ON THE SINE INTEGRAL AND THE CONVOLUTION

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The sine integral $\text{Si}(\lambda x)$ and the cosine integral $\text{Ci}(\lambda x)$ and their associated functions $\text{Si}_+(\lambda x)$, $\text{Si}_-(\lambda x)$, $\text{Ci}_+(\lambda x)$, $\text{Ci}_-(\lambda x)$ are defined as locally summable functions on the real line. Some convolutions of these functions and $\sin(\mu x)$, $\sin_+(\mu x)$, and $\sin_-(\mu x)$ are found.

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The *sine integral* $\text{Si}(x)$ is defined by

$$\int_0^x u^{-1} \sin u \, du, \quad (1)$$

(see Sneddon [6]). This integral is convergent for all x . More generally, for all $\lambda \neq 0$, we define $\text{Si}(\lambda x)$ by

$$\text{Si}(\lambda x) = \int_0^{\lambda x} u^{-1} \sin u \, du = \int_0^x u^{-1} \sin(\lambda u) \, du; \quad (2)$$

and we define $\text{Si}_+(\lambda x)$ and $\text{Si}_-(\lambda x)$ by

$$\text{Si}_+(\lambda x) = H(x) \text{Si}(\lambda x), \quad \text{Si}_-(\lambda x) = H(-x) \text{Si}(\lambda x), \quad (3)$$

(see [1]).

It is easily proved that

$$[\text{Si}_+(\lambda x)]' = \sin(\lambda x) x_+^{-1}. \quad (4)$$

We need the following lemma which was proved in [1].

LEMMA 1. *If $\lambda \neq 0$, then*

$$\int_0^\infty u^{-1} \sin(\lambda u) \, du = \frac{1}{2} \text{sgn } \lambda \cdot \pi. \quad (5)$$

The cosine integral $\text{Ci}(x)$ is defined for $x > 0$ by

$$\text{Ci}(x) = - \int_x^\infty u^{-1} \cos u \, du, \quad (6)$$

(see Sneddon [6]). This integral is divergent for $x \leq 0$; but in [3], $\text{Ci}(\lambda x)$ was defined as a locally summable function on the real line by

$$\text{Ci}(\lambda x) = - \int_{\lambda x}^\infty u^{-1} [\cos u - H(1-u)] \, du + H(1-\lambda x) \ln |\lambda x|, \quad (7)$$

where H denotes Heaviside's function. In particular,

$$\text{Ci}(x) = - \int_x^\infty u^{-1} [\cos u - H(1-u)] du + H(1-x) \ln|x|. \quad (8)$$

It was proved in [4] that the cosine integral is an even function. We can therefore define $\text{Ci}(\lambda x)$ by

$$\text{Ci}(\lambda x) = - \int_{|\lambda x|}^\infty u^{-1} \cos u du = - \int_{|x|}^\infty u^{-1} \cos(\lambda u) du, \quad \lambda, x \neq 0, \quad (9)$$

simplifying the definition given in [3].

The locally summable functions $\text{Ci}_+(\lambda x)$ and $\text{Ci}_-(\lambda x)$ are now defined for $\lambda \neq 0$ by

$$\text{Ci}_+(\lambda x) = H(x) \text{Ci}(\lambda x), \quad \text{Ci}_-(\lambda x) = H(-x) \text{Ci}(\lambda x). \quad (10)$$

It was proved in [3] that

$$[\text{Ci}_+(\lambda x)]' = \cos(\lambda x) x_+^{-1} - (c - \ln|\lambda|) \delta(x), \quad (11)$$

where

$$c = \int_0^\infty u^{-1} [\cos u - H(1-u)] du. \quad (12)$$

We also need the following lemma which was also proved in [4].

LEMMA 2. *If $x > 0$, then*

$$\int_0^x u^{-1} [\cos(\lambda u) - 1] du = c + \text{Ci}(\lambda x) - \ln|\lambda x|. \quad (13)$$

The classical definition of the convolution of two functions f and g is as follows.

