COMPACT HERMITIAN OPERATORS ON PROJECTIVE TENSOR PRODUCTS OF BANACH ALGEBRAS

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Let U and V be, respectively, an infinite- and a finite-dimensional complex Banach algebras, and let $U \otimes_p V$ be their projective tensor product. We prove that (i) every compact Hermitian operator T_1 on U gives rise to a compact Hermitian operator T on $U \otimes_p V$ having the properties that $\|T_1\| = \|T\|$ and $\text{sp}(T_1) = \text{sp}(T)$; (ii) if U and V are separable and U has Hermitian approximation property (HAP), then $U \otimes_p V$ is also separable and has HAP; (iii) every compact analytic semigroup (CAS) on U induces the existence of a CAS on $U \otimes_p V$ having some nice properties. In addition, the converse of the above results are discussed and some open problems are posed.

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1. Introduction. We first introduce the terminologies which are initially needed for our main purpose.

1.1. The projective tensor norm $\|\cdot\|_p$ is defined on the algebraic tensor product $U \otimes V$ of two complex Banach algebras U and V by

$$\|a\|_p = \inf \left\{ \sum_{i=1}^n \|u_i\| \|v_i\| : a = \sum_{i=1}^n u_i \otimes v_i \right\}, \quad (1.1)$$

where the infimum is taken over all (finite) representations of $a \in U \otimes V$. The *projective tensor product* $U \otimes_p V$ is the completion of $U \otimes V$ with this norm. Furthermore, a norm $\|\cdot\|_\pi$ on $U \otimes V$ is said to be a *cross-norm* if $\|u \otimes v\|_\pi = \|u\| \|v\|$. (For detailed discussion of various tensor products, see [1, 3, 6, 7].)

1.2. A *state* on a unital Banach algebra U with the unit e is a continuous linear functional f such that $\|f\| = f(e) = 1$, and an element u in U is *Hermitian* if and only if its numerical range, that is, $N(u) = \{f(u) : f \text{ is a state on } U\}$, is contained in the real line. Equivalently, u is Hermitian if and only if $\lim_{\alpha \rightarrow 0^+} (1/\alpha)[\|e + i\alpha u\| - 1] = 0$. If an operator T on U is such that it is a Hermitian as an element of the operator algebra, then T is called a *Hermitian operator* on U .

1.3. A linear transformation T mapping a normed linear space X into a normed linear space Y is said to be *compact* if, given any sequence $\{x_n\}$ in X such that $\{\|x_n\|\}$ is bounded, the sequence $\{Tx_n\}$ has a convergent subsequence. If T is both compact and Hermitian, it is then termed as a *compact Hermitian operator*. A compact Hermitian operator enjoys many powerful technical results (see [4]).

Throughout this paper, unless stated specifically, U denotes an infinite-dimensional complex Banach algebra with its dual U^* , V is a finite-dimensional complex Banach algebra with $\dim V = k$, $\{e_1, e_2, \dots, e_k\}$ a standardized basis for V with $\|e_j\| = 1$, for all $j = 1, 2, \dots, k$ and $U \otimes_p V$ their projective tensor product. Our primary aim is to study how a compact Hermitian operator on U , the Hermitian approximation property in U and a compact analytic semigroup on U gives rise to the significant existence of those on $U \otimes_p V$. The converse route from $U \otimes_p V$ to U with these properties is also investigated with some fruitful outcomes.

2. The main results. We first prove a simple but illuminating lemma.

LEMMA 2.1. *Every element a of $U \otimes_p V$ can be expressed in the form $\sum_{j=1}^k w_j \otimes e_j$.*

PROOF. We know that $U \otimes_p V$ is a Banach algebra consisting of all elements of the form

$$a = \sum_{i=1}^{\infty} u_i \otimes v_i, \quad \text{where } \sum_{i=1}^{\infty} \|u_i\| \|v_i\| < \infty, \quad (2.1)$$

(see [5]). We can choose v_i such that $\|v_i\| = 1$ for all i . If $\|v_i\| \neq 1$, put $v'_i = v_i / \|v_i\|$ and replace v_i by $\|v_i\| v'_i$ and adjoin $\|v_i\|$ to u_i . Then $v_i = \sum_{j=1}^k \alpha_{ij} e_j$, where α_{ij} 's are scalars.

