ON A SUM ANALOGOUS TO DEDEKIND SUM AND ITS MEAN SQUARE VALUE FORMULA

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The main purpose of this paper is using the mean value theorem of Dirichlet L -functions to study the asymptotic property of a sum analogous to Dedekind sum, and give an interesting mean square value formula.

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1. Introduction. For a positive integer k and an arbitrary integer h , the classical Dedekind sum $S(h, k)$ is defined by

$$S(h, k) = \sum_{a=1}^k \left(\left(\frac{a}{k} \right) \right) \left(\left(\frac{ah}{k} \right) \right), \quad (1.1)$$

where

$$\left((x) \right) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases} \quad (1.2)$$

The various properties of $S(h, k)$ were investigated by many authors. For example, Carlitz [1] obtained a reciprocity theorem of $S(h, k)$. Conrey et al. [2] studied the mean value distribution of $S(h, k)$, and first proved the following important asymptotic formula:

$$\sum_{h=1}^k |S(h, k)|^{2m} = f_m(k) \left(\frac{k}{12} \right)^{2m} + O((k^{9/5} + k^{2m-1+1/(m+1)}) \log^3 k), \quad (1.3)$$

where $\sum'_h$ denotes the summation over all h such that $(k, h) = 1$, and

$$\sum_{n=1}^{\infty} \frac{f_m(n)}{n^s} = 2 \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \zeta(s). \quad (1.4)$$

The author [4] improved the error term of (1.3) for $m = 1$. In October, 2000, Todd Cochrane (personal communication) introduced a sum analogous to Dedekind sum as follows:

$$C(h, k) = \sum_{a=1}^k \left(\left(\frac{\bar{a}}{k} \right) \right) \left(\left(\frac{ah}{k} \right) \right), \quad (1.5)$$

where $\bar{a}$ defined by equation $a\bar{a} \equiv 1 \pmod{k}$. Then he suggested to study the arithmetical properties and mean value distribution properties of $C(h, k)$. About first problem, we have not made any progress at present. But for the second problem, we use the estimates for character sums and the mean value theorem of Dirichlet L -functions to prove the following main conclusion.

THEOREM 1.1. *Let k be any integer with $k > 2$. Then we have the asymptotic formula*

$$\sum_{h=1}^k C^2(h, k) = \frac{5}{144} \phi^2(k) \prod_{p^\alpha \parallel k} \frac{(p+1)^2/(p^2+1) + 1/p^{3\alpha}}{1 + 1/p + 1/p^2} + O\left(k \exp\left(\frac{4 \ln k}{\ln \ln k}\right)\right), \quad (1.6)$$

where $\exp(y) = e^y$, $\phi(k)$ is Euler function, $\prod_{p^\alpha \parallel k}$ denotes the product over all prime divisors of k with $p^\alpha \parallel k$ and $p^{\alpha+1} \nmid k$.

It seems that our methods are useless for mean value $\sum_{h=1}^k C^{2m}(h, k)$ with $m > 1$.

2. Some lemmas. To complete the proof of [Theorem 1.1](#), we need the following lemmas.

LEMMA 2.1. *Let integer $k \geq 3$ and $(h, k) = 1$. Then*

$$S(h, k) = \frac{1}{\pi^2 k} \sum_{d \mid k} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \pmod{d} \\ \chi(-1) = -1}} \chi(h) |L(1, \chi)|^2, \quad (2.1)$$

where χ denotes a Dirichlet character modulo d with $\chi(-1) = -1$, and $L(s, \chi)$ denotes the Dirichlet L -function corresponding to χ .

PROOF. See [\[3\]](#). □

LEMMA 2.2. *Let k be any integer with $k > 2$. Then we have the identity*

$$\sum_{h=1}^k C^2(h, k) = \sum_{d \mid k} \mu(d) \sum_{a=1}^{k'} S\left(a, \frac{k}{d}\right) S(a, k), \quad (2.2)$$

where $\mu(d)$ is Möbius function.

