GRACEFUL NUMBERS

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We construct a labeled graph $D(n)$ that reflects the structure of divisors of a given natural number n . We define the concept of graceful numbers in terms of this associated graph and find the general form of such a number. As a consequence, we determine which graceful numbers are perfect.

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1. Introduction. In [2], Gallian presented a detailed survey of various types of graph labeling, the two best known being graceful and harmonious. Recall that a graph G with q edges is called graceful if one can label its vertices with distinct numbers from the set $\{0, 1, \dots, q\}$ and mark the edges with differences of the labels of the end vertices in such a way that the resulting edge labels are distinct. A number of interesting results on graceful and graceful-like labelings are obtained in [1, 3, 4] and some other works. In this note, we give a description of natural numbers whose associated graph of divisors satisfies certain graceful-like conditions. For any natural number n , we construct a labeled graph $D(n)$ that reflects the structure of divisors of n . We define the concept of graceful number in terms of this associated graph and find the general form of such a number. As a consequence, we determine which graceful numbers are perfect.

2. Main results. Given a natural number n one can generate a graph $D(n)$ that reflects the structure of divisors of n as follows. The vertices of the graph represent all the divisors of the number n , each vertex is labeled by a certain divisor. (In what follows, we refer to the vertex of the graph $D(n)$ with label k as the “vertex k .”) If r and s are two divisors of n and $r > s$, then there is an edge between the vertices s and r if and only if s divides r and the ratio r/s is a prime number. As in the theory of graceful graphs, we label such an edge by the difference $r - s$ of the labels of its vertices. In what follows, the sum of the labels of all edges of the graph $D(n)$ is denoted by $\overline{SD}(n)$ while $SD(n)$ denotes the sum of labels of all edges of $D(n)$ except the edges terminating at n . (Clearly, if $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the prime factorization of a natural number n , then $SD(n) = \overline{SD}(n) - \sum_{i=1}^k (n - n/p_i)$.)

EXAMPLE 2.1. It is easy to see that if $n = p^r$, where p is a prime number and r is any positive integer, then $\overline{SD}(n) = \sum_{i=1}^r (p^i - p^{i-1}) = p^r - 1$ and $SD(n) = \sum_{i=1}^{r-1} (p^i - p^{i-1}) = p^{r-1} - 1$, so that $SD(n) < n$. The graph $D(n)$ is shown in Figure 2.1.

FIGURE 2.1. The graph $D(p^r)$.

The following example shows that there are numbers n such that $SD(n) > n$, as well as numbers that satisfy the condition $SD(n) = n$.

EXAMPLE 2.2. Let $n = 24$ and $m = 12$. Then $SD(n) = (12 - 6) + (12 - 4) + (8 - 4) + (6 - 3) + (6 - 2) + (4 - 2) + (3 - 1) + (2 - 1) = 30 > n$ and $SD(m) = (6 - 3) + (6 - 2) + (4 - 2) + (3 - 1) + (2 - 1) = 12 = m$.

DEFINITION 2.3. A natural number n is called *graceful* if $SD(n) = n$.

In order to obtain the description of graceful numbers, we first find the value of $SD(n)$ when n is a product of powers of two different prime numbers.

EXAMPLE 2.4. Let $n = p^r q^s$ where p and q are different prime numbers, $r \geq 1$, and $s \geq 1$. In this case the graph $D(n)$ is of the form

FIGURE 2.2. The graph $D(p^r q^s)$.

and $\overline{SD}(n) = \sum_{i=0}^r \sum_{j=1}^s (p^i q^j - p^i q^{j-1}) + \sum_{i=1}^r \sum_{j=0}^s (p^i q^j - p^{i-1} q^j) = \sum_{i=0}^r p^i (q^s - 1) + \sum_{j=0}^s q^j (p^r - 1)$ (the first sum corresponds to the differences of the consecutive divisors of n when the exponent of q decreases, and the second sum takes care about the differences of consecutive divisors of n when the exponent of p decreases). Thus,

$$\overline{SD}(n) = (q^s - 1) \sum_{i=0}^r p^i + (p^r - 1) \sum_{j=0}^s q^j = (q^s - 1) \frac{p^{r+1} - 1}{p - 1} + (p^r - 1) \frac{q^{s+1} - 1}{q - 1}, \quad (2.1)$$

so that

$$SD(n) = \overline{SD}(n) - \left[\left(n - \frac{n}{p} \right) + \left(n - \frac{n}{q} \right) \right]. \quad (2.2)$$

It follows from formulas (2.1) and (2.2) that a number $n = p^r q^s$ (p and q are prime, $r \geq 1$, and $s \geq 1$) is graceful if and only if $p = 2$ and $s = 1$, that is, $n = 4q$ for some odd prime number q .