Now

$$\begin{aligned} a_n &= \sum_{i=1}^n u_i \otimes v_i = \sum_{i=1}^n u_i \otimes \sum_{j=1}^k \alpha_{ij} e_j \\ &= \sum_{j=1}^k \left(\sum_{i=1}^n \alpha_{ij} u_i \right) \otimes e_j = \sum_{j=1}^k w_{j,n} \otimes e_j, \quad \text{where } w_{j,n} = \sum_{i=1}^n \alpha_{ij} u_i \in U. \end{aligned} \quad (2.2)$$

If $v = \sum_{j=1}^k \beta_j e_j$ is an arbitrary element of V , then the map $\phi : V \rightarrow \ell_k^\infty$, defined by $\phi(v) = (\beta_1, \beta_2, \dots, \beta_k)$, is a topological isomorphism. Hence there is a positive constant M such that $\|\phi(v)\| \leq M\|v\|$, for all $v \in V$. Substituting v_i in v , $\|\phi(v_i)\| \leq M\|v_i\| = M$, for all i . This gives

$$\max_{1 \leq j \leq k} \{ |\alpha_{ij}| \} \leq M. \quad (2.3)$$

Also,

$$\sum_{i=1}^n |\alpha_{ij}| \|u_i\| \leq M \sum_{i=1}^n \|u_i\| \leq M \sum_{i=1}^{\infty} \|u_i\| < \infty. \quad (2.4)$$

This shows that the partial sums of the series $\sum_{i=1}^{\infty} \alpha_{ij} u_i$ are absolutely uniformly bounded and hence the series $\sum_{i=1}^{\infty} \alpha_{ij} u_i$ is absolutely summable. Consequently, $\sum_{i=1}^{\infty} \alpha_{ij} u_i$ is summable to an element, say w_j in U . Thus, $\lim_{n \rightarrow \infty} w_{j,n} = w_j$ and so $a = \lim_{n \rightarrow \infty} a_n = \sum_{j=1}^k w_j \otimes e_j$. $\square$

Next, our first key result is the following.

THEOREM 2.2. *Every compact Hermitian operator T_1 on U gives rise to a compact Hermitian operator T on $U \otimes_p V$ having the properties that $\|T_1\| = \|T\|$ and $\text{sp}(T_1) = \text{sp}(T)$.*

PROOF. We prove the result in different steps.

STEP 1. One can define a map $T : U \otimes_p V \rightarrow U \otimes_p V$ by the rule

$$T\left(\sum_i u_i \otimes v_i\right) = \sum_i (T_1 u_i) \otimes v_i \quad \forall a = \sum_i u_i \otimes v_i \in U \otimes_p V. \quad (2.5)$$

It is easy to show that the map is well defined. Moreover, the linearity of T follows immediately from its definition.

Next we aim at proving the bound of T . For any arbitrary element $a \in U \otimes_p V$ and $\varepsilon > 0$, the definition of the projective norm provides a finite representation $\sum_{i=1}^n u_i \otimes v_i$ such that

$$\|a\|_p + \varepsilon > \sum_{i=1}^n \|u_i\| \|v_i\|. \quad (2.6)$$

For this representation of a we obtain

$$\|Ta\|_p = \left\| \sum_{i=1}^n (T_1 u_i) \otimes v_i \right\|_p \leq \|T_1\| \sum_{i=1}^n \|u_i\| \|v_i\| < \|T_1\| (\|a\|_p + \varepsilon). \quad (2.7)$$

Since ε is arbitrary, it follows that $\|Ta\|_p \leq \|T_1\| \|a\|_p$ for each $a \in U \otimes_p V$.

Consequently, T is bounded. Furthermore, the compactness of T can be proved without much difficulty.

STEP 2. Our next attempt is to show that T is Hermitian. T_1 is given to be Hermitian and therefore $\lim_{\alpha \rightarrow 0+} (1/\alpha) \{\|I_1 + i\alpha T_1\| - 1\} = 0$, where α is real and I_1 is the identity map on U . Let I be the identity map on $U \otimes_p V$, $\varepsilon > 0$, and let $\sum_{i=1}^n u_i \otimes v_i$ be a finite representation of a such that $\|a\|_p + \varepsilon > \sum_{i=1}^n \|u_i\| \|v_i\|$. Then we can obtain

$$\|(I + i\alpha T)a\|_p \leq \|I_1 + i\alpha T_1\| \|a\|_p \quad \forall a \in U \otimes_p V. \quad (2.8)$$

This gives $\|(I + i\alpha T)\| \leq \|I_1 + i\alpha T_1\|$.

On the other hand, let $u \in U$ with $\|u\| = 1$. Choose $v \in V$ such that $\|v\| = 1$. Then $\|u \otimes v\|_p = 1$.

Now,

$$\begin{aligned} \|(I + i\alpha T)\| &= \sup \{ \|(I + i\alpha T)a\|_p : \|a\|_p = 1 \} \\ &\geq \|(I + i\alpha T)(u \otimes v)\|_p \\ &= \|u \otimes v + (i\alpha T_1 u) \otimes v\|_p \\ &= \|(I_1 + i\alpha T_1)u\| \|v\|_p \\ &= \|(I_1 + i\alpha T_1)u\| \|v\| \\ &= \|(I_1 + i\alpha T_1)u\|. \end{aligned} \quad (2.9)$$

Thus, $\|I + i\alpha T\| \geq \|(I_1 + i\alpha T_1)u\|$, for all $u \in U$ with $\|u\| = 1$. This yields $\|I + i\alpha T\| \geq \|I_1 + i\alpha T_1\|$ and so $\|I + i\alpha T\| = \|I_1 + i\alpha T_1\|$.