PROOF. From the definition of $S(h, k)$ and $C(h, k)$, we have

$$\begin{aligned} \sum_{h=1}^k C^2(h, k) &= \sum_{h=1}^k \left(\sum_{a=1}^{k'} \left(\left(\frac{\bar{a}}{k} \right) \right) \left(\left(\frac{ah}{k} \right) \right) \right)^2 \\ &= \sum_{a=1}^{k'} \sum_{b=1}^{k'} \left(\left(\frac{\bar{a}}{k} \right) \right) \left(\left(\frac{\bar{b}}{k} \right) \right) \sum_{h=1}^k \left(\left(\frac{ah}{k} \right) \right) \left(\left(\frac{bh}{k} \right) \right). \end{aligned} \quad (2.3)$$

Since $(ab, k) = 1$, so if h round through a complete residue system modulo k , then $\bar{b}h$ also round through a complete residue system modulo k . Therefore, note that the identities

$$\sum_{b=1}^k{}' = \sum_{d|k} \mu(d) \sum_{b=1}^{k/d}, \quad S(a, k) = S(\bar{a}, k), \quad (2.4)$$

we have

$$\begin{aligned} \sum_{h=1}^k C^2(h, k) &= \sum_{a=1}^k{}' \sum_{b=1}^k{}' \left(\left(\frac{\bar{a}}{k} \right) \right) \left(\left(\frac{\bar{b}}{k} \right) \right) \sum_{h=1}^k \left(\left(\frac{a\bar{b}h}{k} \right) \right) \left(\left(\frac{h}{k} \right) \right) \\ &= \sum_{a=1}^k{}' \sum_{b=1}^k{}' \left(\left(\frac{\bar{a}}{k} \right) \right) \left(\left(\frac{\bar{b}}{k} \right) \right) S(a\bar{b}, k) \\ &= \sum_{a=1}^k{}' \sum_{b=1}^k{}' \left(\left(\frac{\bar{a}}{k} \right) \right) \left(\left(\frac{\bar{b}}{k} \right) \right) S(ab, k) \\ &= \sum_{d|k} \mu(d) \sum_{a=1}^k{}' \sum_{b=1}^{k/d} \left(\left(\frac{ab}{k/d} \right) \right) \left(\left(\frac{b}{k/d} \right) \right) S(\bar{a}, k) \\ &= \sum_{d|k} \mu(d) \sum_{a=1}^k{}' S\left(a, \frac{k}{d}\right) S(\bar{a}, k) \\ &= \sum_{d|k} \mu(d) \sum_{a=1}^k{}' S\left(a, \frac{k}{d}\right) S(a, k). \end{aligned} \quad (2.5)$$

This proves [Lemma 2.2](#). □

LEMMA 2.3. *Let u and v be integers with $(u, v) = d \geq 2$, χ_u^0 the principal character mod u , and χ_v^0 the principal character mod v . Then we have the asymptotic formula*

$$\begin{aligned} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} |L(1, \chi \chi_u^0)|^2 |L(1, \chi \chi_v^0)|^2 \\ = \frac{5\pi^4}{144} \phi(d) \frac{\prod_{p|uv} (p^2 - 1)^2 / p^2 (p^2 + 1)}{\prod_{p|d} p^2 / (p^2 - 1)} + O\left(\frac{\phi(d)}{d} \exp\left(\frac{3 \ln m}{\ln \ln m}\right)\right), \end{aligned} \quad (2.6)$$

where $\prod_{p|n}$ denotes the product over all prime divisors of n , (u, v) denotes the greatest common divisor of u and v , and $m = \max(u, v)$.

PROOF. Let $r(n) = \sum_{t|n} \chi_u^0(t) \chi_v^0(n/t)$, χ an odd character mod d . Then for parameter $N \geq d$, applying Abel's identity we have

$$\begin{aligned} L(1, \chi \chi_u^0) L(1, \chi \chi_v^0) &= \sum_{n=1}^{\infty} \frac{\chi(n) r(n)}{n} \\ &= \sum_{1 \leq n \leq N} \frac{\chi(n) r(n)}{n} + \int_N^{\infty} \frac{A(y, \chi)}{y^2} dy, \end{aligned} \quad (2.7)$$

where $A(y, \chi) = \sum_{N < n \leq y} \chi(n) r(n)$. Note that the partition identities