Indeed, equality $SD(n) = n$ can hold only for even numbers n (if n is odd, then (2.1) shows that $SD(n)$ is even, whence $SD(n) \neq n$). If $n = 2^r q^s$, where $r \geq 2$, $s \geq 2$, then

$$\begin{aligned} SD(n) - n &= (2^r - 1) \sum_{i=0}^s q^i + (q^s - 1)(2^{r+1} - 1) - 2^{r+1} q^s + 2^{r-1} q^s + 2^r q^{s-1} - 2^r q^s \\ &> (2^{r-1} - 2)q^s + (2^{r+1} q^{s-1} - q^{s-1} - 2^{r+1}) + (2^r - 1) \sum_{i=0}^{s-2} q^i \\ &> 0, \end{aligned} \quad (2.3)$$

so that $SD(n) > n$. Finally, if $n = 2^r q$ ($r \geq 1$), then $SD(n) - n = (q - 1)(2^{r+1} - 1) + (2^r - 1)(q + 1) - 2^{r+1} q + 2^{r-1} q + 2^r - 2^r q = q(2^{r-1} - 2)$, so that $SD(2^r q) = 2^r q$ if and only if $r = 2$. Thus, for any two different prime numbers p and q , $p < q$, and for any two nonnegative integers r and s , the number $p^r q^s$ is graceful if and only if $p = 2$, $r = 2$, and $s = 1$.

Now, we generalize formula (2.1) to the case of arbitrary number n . More precisely, we show that if $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is a prime decomposition of a positive integer n ($p_1, \dots, p_k$ are different primes and $r_1, \dots, r_k$ are positive integers), then

$$\overline{SD}(n) = \sum_{i=1}^k (p_i^{r_i} - 1) \prod_{1 \leq j \leq k, j \neq i} \left(\frac{p_j^{r_j+1} - 1}{p_j - 1} \right). \quad (2.4)$$

We proceed by induction on n . We have seen that the formula is true if n is a power of a prime number or a product of two powers of primes. In order to perform the step of induction, notice that

$$\overline{SD}(n) = \overline{SD}\left(\frac{n}{p_1}\right) + (p_1^{r_1} - p_1^{r_1-1}) \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} p_2^{i_2} \cdots p_k^{i_k} + p_1^{r_1} \overline{SD}\left(\frac{n}{p_1^{r_1}}\right). \quad (2.5)$$

Applying the inductive hypothesis and taking into account that

$$\overline{SD}(n) = \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} p_2^{i_2} \cdots p_k^{i_k} = \prod_{j=2}^k \sum_{i=0}^{r_j} p_j^i = \prod_{j=2}^k \left(\frac{p_j^{r_j+1} - 1}{(p_j - 1)} \right), \quad (2.6)$$

we obtain that

$$\begin{aligned} \overline{SD}(n) &= (p_1^{r_1-1} - 1) \prod_{j=2}^k \left(\frac{p_j^{r_j+1} - 1}{(p_j - 1)} \right) + \sum_{i=2}^k (p_i^{r_i} - 1) \left(\frac{p_1^{r_1} - 1}{(p_1 - 1)} \right) \prod_{2 \leq j \leq k, j \neq i} \left(\frac{p_j^{r_j+1} - 1}{(p_j - 1)} \right) \\ &\quad + (p_1^{r_1} - p_1^{r_1-1}) \prod_{j=2}^k \left(\frac{p_j^{r_j+1} - 1}{(p_j - 1)} \right) + p_1^{r_1} \sum_{i=2}^k (p_i^{r_i} - 1) \prod_{2 \leq j \leq k, j \neq i} \left(\frac{p_j^{r_j+1} - 1}{(p_j - 1)} \right) \end{aligned}$$

$$\begin{aligned}
&= (p_1^{r_1-1}) \prod_{j=2}^k \left(\frac{(p_j^{r_j+1} - 1)}{(p_j - 1)} \right) + \sum_{i=2}^k (p_i^{r_i} - 1) \prod_{1 \leq j \leq k, j \neq i} \left(\frac{(p_j^{r_j+1} - 1)}{(p_j - 1)} \right) \\
&= \sum_{i=1}^k (p_i^{r_i} - 1) \prod_{1 \leq j \leq k, j \neq i} \left(\frac{(p_j^{r_j+1} - 1)}{(p_j - 1)} \right),
\end{aligned} \tag{2.7}$$

so formula (2.4) is proved.

