equiv 10 \mp 6\theta_{t+1} \pmod{\phi_{t+1}}. \end{aligned} \quad (67)$$

In what follows we need the fact that

$$\theta_t \equiv 0 \pmod{8}, \quad \text{for } t \geq 3, \quad (68)$$

which follows by induction using (45) and $\theta_3 = 408$. Now we show that

$$\left(\frac{10 \pm 12\theta_{t+1}}{4\phi_{t+1}^2 - 3} \right) = \left(\frac{5 \pm 6\theta_{t+1}}{59} \right), \quad (69)$$

$$\left(\frac{10 \mp 6\theta_{t+1}}{\phi_{t+1}} \right) = \pm \left(\frac{10\theta_t \pm 3\phi_t}{59} \right). \quad (70)$$

For (69) we have

$$\begin{aligned} \left(\frac{10 \pm 12\theta_{t+1}}{4\phi_{t+1}^2 - 3} \right) &= \left(\frac{2}{4\phi_{t+1}^2 - 3} \right) \left(\frac{5 \pm 6\theta_{t+1}}{4\phi_{t+1}^2 - 3} \right) \\ &= \left(\frac{5 \pm 6\theta_{t+1}}{4\phi_{t+1}^2 - 3} \right), \quad \text{using (59)} \\ &= \left(\frac{5 \pm 6\theta_{t+1}}{8\theta_{t+1}^2 + 1} \right), \quad \text{using (47)} \\ &= \left(\frac{8\theta_{t+1}^2 + 1}{5 \pm 6\theta_{t+1}} \right), \quad \text{since } \theta_t \equiv 0 \pmod{4} \\ &= \left(\frac{36(8\theta_{t+1}^2 + 1)}{5 \pm 6\theta_{t+1}} \right) = \left(\frac{236}{5 \pm 6\theta_{t+1}} \right) \\ &= \left(\frac{59}{5 \pm 6\theta_{t+1}} \right), \quad \text{since } 36\theta_{t+1}^2 \equiv 25 \pmod{5 \pm 6\theta_{t+1}}. \end{aligned} \quad (71)$$

Equation (69) follows since $\theta_t \equiv 0 \pmod{4}$. For (70) we have

$$\begin{aligned} \left(\frac{10 \mp 6\theta_{t+1}}{\phi_{t+1}} \right) &= \left(\frac{5 \mp 3\theta_{t+1}}{\phi_{t+1}} \right) \\ &= \left(\frac{5(\phi_t^2 - 2\theta_t^2) \mp 3\theta_{t+1}}{\phi_t^2 + 2\theta_t^2} \right), \quad \text{using (46) and (47)} \\ &= \left(\frac{-20\theta_t^2 \mp 6\theta_t\phi_t}{\phi_t^2 + 2\theta_t^2} \right), \quad \text{since } \phi_t^2 \equiv -2\theta_t^2 \pmod{\phi_t^2 + 2\theta_t^2} \\ &= \left(\frac{-1}{\phi_t^2 + 2\theta_t^2} \right) \left(\frac{2}{\phi_t^2 + 2\theta_t^2} \right) \left(\frac{\theta_t}{\phi_t^2 + 2\theta_t^2} \right) \left(\frac{10\theta_t \pm 3\phi_t}{\phi_t^2 + 2\theta_t^2} \right) \\ &= (1)(1)(1) \left(\frac{10\theta_t \pm 3\phi_t}{\phi_t^2 + 2\theta_t^2} \right) \\ &= \left(\frac{\phi_t^2 + 2\theta_t^2}{10\theta_t \pm 3\phi_t} \right) = \left(\frac{9\phi_t^2 + 18\theta_t^2}{10\theta_t \pm 3\phi_t} \right) \\ &= \left(\frac{118\theta_t^2}{10\theta_t \pm 3\phi_t} \right), \quad \text{since } 9\phi_t^2 \equiv 100\theta_t^2 \pmod{10\theta_t \pm 3\phi_t} \\ &= \left(\frac{2}{10\theta_t \pm 3\phi_t} \right) \left(\frac{59}{10\theta_t \pm 3\phi_t} \right) = - \left(\frac{59}{10\theta_t \pm 3\phi_t} \right). \end{aligned} \quad (72)$$

Equation (70) follows using (59) and (68).

Since $\theta_3 = 408$, $\phi_3 = 577$, $\phi_{t+1} = 2\phi_t^2 - 1$, and $\theta_{t+1} = 2\theta_t\phi_t$, we can inductively show the following:

$$\begin{aligned}\theta_t &\equiv 12, 5, -12, -5 \pmod{59} & \text{if } t \equiv 0, 1, 2, 3 \pmod{4}, \\ \phi_t &\equiv -17, -13 \pmod{59} & \text{if } t \equiv 0, 1, \pmod{2}, \text{ respectively.}\end{aligned}\tag{73}$$

Using (73) and taking the upper signs in (69) and (70), we get

$$\begin{aligned}\left(\frac{5+6\theta_{t+1}}{59}\right) &= -1 & \text{if } t \equiv 2, 3 \pmod{4}, \\ \left(\frac{10\theta_t+3\phi_t}{59}\right) &= -1 & \text{if } t \equiv 0, 1, 2 \pmod{4}.\end{aligned}\tag{74}$$

Thus this case is always impossible. Using the lower signs in (69) and (70) we get

$$\begin{aligned}\left(\frac{5-6\theta_{t+1}}{59}\right) &= -1 & \text{if } t \equiv 0, 1 \pmod{4}, \\ -\left(\frac{10\theta_t-3\phi_t}{59}\right) &= -1 & \text{if } t \equiv 0, 2, 3 \pmod{4},\end{aligned}\tag{75}$$

and this case is also impossible. Therefore (27) is always impossible.

The only remaining case is $n = 0$. Then $U_0 = V_0 = 0$ and so $X = Y = 0$, a trivial solution and Lemma 7 is proved. $\square$

PROOF OF THEOREM 3. First note that if the pair (X, Y) is a solution of (6), so are $(-X-4, Y)$, $(X, -Y-4)$, and $(-X-4, -Y-4)$. Note also that $-X-4 < -4$ if and only if $X > 0$ and $-Y-4 < -4$ if and only if $Y > 0$. Since $(14, 17)$ is the only solution in positive integers of (6), $(-18, 17)$, $(14, -21)$, $(-18, -21)$ are the only solutions where each of X and Y is either positive or less than -4 . The only remaining possibilities for more solutions are where X or $Y \in \{-4, -3, -2, -1, 0\}$ where there are no nontrivial solutions and the proof is completed. $\square$

Finally note that (6) has 16 trivial solutions and 4 nontrivial solutions of a total of only 20 solutions.

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