$$\begin{aligned}
A(y, \chi) &= \sum_{n \leq \sqrt{y}} \chi(n) \chi_u^0(n) \sum_{m \leq y/n} \chi(m) \chi_v^0(m) \\
&\quad + \sum_{m \leq \sqrt{y}} \chi(m) \chi_v^0(m) \sum_{n \leq y/m} \chi(n) \chi_u^0(n) \\
&\quad - \sum_{n \leq \sqrt{N}} \chi(n) \chi_u^0(n) \sum_{m \leq N/n} \chi(m) \chi_v^0(m) \\
&\quad - \sum_{m \leq \sqrt{N}} \chi(m) \chi_v^0(m) \sum_{n \leq N/m} \chi(n) \chi_u^0(n) \\
&\quad - \left(\sum_{n \leq \sqrt{y}} \chi(n) \chi_u^0(n) \right) \left(\sum_{n \leq \sqrt{y}} \chi(n) \chi_v^0(n) \right) \\
&\quad + \left(\sum_{n \leq \sqrt{N}} \chi(n) \chi_u^0(n) \right) \left(\sum_{n \leq \sqrt{N}} \chi(n) \chi_v^0(n) \right).
\end{aligned} \tag{2.8}$$

Applying Cauchy inequality and the estimates for character sums

$$\begin{aligned}
\sum_{\chi \neq \chi_0} \left| \sum_{N \leq n \leq M} \chi(n) \right|^2 &= \sum_{\chi \neq \chi_0} \left| \sum_{N \leq n \leq M \leq N+d} \chi(n) \right|^2 \\
&= \phi(d) \sum_{N \leq n \leq M \leq N+d} \chi_0(n) - \left| \sum_{N \leq n \leq M \leq N+d} \chi_0(n) \right|^2 \\
&\leq \frac{\phi^2(d)}{4}
\end{aligned} \tag{2.9}$$

and note that the identities

$$\begin{aligned}
\sum_{N \leq n \leq M} \chi(n) \chi_u^0(n) &= \sum_{d|u} \mu(d) \chi(d) \sum_{N/d \leq n \leq M/d} \chi(n), \\
\sum_{d|u} |\mu(d)| &= \prod_{p|u} (1 + |\mu(p)|) = 2^{\omega(u)},
\end{aligned} \tag{2.10}$$

where $\omega(u)$ denotes the number of all different prime divisors of u . We have

$$\begin{aligned}
\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} |A(y, \chi)|^2 &\ll \sqrt{y} \sum_{n \leq \sqrt{y}} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left| \sum_{m \leq y/n} \chi(m) \chi_u^0(m) \right|^2 \\
&\quad + \sqrt{y} \sum_{m \leq \sqrt{y}} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left| \sum_{n \leq y/m} \chi(n) \chi_v^0(n) \right|^2 \\
&\quad + \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left| \sum_{n \leq \sqrt{y}} \chi(n) \chi_u^0(n) \right|^2 \times \left| \sum_{n \leq \sqrt{y}} \chi(n) \chi_v^0(n) \right|^2 \\
&\ll y \phi^2(d) 2^{\omega(u) + \omega(v)}.
\end{aligned} \tag{2.11}$$

Thus from (2.11) and Cauchy inequality we get

$$\begin{aligned}
 \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left| \int_N^\infty \frac{A(y, \chi)}{y^2} dy \right|^2 &\leq \int_N^\infty \int_N^\infty \frac{1}{y^2 z^2} \left(\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} |A(y, \chi)| \cdot |A(z, \chi)| \right) dy dz \\
 &\ll \left(\int_N^\infty \frac{1}{y^2} \left(\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} |A(y, \chi)|^2 \right)^{1/2} dy \right)^2 \\
 &\ll \frac{\phi^2(d)}{N} 2^{\omega(u)+\omega(v)}.
 \end{aligned} \tag{2.12}$$

Note that for $(ab, d) = 1$, from the orthogonality relation for character sums modulo d we have

$$\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) \bar{\chi}(b) = \begin{cases} \frac{1}{2} \phi(d), & \text{if } a \equiv b \bmod d; \\ -\frac{1}{2} \phi(d), & \text{if } a \equiv -b \bmod d; \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