DEFINITION 3. Let f and g be functions. Then the *convolution* $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^\infty f(t) g(x-t) dt, \quad (14)$$

for all points x for which the integral exists.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f, \quad (15)$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \text{ (or } f' * g). \quad (16)$$

Definition 3 can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see Gel'fand and Shilov [5].

DEFINITION 4. Let f and g be distributions in $\mathcal{D}'$. Then the *convolution* $f * g$ is defined by the equation

$$\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x + y) \rangle \rangle \quad (17)$$

for arbitrary ϕ in $\mathcal{D}$, provided that f and g satisfy either of the following conditions:

- (a) either f or g has bounded support,
- (b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition then (15) and (16) are satisfied.

In the following, the locally summable functions $\sin_{\pm}(\lambda x)$ and $\cos_{\pm}(\lambda x)$ are defined by

$$\begin{aligned} \sin_+(\lambda x) &= H(x) \sin(\lambda x), & \sin_-(\lambda x) &= H(-x) \sin(\lambda x), \\ \cos_+(\lambda x) &= H(x) \cos(\lambda x), & \cos_-(\lambda x) &= H(-x) \cos(\lambda x). \end{aligned} \quad (18)$$

THEOREM 5. If $\lambda, \mu \neq 0$, then the convolution $\text{Si}_+(\lambda x) * \sin_+(\mu x)$ exists and

$$\begin{aligned} \text{Si}_+(\lambda x) * \sin_+(\mu x) &= -\frac{1}{2} \mu^{-1} \sin(\mu x) \{ \text{Ci}_+[(\lambda - \mu)x] - \text{Ci}_+[(\lambda + \mu)x] \} \\ &\quad + \mu^{-1} \text{Si}_+(\lambda x) - \frac{1}{2} \mu^{-1} \cos(\mu x) \{ \text{Si}_+[(\lambda - \mu)x] + \text{Si}_+[(\lambda + \mu)x] \} \\ &\quad - \frac{1}{2} \mu^{-1} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \sin_+(\mu x) \end{aligned} \quad (19)$$

if $\lambda \neq \pm \mu$; and

$$\begin{aligned} \text{Si}_+(\lambda x) * \sin_+(\lambda x) &= \lambda^{-1} \text{Si}_+(\lambda x) - \frac{1}{2} \lambda^{-1} \left[\ln \left| \frac{1}{2} \lambda \right| - c \right] \sin_+(\lambda x) \\ &\quad - \frac{1}{2} \lambda^{-1} \cos(\lambda x) \text{Si}_+(2\lambda x) \\ &\quad + \frac{1}{2} \lambda^{-1} \sin(\lambda x) [\text{Ci}_+(2\lambda x) - \ln x_+] \end{aligned} \quad (20)$$

if $\lambda = \pm \mu$.

PROOF. It is obvious that $\text{Si}_+(\lambda x) * \sin_+(\mu x) = 0$ if $x < 0$. If $x > 0$ and $\lambda \neq \pm \mu$, we have

$$\begin{aligned} \text{Si}_+(\lambda x) * \sin_+(\mu x) &= \int_0^x \sin[\mu(x-t)] \int_0^t u^{-1} \sin(\lambda u) du dt \\ &= \int_0^x u^{-1} \sin(\lambda u) \int_u^x \sin[\mu(x-t)] dt du \\ &= \mu^{-1} \int_0^x u^{-1} \sin(\lambda u) \{1 - \cos[\mu(x-u)]\} du \end{aligned} \quad (21)$$

$$= \mu^{-1} \text{Si}(\lambda x) - \mu^{-1} I, \quad (22)$$

where

$$\begin{aligned}
 I &= \int_0^x u^{-1} \sin(\lambda u) \cos[\mu(x-u)] du \\
 &= \cos(\mu x) \int_0^x u^{-1} \sin(\lambda u) \cos(\mu u) du + \sin(\mu x) \int_0^x u^{-1} \sin(\lambda u) \sin(\mu u) du \\
 &= \frac{1}{2} \cos(\mu x) \int_0^x u^{-1} \{ \sin[(\lambda - \mu)u] + \sin[(\lambda + \mu)u] \} du \\
 &\quad + \frac{1}{2} \sin(\mu x) \int_0^x u^{-1} \{ \cos[(\lambda - \mu)u] - \cos[(\lambda + \mu)u] \} du \\
 &= \frac{1}{2} \cos(\mu x) \{ \text{Si}[(\lambda - \mu)x] + \text{Si}[(\lambda + \mu)x] \} \\
 &\quad + \frac{1}{2} \sin(\mu x) \left\{ \text{Ci}[(\lambda - \mu)x] - \text{Ci}[(\lambda + \mu)x] + \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \right\}
 \end{aligned} \tag{23}$$

on using [Lemma 2](#); and (19) follows from (22) and (23).