Therefore, $\lim_{\alpha \rightarrow 0+} ((\|I + i\alpha T\| - 1)/\alpha) = \lim_{\alpha \rightarrow 0+} ((\|I_1 + i\alpha T_1\| - 1)/\alpha) = 0$ and hence T is Hermitian.

STEP 3. From [Step 1](#) we can obtain that $\|Ta\|_p \leq \|T_1\| \|a\|_p$, for each $a \in U \otimes_p V$. This implies that $\|T\| \leq \|T_1\|$. The converse inequality $\|T_1\| \leq \|T\|$ can also be established without much difficulty. This establishes that $\|T\| = \|T_1\|$.

STEP 4. We now concentrate on the result $\text{sp}(T_1) = \text{sp}(T)$.

Let

$$\begin{aligned} \lambda_1 \in \text{sp}(T_1) &\Rightarrow T_1 - \lambda_1 I_1 \text{ is singular} \\ &\Rightarrow \exists \text{ a nonzero vector } u \in U \text{ such that } (T_1 - \lambda_1 I_1)u = 0. \end{aligned} \quad (2.10)$$

Let $v \in V$ be a vector with $v \neq 0$. Then $u \otimes v \in U \otimes_p V$ with $u \otimes v \neq 0$.

Now,

$$\begin{aligned} (T - \lambda_1 I)(u \otimes v) &= T(u \otimes v) - \lambda_1 I(u \otimes v) \\ &= (T_1 u) \otimes v - (\lambda_1 u) \otimes v \\ &= (T_1 - \lambda_1 I_1)u \otimes v = 0. \end{aligned} \quad (2.11)$$

Consequently, $\lambda_1 \in \text{sp}(T)$ and thus $\text{sp}(T_1) \subseteq \text{sp}(T)$. On the other hand, let $\lambda \in \text{sp}(T)$. Then there exists a nonzero vector $a = \sum_{j=1}^k u_j \otimes e_j \in U \otimes_p V$, such that

$$\begin{aligned} (T - \lambda I)a = 0 &\Rightarrow T\left(\sum_{j=1}^k u_j \otimes e_j\right) - \lambda \sum_{j=1}^k u_j \otimes e_j = 0 \\ &\Rightarrow \sum_{j=1}^k (T_1 u_j) \otimes e_j - \sum_{j=1}^k (\lambda u_j) \otimes e_j = 0 \\ &\Rightarrow \sum_{j=1}^k (T_1 - \lambda I_1)u_j \otimes e_j = 0 \\ &\Rightarrow (T_1 - \lambda I_1)u_j = 0 \quad \forall j = 1, 2, \dots, k. \end{aligned} \quad (2.12)$$

Since $u \neq 0$, there exists at least one j such that $u_j \neq 0$. Hence, λ is an eigenvalue of T_1 . So, $\text{sp}(T) \subseteq \text{sp}(T_1)$. Ultimately we have $\text{sp}(T) = \text{sp}(T_1)$. $\square$

REMARK 2.3. (i) If both U and V are infinite-dimensional Banach algebras, then the compactness of T_1 may not imply that of T . For example, choose a sequence $\{v_n\}$ in V with $\|v_n\| = 1$ so that $\{v_n\}$ has no convergent subsequence. (To wit, let $v = 1_2$, $v_n = e_n$, where e_n is a sequence with n th place equal to 1 and 0 elsewhere. Then $\{e_n\}$ cannot have a convergent subsequence.)

Let $u \in U$ with $\|u\| = 1$. Then $\{u \otimes v_n\}$ is a bounded sequence in $U \otimes_p V$. Now, $T(u \otimes v_n) = T_1 u \otimes v_n$ and

$$\|T(u \otimes v_n) - T(u \otimes v_m)\|_p = \|T_1 u \otimes v_n - T_1 u \otimes v_m\|_p = \|T_1 u\| \|v_n - v_m\|. \quad (2.13)$$

Equation (2.13) shows that $\{T(u \otimes v_n)\}$ has a convergent subsequence only if the sequence $\{v_n\}$ has a convergent subsequence. This ascertains that T cannot be compact. So, we are forced to consider V to be finite dimensional.

(ii) Although $\text{sp}(T)$ is, in general, a larger set than $\text{sp}(T_1)$, the result $\text{sp}(T) = \text{sp}(T_1)$, indicates that many eigenvalues of T repeat the same eigenvalue. For the sake of completeness, we illustrate the situation *with an example*.

Let $\mathcal{A}$ be the Banach algebra of 2×2 real square matrices with $\|A\| = \max\{|u_i| : i = 1, 2, 3, 4\}$, where

$$A = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \in \mathcal{A}. \quad (2.14)$$

Let $U = V = \mathcal{A}$. Put

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (2.15)$$

Then $\beta = \{e_1, e_2, e_3, e_4\}$ is a basis of $\mathcal{A}$.