So that

$$\begin{aligned}
 \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left| \sum_{1 \leq n \leq N} \frac{\chi(n) r(n)}{n} \right|^2 &= \frac{1}{2} \phi(d) \sum_{\substack{1 \leq a, b \leq N \\ (ab, d)=1 \\ a \equiv b(d)}} \frac{r(a) r(b)}{ab} - \frac{1}{2} \phi(d) \sum_{\substack{1 \leq a, b \leq N \\ (ab, d)=1 \\ a \equiv -b(d)}} \frac{r(a) r(b)}{ab} \\
 &= \frac{1}{2} \phi(d) \sum_{\substack{1 \leq a \leq N \\ (a, d)=1}} \frac{|r(a)|^2}{a^2} + O \left(\phi(d) \sum_{b=1}^N \sum_{\ell=1}^{[N/d]} \frac{\tau(b) \tau(\ell d + b)}{(\ell d + b) b} \right) \\
 &\quad + O \left(\phi(d) \sum_{a=1}^{d-1} \frac{\tau(a) \tau(d-a)}{a(d-a)} \right) \\
 &\quad + O \left(\phi(d) \sum_{1 \leq a \leq N} \sum_{(1+a/d) \leq \ell \leq N/d} \frac{\tau(a) \tau(\ell d - a)}{a(\ell d - a)} \right) \\
 &= \frac{1}{2} \phi(d) \sum_{\substack{n=1 \\ (n, d)=1}}^\infty \frac{|r(n)|^2}{n^2} + O \left(\frac{\phi(d)}{d} \exp \left(\frac{\ln N}{\ln \ln N} \right) \right),
 \end{aligned} \tag{2.14}$$

where $\tau(n)$ is the divisor function and $r(n) \leq \tau(n) \ll \exp((1+\epsilon) \ln 2 \ln n / \ln \ln n)$,

$$\begin{aligned}
 & \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left(\sum_{1 \leq n \leq N} \frac{\chi(n)r(n)}{n} \right) \left(\int_N^\infty \frac{\overline{A(y, \chi)}}{y^2} dy \right) \\
 & \ll (\ln N)^2 \int_N^\infty \frac{1}{y^2} \left(\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} |A(y, \bar{\chi})| \right) dy \\
 & \ll \phi^{3/2}(d) (\ln N)^2 N^{-1/2} 2^{\omega(u)+\omega(v)}.
 \end{aligned} \tag{2.15}$$

Taking parameter $N = d^3$ and note the identity

$$\sum_{\substack{n=1 \\ (n,d)=1}}^\infty \frac{|r(n)|^2}{n^2} = \frac{5\pi^4}{72} \frac{\prod_{p|uv} (p^2-1)^2 / p^2(p^2+1)}{\prod_{p|d} p^2 / (p^2-1)}. \tag{2.16}$$

From (2.11), (2.12), (2.14), and (2.15) we obtain

$$\begin{aligned}
 & \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} |L(1, \chi \chi_u^0)|^2 |L(1, \chi \chi_v^0)|^2 \\
 & = \frac{5\pi^4}{144} \phi(d) \frac{\prod_{p|uv} (p^2-1)^2 / p^2(p^2+1)}{\prod_{p|d} p^2 / (p^2-1)} + O\left(\frac{\phi(d)}{d} \exp\left(\frac{3 \ln m}{\ln \ln m}\right)\right).
 \end{aligned} \tag{2.17}$$

This proves [Lemma 2.3](#). □

LEMMA 2.4. *Let p be a prime, and let α, β be nonnegative integers with $\beta \geq \alpha$. Then we have the identity*

$$\begin{aligned}
 & \sum_{d_1|p^\beta} \sum_{d_2|p^\alpha} \frac{d_1^2 d_2^2}{\phi(d_1)\phi(d_2)} \phi(d) \frac{\prod_{p_1|d_1 d_2} (p_1^2-1)^2 / p_1^2(p_1^2+1)}{\prod_{p_1|d} p_1^2 / (p_1^2-1)} \\
 & = p^{3\alpha} \frac{(1+1/p)^2 - 1/p^{3\alpha+1}}{1+1/p+1/p^2} + \frac{(p^2-1)^2 p^{2\alpha} (p^\beta - p^\alpha)}{(p-1)^2 (p^2+1)},
 \end{aligned} \tag{2.18}$$

where $d = (d_1, d_2)$ denotes the greatest common divisors of d_1 and d_2 .