If $\lambda = \pm\mu$, (21) is replaced by

$$\begin{aligned}
 \text{Si}_+(\lambda x) * \sin_+(\lambda x) &= \lambda^{-1} \int_0^x u^{-1} \sin(\lambda u) \{1 - \cos[\lambda(x-u)]\} du \\
 &= \lambda^{-1} \text{Si}(\lambda x) - \lambda^{-1} J,
 \end{aligned} \tag{24}$$

where

$$\begin{aligned}
 J &= \int_0^x u^{-1} \sin(\lambda u) \cos[\lambda(x-u)] du \\
 &= \frac{1}{2} \cos(\lambda x) \int_0^x u^{-1} \sin(2\lambda u) du - \frac{1}{2} \sin(\lambda x) \int_0^x u^{-1} [\cos(2\lambda u) - 1] du \\
 &= \frac{1}{2} \cos(\lambda x) \text{Si}(2\lambda x) + \frac{1}{2} [\ln|2\lambda x| - c] \sin(\lambda x) - \frac{1}{2} \sin(\lambda x) \text{Ci}(2\lambda x)
 \end{aligned} \tag{25}$$

on using [Lemma 2](#); and (20) follows from (24) and (25). $\square$

COROLLARY 6. *If $\lambda, \mu \neq 0$, then the convolution $\text{Si}_+(\lambda x) * \cos_+(\mu x)$ exists and*

$$\begin{aligned}
 \text{Si}_+(\lambda x) * \cos_+(\mu x) &= \frac{1}{2} \mu^{-1} \sin(\mu x) \{ \text{Si}_+[(\lambda - \mu)x] + \text{Si}_+[(\lambda + \mu)x] \} \\
 &\quad - \frac{1}{2} \mu^{-1} \cos(\mu x) \{ \text{Ci}_+[(\lambda - \mu)x] - \text{Ci}_+[(\lambda + \mu)x] \} \\
 &\quad - \frac{1}{2} \mu^{-1} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \cos_+(\mu x)
 \end{aligned} \tag{26}$$

if $\lambda \neq \pm\mu$; and

$$\begin{aligned}
 \text{Si}_+(\lambda x) * \cos_+(\lambda x) &= \frac{1}{2} \lambda^{-1} \sin(\lambda x) \text{Si}_+(2\lambda x) - \frac{1}{2} \lambda^{-1} \left[\ln \left| \frac{1}{2} \lambda \right| - c \right] \cos_+(\lambda x) \\
 &\quad + \frac{1}{2} \lambda^{-1} \cos(\lambda x) [\text{Ci}_+(2\lambda x) - \ln x_+]
 \end{aligned} \tag{27}$$

if $\lambda = \pm\mu$.