Define a map $T_1 : \mathcal{A} \rightarrow \mathcal{A}$ by

$$T_1 A = \begin{bmatrix} \sum_{i=1}^4 \alpha_i u_i & \sum_{i=1}^4 \beta_i u_i \\ \sum_{i=1}^4 \gamma_i u_i & \sum_{i=1}^4 \delta_i u_i \end{bmatrix}. \quad (2.16)$$

Then it is evident that T_1 is a linear operator on $\mathcal{A}$ and the matrix representation of T_1 with respect to the basis β is

$$[T_1]_\beta = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{bmatrix}. \quad (2.17)$$

Next, $\{e_i \otimes e_j : i, j = 1, 2, 3, 4\}$ is a basis for $\mathcal{A} \otimes_p \mathcal{A}$. If we define an operator $T : \mathcal{A} \otimes_p \mathcal{A} \rightarrow \mathcal{A} \otimes_p \mathcal{A}$ by $T(\sum_i u_i \otimes v_i) = \sum_i (T_1 u_i) \otimes v_i$, then the matrix representation of T is a 16×16 square matrix, that is, exhibited below:

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_4 \\ b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_4 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_4 & 0 & 0 \\ 0 & 0 & b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_4 \\ c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & c_4 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & c_4 & 0 & 0 \\ 0 & 0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & c_4 & 0 \\ 0 & 0 & 0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & c_4 \\ d_1 & 0 & 0 & 0 & d_2 & 0 & 0 & 0 & d_3 & 0 & 0 & 0 & d_4 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 & d_2 & 0 & 0 & 0 & d_3 & 0 & 0 & 0 & d_4 & 0 & 0 \\ 0 & 0 & d_1 & 0 & 0 & 0 & d_2 & 0 & 0 & 0 & d_3 & 0 & 0 & 0 & d_4 & 0 \\ 0 & 0 & 0 & d_1 & 0 & 0 & 0 & d_2 & 0 & 0 & 0 & d_3 & 0 & 0 & 0 & d_4 \end{bmatrix}. \quad (2.18)$$

For the matrix representation of T_1 and T , we found that if T_1 is a Hermitian operator, that is, T_1 has a diagonal matrix representation, then so is T . Moreover, putting different values of α_i 's, β_i 's, γ_i 's, and δ_i 's, we can see that T_1 and T have the same eigenvalues. For instance, if we substitute

$$\begin{aligned} \alpha_1 &= 1, & \alpha_2 &= -1, & \alpha_3 &= 2, & \alpha_4 &= -2; \\ \beta_1 &= -1, & \beta_2 &= 2, & \beta_3 &= 3, & \beta_4 &= -3; \\ \gamma_1 &= 2, & \gamma_2 &= 3, & \gamma_3 &= 3, & \gamma_4 &= 1; \\ \delta_1 &= -2, & \delta_2 &= -3, & \delta_3 &= 1, & \delta_4 &= 4; \end{aligned} \quad (2.19)$$

then the eigenvalues of T_1 are 4.456658, 6.930083, 2.234990, and -3.621735 (correct up to six decimal places). Then the eigenvalues of T are also the same each repeating four times (verified by computer).

To show some applications of [Theorem 2.2](#), we want to concentrate on the study of the Hermitian approximation property and compact analytic semigroups on $U \otimes_p V$. We first recall some definitions.

(I) A Banach space X is said to have the *Hermitian approximation property* (HAP), if for each compact subset C of X and every $\varepsilon > 0$, there is a compact Hermitian operator H on X such that $\|Hx - x\| < \varepsilon$ for every $x \in C$, and $\|H\| \leq 1$.

To wit, the spaces l_2 , c_0 , and so forth have the HAP.

If X is separable, HAP is equivalent to the existence of a sequence $\{h_m\}$ with $\|h_m\| \leq 1$, for every $m \in \mathbb{N}$ of compact Hermitian operators on X such that

$$\|h_m x - x\| \rightarrow 0 \quad \text{as } m \rightarrow \infty \quad \forall x \in X. \quad (2.20)$$

(II) Let $S_\alpha = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } |\operatorname{Arg}(z)| < \alpha\}$ be a sector in $\mathbb{C}$, where α lies in $(0, \pi/2]$. An *analytic semigroup* T^z on X is a family of bounded linear operators $T^z : X \rightarrow X$ defined for $z \in S_\alpha$, where α is fixed, satisfying the following conditions:

- (i) $T^{z_1} T^{z_2} = T^{z_1 + z_2}$ for all z_1, z_2 in S_α ;
- (ii) T^z is an analytic-valued function of $z \in S_\alpha$;
- (iii) if $x \in X$ and $\varepsilon > 0$, then $\lim_{z \rightarrow 0} T^z x = x$ provided z remains within $S_{\alpha-\varepsilon}$.