PROOF. Note that $d = (d_1, d_2)$, we have

$$\begin{aligned}
 & \sum_{d_1|p^\alpha} \sum_{d_2|p^\alpha} \frac{d_1^2 d_2^2}{\phi(d_1)\phi(d_2)} \phi(d) \frac{\prod_{p_1|d_1 d_2} (p_1^2-1)^2 / p_1^2(p_1^2+1)}{\prod_{p_1|d} p_1^2 / (p_1^2-1)} \\
 & = \sum_{u=0}^\alpha \sum_{v=0}^\alpha \frac{p^{2u+2v}}{\phi(p^u)\phi(p^v)} \phi((p^u, p^v)) \frac{\prod_{p_1|p^{u+v}} (p_1^2-1)^2 / p_1^2(p_1^2+1)}{\prod_{p_1|(p^u, p^v)} p_1^2 / (p_1^2-1)}
 \end{aligned}$$

$$\begin{aligned}
&= 1 + 2 \sum_{\beta=1}^{\alpha} \frac{p^{2\beta}}{\phi(p^\beta)} \frac{(p^2-1)^2}{p^2(p^2+1)} + \sum_{\beta=1}^{\alpha} \frac{p^{4\beta}}{\phi(p^\beta)} \frac{(p^2-1)^3}{p^4(p^2+1)} \\
&\quad + 2 \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=\beta+1}^{\alpha} \frac{p^{2\beta} p^{2\gamma}}{\phi(p^\gamma)} \frac{(p^2-1)^3}{p^4(p^2+1)} \\
&= 1 + 2 \frac{(p^\alpha-1)(p^2-1)^2}{(p-1)^2(p^2+1)} + \frac{(p^{3\alpha}-1)(p^2-1)^3}{(p^3-1)(p-1)(p^2+1)} \\
&\quad + 2 \sum_{\beta=1}^{\alpha-1} \frac{p^{3\beta}(p^{\alpha-\beta}-1)}{(p-1)^2} \frac{(p^2-1)^3}{p^2(p^2+1)} \\
&= 1 + 2 \frac{(p^\alpha-1)(p^2-1)^2}{(p-1)^2(p^2+1)} + \frac{(p^{3\alpha}-1)(p^2-1)^3}{(p^3-1)(p-1)(p^2+1)} \\
&\quad + 2 \frac{(p^{3\alpha-2}-p^\alpha)(p^2-1)^2}{(p-1)^2(p^2+1)} - 2 \frac{p(p^{3\alpha-3}-1)(p^2-1)^3}{(p^3-1)(p-1)^2(p^2+1)} \tag{2.19}
\end{aligned}$$

$$\begin{aligned}
&= \frac{p^{4\alpha}}{\phi(p^\alpha)} \left(1 - \frac{1}{p^3}\right)^{-1} \left[\left(1 - \frac{1}{p^2}\right)^2 - \frac{1}{p^{3\alpha+1}} \left(1 - \frac{1}{p}\right)^2 \right] \\
&= p^{3\alpha} \left(1 + \frac{1}{p} + \frac{1}{p^2}\right)^{-1} \left[\left(1 + \frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}} \right], \\
&\sum_{u=0}^{\alpha} \sum_{v=\alpha+1}^{\beta} \frac{p^{2u+2v}}{\phi(p^u)\phi(p^v)} \phi((p^u, p^v)) \frac{\prod_{p_1|p^{u+v}} (p_1^2-1)/p_1^2 (p_1^2+1)}{\prod_{p_1|(p^u, p^v)} p_1^2/(p_1^2-1)} \tag{2.20} \\
&= \sum_{j=0}^{\beta-\alpha-1} \frac{p^{j+\alpha+2}}{p-1} \frac{(p^2-1)^2}{p^2(p^2+1)} + \sum_{i=1}^{\alpha} p^{2i} \sum_{j=0}^{\beta-\alpha-1} \frac{p^{j+\alpha+2}}{p-1} \frac{(p^2-1)^3}{p^4(p^2+1)} \\
&= \frac{p^{\alpha+2}(p^{\beta-\alpha}-1)}{(p-1)^2} \frac{(p^2-1)^2}{p^2(p^2+1)} + \frac{p^2(p^{2\alpha}-1)}{p^2-1} \frac{p^{\alpha+2}(p^{\beta-\alpha}-1)}{(p-1)^2} \frac{(p^2-1)^3}{p^4(p^2+1)} \\
&= \frac{(p^2-1)^2 p^{2\alpha}(p^\beta-p^\alpha)}{(p-1)^2(p^2+1)}. \tag{2.21}
\end{aligned}$$