PROOF. It follows from (4), (11), (16), and (19) that

$$\begin{aligned}
 [\text{Si}_+(\lambda x) * \sin_+(\mu x)]' &= \mu \text{Si}_+(\lambda x) * \cos_+(\mu x) \\
 &= \mu^{-1} \sin(\lambda x) x_+^{-1} - \frac{1}{2} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \cos_+(\mu x) \\
 &\quad - \frac{1}{2} \cos(\mu x) \{ \text{Ci}_+[(\lambda - \mu)x] - \text{Ci}_+[(\lambda + \mu)x] \} \\
 &\quad - \frac{1}{2} \mu^{-1} \sin(\mu x) \{ \cos[(\lambda - \mu)x] - \cos[(\lambda + \mu)x] \} x_+^{-1} \\
 &\quad - \frac{1}{2} \mu^{-1} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \sin(\mu x) \delta(x) \\
 &\quad + \frac{1}{2} \sin(\mu x) \{ \text{Si}_+[(\lambda - \mu)x] + \text{Si}_+[(\lambda + \mu)x] \} \\
 &\quad - \frac{1}{2} \mu^{-1} \cos(\mu x) \{ \sin[(\lambda + \mu)x] + \sin[(\lambda - \mu)x] \} x_+^{-1} \\
 &= -\frac{1}{2} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \cos_+(\mu x) \\
 &\quad - \frac{1}{2} \cos(\mu x) \{ \text{Ci}_+[(\lambda + \mu)x] - \text{Ci}_+[(\lambda - \mu)x] \} \\
 &\quad + \frac{1}{2} \sin(\mu x) \{ \text{Si}_+[(\lambda + \mu)x] + \text{Si}_+[(\lambda - \mu)x] \}
 \end{aligned} \tag{28}$$

and (26) follows.

If $\lambda = \pm\mu$, it follows from (4), (11), (16), and (20) that

$$\begin{aligned}
 \lambda \text{Si}_+(\lambda x) * \cos_+(\lambda x) &= \lambda^{-1} \sin(\lambda x) x_+^{-1} - \frac{1}{2} \left[\ln \left| \frac{1}{2} \lambda \right| - c \right] \cos_+(\lambda x) \\
 &\quad + \frac{1}{2} \sin(\lambda x) \text{Si}_+(2\lambda x) - \frac{1}{2} \lambda^{-1} \cos(\lambda x) \sin(2\lambda x) x_+^{-1} \\
 &\quad + \frac{1}{2} \cos(\lambda x) [\text{Ci}_+(2\lambda x) - \ln x_+] \\
 &\quad + \frac{1}{2} \lambda^{-1} \sin(\lambda x) [\cos(2\lambda x) - 1] x_+^{-1} \\
 &= -\frac{1}{2} \left[\ln \left| \frac{1}{2} \lambda \right| - c \right] \cos_+(\lambda x) + \frac{1}{2} \sin(\lambda x) \text{Si}_+(2\lambda x) \\
 &\quad + \frac{1}{2} \cos(\lambda x) [\text{Ci}_+(2\lambda x) - \ln x_+]
 \end{aligned} \tag{29}$$

and (27) follows. $\square$

COROLLARY 7. If $\lambda, \mu \neq 0$, then the convolutions $\text{Si}_-(\lambda x) * \sin_-(\mu x)$ and $\text{Si}_-(\lambda x) * \cos_-(\mu x)$ exist and

$$\begin{aligned}
 \text{Si}_-(\lambda x) * \sin_-(\mu x) &= \frac{1}{2} \mu^{-1} \sin(\mu x) \{ \text{Ci}_-[(\lambda - \mu)x] - \text{Ci}_-[(\lambda + \mu)x] \} \\
 &\quad - \mu^{-1} \text{Si}_-(\lambda x) + \frac{1}{2} \mu^{-1} \cos(\mu x) \{ \text{Si}_-[(\lambda - \mu)x] + \text{Si}_-[(\lambda + \mu)x] \} \\
 &\quad + \frac{1}{2} \mu^{-1} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \sin_-(\mu x),
 \end{aligned} \tag{30}$$

$$\begin{aligned}
\text{Si}_-(\lambda x) * \cos_-(\mu x) &= -\frac{1}{2}\mu^{-1} \sin(\mu x) \{ \text{Si}_-[(\lambda - \mu)x] + \text{Si}_-[(\lambda + \mu)x] \} \\
&\quad + \frac{1}{2}\mu^{-1} \cos(\mu x) \{ \text{Ci}_-[(\lambda - \mu)x] - \text{Ci}_-[(\lambda + \mu)x] \} \quad (31) \\
&\quad + \frac{1}{2}\mu^{-1} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \sin_-(\mu x)
\end{aligned}$$