We define the generator Z of T^z by $Zx = \lim_{t \downarrow 0} t^{-1}(T^t x - x)$ where $t > 0$ and $\operatorname{Dom}(Z)$ is the set of x for which the limit exists. If all T^z are compact operators, then we call it a *compact analytic semigroup* on X . In the following, $\mathbb{C}_+$ denotes the right-hand half plane of the complex plane, $\operatorname{KL}(X)$ the algebra of compact operators on X and “ $-$ ” the closure of a set. As a first shot, a little but interesting lemma is set forth.

LEMMA 2.4. *If U and V are arbitrary separable Banach algebras, then $U \otimes_p V$ is also separable.*

PROOF. The proof is straightforward. □

The principal theorem in this section is the following.

THEOREM 2.5. *If U and V are separable Banach algebras, then the following results are true:*

- (i) *if U has the HAP, then $U \otimes_p V$ has the HAP,*

(ii) if there is a compact analytic semigroup

$$z \rightarrow T_1^z : \mathbb{C}_+ \rightarrow \text{KL}(U) \quad (2.21)$$

such that

$$(T_1^z U)^- = U, \quad \|T_1^z\| \leq 1 \quad \forall z \in \mathbb{C}_+, \quad (2.22)$$

then there exists a compact analytic semigroup

$$z \rightarrow T^z : \mathbb{C}_+ \rightarrow \text{KL}(U \otimes_p V) \quad (2.23)$$

having the same kind of properties, viz,

$$(T^z(U \otimes_p V))^- = U \otimes_p V, \quad \|T^z\| \leq 1 \quad \forall z \in \mathbb{C}_+. \quad (2.24)$$

PROOF. (i) Let U have the Hermitian approximation property. Then there exists a sequence $\{T_m^1\}$ with $\|T_m^1\| \leq 1$ for all $m \in \mathbb{N}$ of a compact Hermitian operators on U such that $\|T_m^1 u - u\| \rightarrow 0$ for all $u \in U$. Then as usual for every $m \in \mathbb{N}$, define

$$T_m : U \otimes_p V \rightarrow U \otimes_p V \quad \text{by } T_m \left(\sum_i u_i \otimes v_i \right) = \sum_i (T_m^1 u_i) \otimes v_i. \quad (2.25)$$

Then by [Theorem 2.2](#), T_m is a compact Hermitian operator on $U \otimes_p V$ such that

$$\|T_m\| = \|T_m^1\| \leq 1 \quad \forall m \in \mathbb{N}. \quad (2.26)$$

Further,

$$\begin{aligned} \left\| T_m \left(\sum_i u_i \otimes v_i \right) - \sum_i (u_i \otimes v_i) \right\|_p &= \left\| \sum_i (T_m^1 u_i) \otimes v_i - \sum_i u_i \otimes v_i \right\|_p \\ &= \left\| (T_m^1 u_i - u_i) \otimes v_i \right\|_p \\ &\leq \sum_i \|T_m^1 u_i - u_i\| \|v_i\| \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty, \end{aligned} \quad (2.27)$$

(we can choose $\|v_i\| = 1$, for all i as in [Lemma 2.1](#)). Since by [Lemma 2.4](#), $U \otimes_p V$ is separable, $U \otimes_p V$ has the HAP.

(ii) For each $z \in \mathbb{C}_+$, define

$$T^z \left(\sum_i u_i \otimes v_i \right) = \sum_i (T_1^z u_i) \otimes v_i. \quad (2.28)$$

We first show that T^z is an analytic semigroup.

(1) Let $z_1, z_2 \in \mathbb{C}_+$. Then

$$\begin{aligned} (T^{z_1} T^{z_2}) a &= T^{z_1} \left(\sum_i (T_1^{z_2} u_i) \otimes v_i \right) = \sum_i T_1^{z_1} (T_1^{z_2} u_i) \otimes v_i \\ &= \sum_i T_1^{z_1+z_2} u_i \otimes v_i \quad (\text{because } T_1 \text{ is an analytic semigroup}) \\ &= T^{z_1+z_2}(a). \end{aligned} \quad (2.29)$$

So, $T^{z_1} T^{z_2} = T^{z_1+z_2}$.

(2) Let $a = \sum_i u_i \otimes v_i \in U \otimes_p V$.

Since T_1^z is analytic,

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{T_1^{z+h} u_i - T_1^z u_i}{h} & \text{ exists for every } u_i \text{ (w.r.t. the norm on } U) \\
 \Rightarrow \lim_{h \rightarrow 0} \left(\frac{T_1^{z+h} - T_1^z}{h} \right) u_i \otimes v_i & \text{ exists for every pair } u_i \in U, v_i \in V \\
 & \text{ (w.r.t. the norm on } U \otimes_p V) \\
 \Rightarrow \sum_i \left[\lim_{h \rightarrow 0} \left(\frac{T_1^{z+h} - T_1^z}{h} \right) \right] u_i \otimes v_i & \text{ exists} \\
 \Rightarrow \lim_{h \rightarrow 0} \left(\frac{T^{z+h} - T^z}{h} \right) a & \text{ exists.}
 \end{aligned} \tag{2.30}$$

So, T^z is an analytic semigroup of $z \in \mathbb{C}_+$.