Now, formulas (2.2) and (2.4) imply that

$$SD(n) = \sum_{i=1}^k (p_i^{r_i} - 1) \prod_{1 \leq j \leq k, j \neq i} \left(\frac{(p_j^{r_j+1} - 1)}{(p_j - 1)} \right) - \sum_{i=1}^k \left(n - \frac{n}{p_i} \right). \tag{2.8}$$

Formula (2.8) shows, in particular, that if a number n is odd, then $SD(n)$ is even (it is easily seen that both sums in the right side of the formula are even if n is odd). Therefore, every graceful number must be even, that is,

$$n = 2^r q_1^{s_1} \cdots q_m^{s_m} \tag{2.9}$$

for some odd primes $q_1, \dots, q_m$ ($m \geq 1, s_i \geq 1$ for $i = 1, \dots, m$). As we have seen, if $m = 1$, then the number n is graceful if and only if $s_1 = 1$ and $r = 2$, that is, $n = 4q_1$. We show that if $m \geq 2$, then $SD(n) > n$, so the only graceful numbers are the numbers of the form $4q$ where q is an odd prime.

First of all, notice that $SD(2^r q_1^{s_1}) \geq 2^r q_1^{s_1}$ for $r \geq 1, s \geq 2$ (see Example 2.4) and $SD(2q_1 q_2) \geq 2q_1 q_2$ for any two different primes q_1 and q_2 (applying formula (2.1) we obtain that $SD(2q_1 q_2) = (q_1 + 1)(q_2 + 1) + 3(q_1 - 1)(q_2 + 1) + 3(q_2 - 1)(q_1 + 1) - 6q_1 q_2 + q_1 q_2 + 2q_1 + 2q_2 = 2q_1 q_2 + 3(q_1 + q_2) - 5 > 2q_1 q_2$). Therefore, in order to prove that $SD(n) > n$ for any number n of the form (2.9) with $m \geq 2$, it is sufficient to prove that $SD(n) > q_m^{s_m} SD(n/q_m^{s_m})$. But the last inequality is a consequence of equality (2.5). Indeed,

$$\begin{aligned}
SD(n) &= \overline{SD}(n) - n = \overline{SD}\left(\frac{n}{q_m}\right) + q^{s_m} - q^{s_m-1} \sum_{i=0}^r \sum_{i_1=0}^{s_1} \cdots \sum_{i_{m-1}=0}^{s_{m-1}} 2^i q_1^{i_1} \cdots q_{m-1}^{i_{m-1}} \\
&\quad + q_m^{s_m} \overline{SD}\left(\frac{n}{q_m^{s_m}}\right) - n > q_m^{s_m} \left(\overline{SD}\left(\frac{n}{q_m}\right) - \frac{n}{q_m} \right) = q_m^{s_m} SD\left(\frac{n}{q_m^{s_m}}\right).
\end{aligned} \tag{2.10}$$

We arrive at the following result.

THEOREM 2.5. *A natural number n is graceful if and only if $n = 4q$ where q is an odd prime.*

Recall that a positive integer m is called a *perfect number* if it is equal to the sum of all its proper divisors (i.e., of all divisors of m except of the number m itself). It is known (cf. [4, Theorem 5.10]) that every even perfect number is of the form $2^{k-1}(2^k - 1)$, where the number $2^k - 1$ is prime. Thus, our theorem implies the following result.

COROLLARY 2.6. *The only perfect graceful number is 28.*

REFERENCES

- [1] R. W. Frucht, *Nearly graceful labelings of graphs*, Sci. Ser. A Math. Sci. (N.S.) **5** (1992/93), 47–59.
- [2] J. A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. **5** (1998), no. 1, Dynamic Survey 6, 1–43.
- [3] D. Moulton, *Graceful labelings of triangular snakes*, Ars Combin. **28** (1989), 3–13.
- [4] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (Internat. Sympos., Rome, 1966), Gordon and Breach, New York, 1967, pp. 349–355.

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