if $\lambda \neq \pm\mu$; and

$$\begin{aligned}
\text{Si}_-(\lambda x) * \sin_-(\lambda x) &= -\lambda^{-1} \text{Si}_-(\lambda x) + \frac{1}{2}\lambda^{-1} \left[\ln \left| \frac{1}{2}\lambda \right| - c \right] \sin_-(\lambda x) \\
&\quad + \frac{1}{2}\lambda^{-1} \cos(\lambda x) \text{Si}_-(2\lambda x) \quad (32) \\
&\quad - \frac{1}{2}\lambda^{-1} \sin(\lambda x) [\text{Ci}_-(2\lambda x) - \ln x_-],
\end{aligned}$$

$$\begin{aligned}
\text{Si}_-(\lambda x) * \cos_-(\lambda x) &= -\frac{1}{2}\lambda^{-1} \sin(\lambda x) \text{Si}_-(2\lambda x) + \frac{1}{2}\lambda^{-1} \left[\ln \left| \frac{1}{2}\lambda \right| - c \right] \cos_-(\lambda x) \quad (33) \\
&\quad - \frac{1}{2}\lambda^{-1} \cos(\lambda x) [\text{Ci}_-(2\lambda x) - \ln x_-]
\end{aligned}$$

if $\lambda = \pm\mu$.

PROOF. Equations (30) and (31) follow on replacing x by $-x$ in (19) and (26), respectively. Equations (32) and (33) follow on replacing x by $-x$ in (20) and (27), respectively.

Definition 4 of the convolution is rather restrictive and so a neutrix convolution was introduced in [2]. In order to define the neutrix convolution we, first of all, let τ be a function in $\mathcal{D}$ satisfying the following properties:

- (i) $\tau(x) = \tau(-x)$,
- (ii) $0 \leq \tau(x) \leq 1$,
- (iii) $\tau(x) = 1$ for $|x| \leq 1/2$,
- (iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_ν is now defined by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \leq \nu, \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu, \end{cases} \quad (34)$$

for $\nu > 0$. □

The following definition of the neutrix convolution was given in [2].

DEFINITION 8. Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. Then the *neutrix convolution* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided that the limit h exists in the sense that

$$\text{N-}\lim_{\nu \rightarrow \infty} \langle f_\nu * g, \varphi \rangle = \langle h, \varphi \rangle, \quad (35)$$

for all φ in $\mathcal{D}$, where N is the neutrix (see van der Corput [7]), having domain N' the positive reals, range N'' the real numbers and with negligible functions finite linear sums of the functions

$$v^\lambda \ln^{r-1} v, \quad \ln^r v, \quad (\lambda \neq 0, r = 1, 2, \dots) \quad (36)$$

and all functions which converge to zero in the usual sense as v tends to infinity.

Note that in this definition, the convolution $f_v * g$ is defined in Gel'fand's and Shilov's sense, the distribution f_v having bounded support.

It is easily seen that any results proved with the original definition hold with the new definition. The following theorem (proved in [2]) therefore holds, showing that the neutrix convolution is a generalization of the convolution.

THEOREM 9. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Definition 4 (Gel'fand's and Shilov's [5]). Then the neutrix convolution $f \odot g$ exists and*

$$f \odot g = f * g. \quad (37)$$

The next theorem was also proved in [2].

THEOREM 10. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution $f \odot g$ exists. Then the neutrix convolution $f \odot g'$ exists and*

$$(f \odot g)' = f \odot g'. \quad (38)$$

Note, however, that the neutrix convolution $(f \odot g)'$ is not necessarily equal to $f' \odot g$.

We now increase the set of negligible functions given here to include finite linear sums of the functions

$$\cos(\lambda v), \quad \sin(\lambda v), \quad (\lambda \neq 0). \quad (39)$$

THEOREM 11. *If $\lambda, \mu \neq 0$, then the neutrix convolution $\text{Si}_+(\lambda x) \odot \sin(\mu x)$ exists and*

$$\begin{aligned} \text{Si}_+(\lambda x) \odot \sin(\mu x) = & -\frac{1}{4} \mu^{-1} [\text{sgn}(\lambda + \mu) + \text{sgn}(\lambda - \mu)] \pi \cos(\mu x) \\ & - \frac{1}{2} \mu^{-1} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \sin(\mu x) \end{aligned} \quad (40)$$

if $\lambda \neq \pm \mu$; and

$$\begin{aligned} \text{Si}_+(\lambda x) \odot \sin(\lambda x) = & -\frac{1}{4} \lambda^{-1} \text{sgn} \lambda \cdot \pi \cos(\lambda x) \\ & + \frac{1}{2} \lambda^{-1} [c - \ln |2\lambda|] \sin(\lambda x) \end{aligned} \quad (41)$$

if $\lambda = \pm \mu$.