Also,

$$\begin{aligned}
 \lim_{z \rightarrow 0} T^z a &= \lim_{z \rightarrow 0} \sum_i (T_1^z u_i) \otimes v_i = \sum_i \left(\lim_{z \rightarrow 0} T_1^z u_i \right) \otimes v_i \\
 &= \sum_i (u_i \otimes v_i) \quad \text{because } T_1^z \text{ is analytic} \\
 &= a.
 \end{aligned} \tag{2.31}$$

Consequently, T^z is an analytic semigroup. By [Theorem 2.2](#), the compactness of T_1^z implies that of T^z . To show that $(T^z(U \otimes_p V))^- = U \otimes_p V$, let $a = \sum_{j=1}^k u_j \otimes e_j$ by any element of $U \otimes_p V$ and $\varepsilon > 0$. $(T_1^z U)^- = U$ provides an element $u'_j \in U$ such that $\|T_1^z u'_j - u_j\| < \varepsilon/k$.

Let $a' = \sum_{j=1}^k u'_j \otimes e_j$. Then

$$\begin{aligned}
 \|T^z a' - a\|_p &= \left\| \sum_{j=1}^k T_1 u'_j \otimes e_j - \sum_{j=1}^k T_1 u_j \otimes e_j \right\|_p \\
 &\leq \sum_{j=1}^k \|T_1 u'_j - u_j\| \|e_j\| \\
 &< \varepsilon.
 \end{aligned} \tag{2.32}$$

This guarantees that $(T^z(U \otimes_p V))^- = U \otimes_p V$, for all $z \in \mathbb{C}_+$. Further, $\|T_1^z\| = \|T^z\|$ ensures that $\|T^z\| \leq 1$, for all $z \in \mathbb{C}_+$. $\square$

Our next goal is to study the orthogonality of the null space of T with its range.

We again recall some definitions. Let X be a normed linear space and $x, y \in X$. If $\|x - \lambda y\| \geq \|\lambda y\|$ for all $\lambda \in \mathbb{C}$, then x is said to be *orthogonal* to y . Let M and N be two subspaces of X . If $\|m + n\| \geq \|n\|$ for all $m \in M$ and for all $n \in N$, then M is said to be *orthogonal* to N , and then we write $M \perp N$ (for details, see [2]).

THEOREM 2.6. (i) If N_1 and R_1 are the null space and the range of T_1 , then $N = N_1 \otimes V$ and $R = R_1 \otimes V$ are the respective null space N and the range R of T .

(ii) If $N_1 \perp R_1$, then $N \perp R$, provided the basis $\{e_j\}$ is chosen in such a way that

$$\|a\| = \left\| \sum_{j=1}^k u_j \otimes e_j \right\| = \sum_{j=1}^k \|u_j\| \quad \forall a \in U \otimes_p V. \quad (2.33)$$

PROOF. It is not so hard to establish the results. $\square$

THE CONVERSE PROBLEMS. The main objective here is to investigate the possibility of studying the converse of the above results. To be precise, for a given compact Hermitian operator T on $U \otimes_p V$, can we obtain a compact Hermitian operator T_1 on U such that $\|T\| = \|T_1\|$ and $\text{sp}(T) = \text{sp}(T_1)$? Some possibilities are highlighted below. We first state a lemma whose proof is straightforward, and hence omitted.

LEMMA 2.7. For each j ($j = 1, 2, \dots, k$), let T_j be a map from $U \otimes_p V$ into U defined by $T_j(a) = u_j$ for every element $a = \sum_{j=1}^k u_j \otimes e_j$ in $U \otimes_p V$. Then T_j 's are continuous linear transformation.

THEOREM 2.8. Every compact Hermitian bounded operator T on $U \otimes_p V$ gives rise to k^2 number of linear operators $T^{i,j}$ ($i, j = 1, 2, \dots, k$) on U such that

- (i) $T^{i,j}$ is compact for all i, j .
- (ii) If $T^{i,j} = 0$ for $i \neq j$, then $T^{i,j}$ is Hermitian, $\|T^{i,j}\| \leq \|T\|$ for all $j = 1, 2, \dots, k$ and $\bigcup_{j=1}^k \text{sp}(T^{j,j}) = \text{sp}(T)$.

PROOF. Let T be a compact Hermitian operator on $U \otimes_p V$. For fixed i, j ($i, j = 1, 2, \dots, k$), we define an operator $T^{i,j}$ as follows.