Now combining (2.19) and (2.20) we have

$$\begin{aligned}
&\sum_{d_1|p^\alpha} \sum_{d_2|p^\beta} \frac{d_1^2 d_2^2}{\phi(d_1)\phi(d_2)} \phi((d_1, d_2)) \frac{\prod_{p_1|d_1 d_2} (p_1^2-1)/p_1^2 (p_1^2+1)}{\prod_{p_1|(d_1, d_2)} p_1^2/(p_1^2-1)} \\
&= \sum_{u=0}^{\alpha} \sum_{v=0}^{\alpha} \frac{p^{2u+2v}}{\phi(p^u)\phi(p^v)} \phi((p^u, p^v)) \frac{\prod_{p_1|p^{u+v}} (p_1^2-1)/p_1^2 (p_1^2+1)}{\prod_{p_1|(p^u, p^v)} p_1^2/(p_1^2-1)} \\
&\quad + \sum_{u=0}^{\alpha} \sum_{v=\alpha+1}^{\beta} \frac{p^{2u+2v}}{\phi(p^u)\phi(p^v)} \phi((p^u, p^v)) \frac{\prod_{p_1|p^{u+v}} (p_1^2-1)/p_1^2 (p_1^2+1)}{\prod_{p_1|(p^u, p^v)} p_1^2/(p_1^2-1)} \\
&= p^{3\alpha} \frac{(1+1/p)^2 - 1/p^{3\alpha+1}}{1+1/p+1/p^2} + \frac{(p^2-1)^2 p^{2\alpha}(p^\beta-p^\alpha)}{(p-1)^2(p^2+1)}. \tag{2.22}
\end{aligned}$$

This proves Lemma 2.4. $\square$

3. Proof of the theorem. In this section, we complete the proof of [Theorem 1.1](#). Let k be an integer with $k \geq 3$. Then applying [Lemmas 2.1](#) and [2.2](#) we have

$$\begin{aligned}
 \sum_{h=1}^k C^2(h, k) &= \sum_{d|k} \mu(d) \sum_{a=1}^{k'} S\left(a, \frac{k}{d}\right) S(a, k) \\
 &= \sum_{d|k} \mu(d) \sum_{a=1}^{k'} \left(\frac{d}{\pi^2 k} \sum_{u|k/d} \frac{u^2}{\phi(u)} \sum_{\substack{\chi \bmod u \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2 \right) \\
 &\quad \times \left(\frac{1}{\pi^2 k} \sum_{v|k} \frac{v^2}{\phi(v)} \sum_{\substack{\chi \bmod v \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2 \right) \\
 &= \frac{1}{k^2 \pi^4} \sum_{d|k} \mu(d) d \sum_{u|k/d} \sum_{v|k} \frac{u^2 v^2}{\phi(u) \phi(v)} \\
 &\quad \times \sum_{\substack{\chi_1 \bmod u \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod v \\ \chi_2(-1)=-1}} \sum_{a=1}^{k'} \chi_1(a) \chi_2(a) |L(1, \chi_1)|^2 |L(1, \chi_2)|^2.
 \end{aligned} \tag{3.1}$$

For each $\chi_1 \bmod u$, it is clear that there exists one and only one $k_1 | u$ with a unique primitive character $\chi_{k_1}^1 \bmod k_1$ such that $\chi_1 = \chi_{k_1}^1 \chi_u^0$, here χ_u^0 denotes the principal character mod u . Similarly, we also have $\chi_2 = \chi_{k_2}^2 \chi_v^0$, here $k_2 | v$ and $\chi_{k_2}^2$ is a primitive character mod k_2 . Note that $u | k$ and $v | k$, from the orthogonality of characters we have

$$\begin{aligned}
 \sum_{a=1}^{k'} \chi_1(a) \chi_2(a) &= \sum_{a=1}^k [\chi_{k_1}^1(a) \chi_k^0(a)] [\chi_{k_2}^2(a) \chi_k^0(a)] \\
 &= \begin{cases} \phi(k), & \text{if } k_1 = k_2, \chi_{k_1}^1 = \overline{\chi_{k_2}^2}; \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned} \tag{3.2}$$