PROOF. We put $[\text{Si}_+(\lambda x)]_\nu = \text{Si}_+(\lambda x)\tau_\nu(x)$. Then the convolution $[\text{Si}_+(\lambda x)]_\nu * \sin(\mu x)$ exists by [Definition 3](#) and

$$\begin{aligned} [\text{Si}_+(\lambda x)]_\nu * \sin(\mu x) &= \int_0^\nu \text{Si}_+(\lambda t) \sin[\mu(x-t)] dt \\ &\quad + \int_\nu^{\nu+\nu^{-\nu}} \text{Si}_+(\lambda t) \sin[\mu(x-t)] \tau_\nu(t) dt \\ &= I_1 + I_2, \end{aligned} \quad (42)$$

where it is easily seen that

$$\lim_{n \rightarrow \infty} I_2 = 0. \quad (43)$$

Further,

$$\begin{aligned} I_1 &= \int_0^\nu u^{-1} \sin(\lambda u) \int_u^\nu \sin[\mu(x-t)] dt du \\ &= \mu^{-1} \int_0^\nu u^{-1} \sin(\lambda u) \{ \cos[\mu(x-\nu)] - \cos[\mu(x-u)] \} du \\ &= \mu^{-1} \cos[\mu(x-\nu)] \text{Si}(\lambda \nu) \\ &\quad - \frac{1}{2} \mu^{-1} \cos(\mu x) \int_0^\nu u^{-1} \{ \sin[(\lambda+\mu)u] + \sin[(\lambda-\mu)u] \} du \\ &\quad + \frac{1}{2} \mu^{-1} \sin(\mu x) \int_0^\nu u^{-1} \{ \cos[(\lambda+\mu)u] - \cos[(\lambda-\mu)u] \} du \\ &= \mu^{-1} \cos[\mu(x-\nu)] \text{Si}(\lambda \nu) - \frac{1}{2} \mu^{-1} \cos(\mu x) \{ \text{Si}[(\lambda+\mu)\nu] + \text{Si}[(\lambda-\mu)\nu] \} \\ &\quad + \frac{1}{2} \mu^{-1} \sin(\mu x) \left\{ \text{Ci}[(\lambda+\mu)\nu] - \text{Ci}[(\lambda-\mu)\nu] - \ln \left| \frac{\lambda+\mu}{\lambda-\mu} \right| \right\} \end{aligned} \quad (44)$$

on using [Lemma 2](#). It follows that

$$\text{N-}\lim_{\nu \rightarrow \infty} I_1 = -\frac{1}{4} \mu^{-1} [\text{sgn}(\lambda+\mu) + \text{sgn}(\lambda-\mu)] \pi \cos(\mu x) - \frac{1}{2} \mu^{-1} \ln \left| \frac{\lambda+\mu}{\lambda-\mu} \right| \sin(\mu x) \quad (45)$$

on using [Lemma 1](#). Equation (40) now follows from (42), (43), and (45).

If $\lambda = \pm\mu$, we have

$$\begin{aligned} [\text{Si}_+(\lambda x)]_\nu * \sin(\lambda x) &= \int_0^\nu \text{Si}_+(\lambda t) \sin[\lambda(x-t)] dt \\ &\quad + \int_\nu^{\nu+\nu^{-\nu}} \text{Si}_+(\lambda t) \sin[\lambda(x-t)] \tau_\nu(t) dt \\ &= J_1 + J_2, \end{aligned} \quad (46)$$