Let u be an arbitrary element of U and let $T(u \otimes e_i) = \sum_{\ell=1}^k u_\ell \otimes e_\ell$. Then $T^{i,j} : U \rightarrow U$ is a map defined by $T^{i,j}(u) = u_j$. Since the expression $\sum_{\ell=1}^k u_\ell \otimes e_\ell$ for every element $a \in U \otimes_p V$ is unique, $T^{i,j}$ is well defined.

(1) The linearity of $T^{i,j}$ is obvious.

(2) We wish to show that $T^{i,j}$ is bounded.

Now, for each $u \in U$, $T(u \otimes e_i) = \sum_{\ell=1}^k T^{i,\ell}(u) \otimes e_\ell$. For a fixed j ($j = 1, 2, 3, \dots, k$), we define a map f_j by $f_j : U \otimes_p V \rightarrow U$ and $f_j(a) = u_j$, where $a = \sum_{\ell=1}^k u_\ell \otimes e_\ell$. Then f_j is a bounded linear operator by Lemma 2.7.

So,

$$\begin{aligned} \|T^{i,j}u\| &= \|f_j(T(u \otimes e_i))\| \\ &\leq \|f_j\| \|T(u \otimes e_i)\| \\ &\leq \|f_j\| \|T\| \|u \otimes e_i\|_p \\ &= K \|u\|, \quad \text{where } K = \|f_j\| \|T\|. \end{aligned} \quad (2.34)$$

Hence, $T^{i,j}$ is bounded.

(i) Next, our purpose is to prove the compactness of $T^{i,j}$. Let $\{u_n\}$ be a sequence in U with $\|u_n\| \leq 1$, for all n . Then for a fixed i ($i = 1, 2, \dots, k$),

$$\|u_n \otimes e_i\|_p = \|u_n\| \|e_i\| = \|u_n\| \leq 1 \quad \forall n. \quad (2.35)$$

Hence for a fixed i , $\{u_n \otimes e_i\}_{n=1}^\infty$ is a bounded sequence in $U \otimes_p V$. The compactness of T yields that $\{T(u_n \otimes e_i)\}$ has a convergent subsequence, say $\{T(u_{n_m} \otimes e_i)\}$

converging to

$$x = \sum_{\ell=1}^k x_{\ell} \otimes e_{\ell} \quad \text{in } U \otimes_p V. \quad (2.36)$$

Now,

$$\begin{aligned} & \lim_{m \rightarrow \infty} \left\| T(u_{n_m} \otimes e_i) - \sum_{\ell=1}^k x_{\ell} \otimes e_{\ell} \right\|_p = 0 \\ \Rightarrow & \lim_{m \rightarrow \infty} \left\| \sum_{\ell=1}^k (T^{i,\ell} u_{n_m}) - \sum_{\ell=1}^k x_{\ell} \otimes e_{\ell} \right\|_p = 0 \\ \Rightarrow & \left\| \sum_{\ell=1}^k \left[\left(\lim_{m \rightarrow \infty} T^{i,\ell} u_{n_m} - x_{\ell} \right) \otimes e_{\ell} \right] \right\|_p = 0 \quad \text{because } T^{i,\ell} \text{ are continuous} \\ \Rightarrow & \sum_{\ell=1}^k \lim_{m \rightarrow \infty} (T^{i,\ell} u_{n_m} - x_{\ell}) \otimes e_{\ell} = 0. \end{aligned} \quad (2.37)$$

Hence $\lim_{m \rightarrow \infty} T^{i,\ell} u_{n_m} = x_{\ell}$, for all $\ell = 1, 2, \dots, k$. So, for $\ell = j$, we have that $T^{i,j}$ is compact.

(ii) Now suppose that $T^{i,j} = 0$ for $i \neq j$.

(I) We want to show that $T^{j,j}$ is Hermitian for each $j = 1, 2, \dots, k$.

T is Hermitian and so $\|e^{i\alpha T}\| = 1$, for all $\alpha \in \mathbb{R}$. Let $u \in U$ with $\|u\| = 1$. Then for fixed j , $\|u \otimes e_j\|_p = 1$.

Now,

$$\begin{aligned} 1 &= \|e^{i\alpha T}\| = \sup \{ \|e^{i\alpha T} a\|_p : \|a\|_p = 1 \} \geq \|e^{i\alpha T} (u \otimes e_j)\|_p \\ &= \left\| \left[1 + i\alpha T + \frac{(i\alpha T)^2}{2!} + \frac{(i\alpha T)^3}{3!} + \dots \right] (u \otimes e_j) \right\|_p \\ &= \left\| \left[u + i\alpha T^{i,j} u + \frac{(i\alpha T^{i,j})^2}{2!} u + \frac{(i\alpha T^{j,j})^3}{3!} u + \dots \right] \otimes e_j \right\|_p \\ &= \|e^{i\alpha T^{j,j}} u\| \quad \text{because } \|e_j\| = 1. \end{aligned} \quad (2.38)$$

Thus, $\|e^{i\alpha T^{j,j}} u\| \leq 1$, for all $u \in U$ with $\|u\| = 1$. This gives $\|e^{i\alpha T^{j,j}}\| \leq 1$. Again, $e^{i\alpha T^{j,j}} e^{-i\alpha T^{j,j}} = I$, so, $1 = \|I\| = \|e^{i\alpha T^{j,j}} e^{-i\alpha T^{j,j}}\|_p \leq \|e^{i\alpha T^{j,j}}\|_p \|e^{-i\alpha T^{j,j}}\|_p$.