Let $d_1 = (u, v)$. If $k_1 = k_2$ and $\chi_{k_1}^1 = \overline{\chi_{k_2}^2}$, then $\chi_{k_1}^1 \chi_{d_1}^0$ is also a character mod d_1 . So from [\(3.1\)](#), [\(3.2\)](#), and [Lemma 2.3](#) we have

$$\begin{aligned}
 \sum_{h=1}^k C^2(h, k) &= \frac{\phi(k)}{k^2 \pi^4} \sum_{d|k} \mu(d) d \sum_{u|k/d} \sum_{v|k} \frac{u^2 v^2}{\phi(u) \phi(v)} \sum_{\substack{\chi \bmod (u, v) \\ \chi(-1)=-1}} |L(1, \chi \chi_u^0)|^2 |L(1, \chi \chi_v^0)|^2 \\
 &= \frac{\phi(k)}{k^2 \pi^4} \sum_{d|k} \sum_{u|k/d} \sum_{v|k} \frac{\mu(d) du^2 v^2}{\phi(u) \phi(v)} \left\{ \frac{5\pi^4}{144} \phi((u, v)) \frac{\prod_{p|uv} (p^2 - 1)^2 / p^2 (p^2 + 1)}{\prod_{p|(u, v)} p^2 / (p^2 - 1)} \right. \\
 &\quad \left. + O\left(\exp\left(\frac{3 \ln k}{\ln \ln k}\right)\right) \right\} \\
 &= \frac{5\phi(k)}{144 k^2} \sum_{d|k} \sum_{u|k/d} \sum_{v|k} \frac{\mu(d) du^2 v^2}{\phi(u) \phi(v)} \phi((u, v)) \frac{\prod_{p|uv} (p^2 - 1)^2 / p^2 (p^2 + 1)}{\prod_{p|(u, v)} p^2 / (p^2 - 1)} \\
 &\quad + O\left(k \exp\left(\frac{4 \ln k}{\ln \ln k}\right)\right).
 \end{aligned} \tag{3.3}$$

Since $\phi(n)$ and $\mu(n)$ are multiplicative functions, so from the multiplicative properties of these functions, (3.3) and Lemma 2.4 and note that the identities (for any multiplicative functions $f(u)$ and $g(v)$)

$$\begin{aligned} \sum_{d|k} \mu(d) d \sum_{u|k/d} \sum_{v|k} f(u) g(v) &= \prod_{p^\alpha || k} \left[\sum_{u|p^\alpha} \sum_{v|p^\alpha} f(u) g(v) - p \sum_{u|p^{\alpha-1}} \sum_{v|p^\alpha} f(u) g(v) \right], \\ p^{3\alpha} \frac{(1+1/p)^2 - 1/p^{3\alpha+1}}{1+1/p+1/p^2} - p \left[p^{3\alpha-3} \frac{(1+1/p)^2 - 1/p^{3\alpha-2}}{1+1/p+1/p^2} + \frac{(p^2-1)^2 p^{2\alpha-2} (p-p^{\alpha-1})}{(p-1)^2 (p^2+1)} \right] \\ &= \frac{p^{3\alpha} (1-1/p)}{1+1/p+1/p^2} \left(\frac{(p+1)^2}{p^2+1} + \frac{1}{p^{3\alpha}} \right), \end{aligned} \quad (3.4)$$

we have

$$\begin{aligned} \sum_{h=1}^k C^2(h, k) &= \sum_{d|k} \mu(d) \sum_{a=1}^k S\left(a, \frac{k}{d}\right) S(a, k) \\ &= \frac{5}{144} \frac{\phi(k)}{k^2} \prod_{p^\alpha || k} \left\{ \sum_{d|p^\alpha} \mu(d) d \sum_{u|p^\alpha/d} \sum_{v|p^\alpha} \frac{uv}{\phi(u)\phi(v)} \phi((u, v)) \right. \\ &\quad \times \frac{\prod_{p_1|uv} (p_1^2-1)^2 / p_1^2 (p_1^2+1)}{\prod_{p_1|(u,v)} p_1^2 / (p_1^2-1)} \left. \right\} \\ &\quad + O\left(k \exp\left(\frac{4 \ln k}{\ln \ln k}\right)\right) \\ &= \frac{5}{144} \phi^2(k) \prod_{p^\alpha || k} \frac{(p+1)^2 / (p^2+1) + 1/p^{3\alpha}}{1+1/p+1/p^2} + O\left(k \exp\left(\frac{4 \ln k}{\ln \ln k}\right)\right). \end{aligned} \quad (3.5)$$

This completes the proof of Theorem 1.1.

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