This yields $\|e^{-i\alpha T^{j,j}}\| \geq 1/\|e^{i\alpha T^{j,j}}\| \geq 1$. Since $\alpha \in \mathbb{R}$ then $-\alpha \in \mathbb{R}$, we can obtain $\|e^{i\alpha T^{j,j}}\| \geq 1$ and hence $\|e^{i\alpha T^{j,j}}\| = 1$. This implies that $T^{j,j}$ is Hermitian.

(II) Again for fixed j ,

$$\|u \otimes e_j\|_p = 1, \quad \text{whenever } u \in U, \|u\| = 1. \quad (2.39)$$

Now, $T(u \otimes e_j) = (T^{j,j} u) \otimes e_j$. So, $\|T\| \geq \|T(u \otimes e_j)\|_p = \|T^{j,j}(u \otimes e_j)\|_p = \|T^{j,j} u\|$. This gives $\|T^{j,j}\| \leq \|T\|$, $j = 1, 2, \dots, k$.

(III) We wish to show that $\bigcup_{j=1}^k \text{sp}(T^{j,j}) = \text{sp}(T)$.

Let

$$\begin{aligned}
 \lambda \in \bigcup_{j=1}^k \text{sp}(T^{j,j}) &\Rightarrow \lambda \in \text{sp}(T^{j,j}) \text{ for some } j \\
 &\Rightarrow (T^{j,j} - \lambda I) \text{ is singular} \\
 &\Rightarrow \text{there exists a nonzero vector } u \in U \text{ such that } (T^{j,j} - \lambda I)u = 0 \\
 &\Rightarrow (T^{j,j} - \lambda I)u \otimes e_j = 0 \\
 &\Rightarrow (T - \lambda I)u \otimes e_j = 0 \text{ because } u \neq 0, e_j \neq 0 \Rightarrow u \otimes e_j \neq 0 \\
 &\Rightarrow \lambda \in \text{sp}(T).
 \end{aligned} \tag{2.40}$$

This gives $\bigcup_{j=1}^k \text{sp}(T^{j,j}) \subseteq \text{sp}(T)$.

On the other hand,

$\lambda \in \text{sp}(T) \Rightarrow T - \lambda I$ is singular

$$\begin{aligned}
 &\Rightarrow \exists a \text{ nonzero vector } a = \sum_{\ell=1}^k u_\ell \otimes e_\ell \text{ such that } (T - \lambda I)a = 0 \\
 &\Rightarrow \sum_{\ell=1}^k T(u_\ell \otimes e_\ell) - \lambda \sum_{\ell=1}^k u_\ell \otimes e_\ell = 0 \\
 &\Rightarrow \sum_{\ell=1}^k T^{\ell,\ell} u_\ell \otimes e_\ell - \lambda \sum_{\ell=1}^k u_\ell \otimes e_\ell = 0 \\
 &\Rightarrow \sum_{\ell=1}^k (T^{\ell,\ell} - \lambda I) u_\ell \otimes e_\ell = 0 \\
 &\Rightarrow (T^{\ell,\ell} - \lambda I) u_\ell = 0; \quad \forall \ell = 1, 2, \dots, k.
 \end{aligned} \tag{2.41}$$

Since $a \neq 0$, there exists at least one j such that $u_j \neq 0$. For this u_j , we have

$$\begin{aligned}
 (T^{j,j} - \lambda I)u_j &= 0 \Rightarrow \lambda \in \text{sp}(T^{j,j}) \\
 &\Rightarrow \lambda \in \bigcup_{j=1}^k \text{sp}(T^{j,j}).
 \end{aligned} \tag{2.42}$$

Thus,

$$\bigcup_{j=1}^k \text{sp}(T^{j,j}) = \text{sp}(T). \tag{2.43}$$

This completes the proof of the theorem. $\square$

We are now in a position to pose some open problems.

- (i) Does the HAP in $U \otimes_p V$ give the existence of the HAP in U ?
- (ii) If the null space of T is orthogonal to its range, is the null space $T^{i,j}$ orthogonal to its range? What are the orthogonal complements of the null space and the range of T ?

(iii) Suppose that the operator T is the derivation (for definition of a derivation, see [5]). Is $T^{i,j}$ a derivation?

(iv) Can we make the analogous study in case of other tensor products, viz, the injective tensor products, the Haagerup tensor product, and so forth, [1]?

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