where it is easily seen that

$$\lim_{n \rightarrow \infty} J_2 = 0. \quad (47)$$

Further,

$$\begin{aligned} J_1 &= \int_0^v u^{-1} \sin(\lambda u) \int_u^v \sin[\lambda(x-t)] dt du \\ &= \lambda^{-1} \int_0^v u^{-1} \sin(\lambda u) \{ \cos[\lambda(x-v)] - \cos[\lambda(x-u)] \} du \\ &= \lambda^{-1} \cos[\lambda(x-v)] \text{Si}(\lambda v) \\ &\quad - \frac{1}{2} \lambda^{-1} \cos(\lambda x) \int_0^v u^{-1} \sin(2\lambda u) du \\ &\quad + \frac{1}{2} \lambda^{-1} \sin(\lambda x) \int_0^v u^{-1} [\cos(2\lambda u) - 1] du \\ &= \lambda^{-1} \cos[\lambda(x-v)] \text{Si}(\lambda v) - \frac{1}{2} \lambda^{-1} \cos(\lambda x) \text{Si}(2\lambda v) \\ &\quad + \frac{1}{2} \lambda^{-1} \sin(\lambda x) [c + \text{Ci}(2\lambda v) - \ln |2\lambda v|] \end{aligned} \quad (48)$$

on using [Lemma 2](#). It follows that

$$\text{N-}\lim_{v \rightarrow \infty} J_1 = -\frac{1}{4} \lambda^{-1} \text{sgn} \lambda \cdot \pi \cos(\lambda x) + \frac{1}{2} \lambda^{-1} [c - \ln |2\lambda|] \sin(\lambda x) \quad (49)$$

on using [Lemma 1](#). Equation (41) now follows from (46), (47), and (49). $\square$

COROLLARY 12. *If $\lambda, \mu \neq 0$, then the neutrix convolution $\text{Si}_+(\lambda x) \odot \cos(\mu x)$ exists and*

$$\begin{aligned} \text{Si}_+(\lambda x) \odot \cos(\mu x) &= \frac{1}{4} \pi \mu^{-1} [\text{sgn}(\lambda + \mu) + \text{sgn}(\lambda - \mu)] \sin(\mu x) \\ &\quad - \frac{1}{2} \mu^{-1} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \cos(\mu x) \end{aligned} \quad (50)$$

if $\lambda \neq \pm \mu$; and

$$\text{Si}_+(\lambda x) \odot \cos(\lambda x) = \frac{1}{4} \lambda^{-1} \text{sgn} \lambda \cdot \pi \sin(\lambda x) + \frac{1}{2} \lambda^{-1} [c - \ln |2\lambda|] \cos(\lambda x) \quad (51)$$

if $\lambda = \pm \mu$.

PROOF. It follows from (38) and (40) that

$$\begin{aligned} [\text{Si}_+(\lambda x) \odot \sin(\mu x)]' &= \mu \text{Si}_+(\lambda x) \odot \cos(\mu x) \\ &= \frac{1}{4} \pi [\text{sgn}(\lambda + \mu) + \text{sgn}(\lambda - \mu)] \sin(\mu x) \\ &\quad - \frac{1}{2} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \cos(\mu x) \end{aligned} \quad (52)$$

and (50) follows.

If $\lambda = \pm\mu$, it follows from (38) and (41) that

$$\lambda \text{Si}_+(\lambda x) \odot \cos(\lambda x) = \frac{1}{4} \operatorname{sgn} \lambda \cdot \pi \sin(\lambda x) + \frac{1}{2} [c - \ln |2\lambda|] \cos(\lambda x) \quad (53)$$

and (51) follows. $\square$

COROLLARY 13. *If $\lambda, \mu \neq 0$, then the neutrix convolutions $\text{Si}_-(\lambda x) \odot \sin(\mu x)$ and $\text{Si}_-(\lambda x) \odot \cos(\mu x)$ exist and*

$$\begin{aligned} \text{Si}_-(\lambda x) \odot \sin(\mu x) &= -\frac{1}{4} \pi \mu^{-1} [\operatorname{sgn}(\lambda + \mu) + \operatorname{sgn}(\lambda - \mu)] \cos(\mu x) \\ &\quad + \frac{1}{2} \mu^{-1} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \sin(\mu x), \end{aligned} \quad (54)$$

$$\begin{aligned} \text{Si}_-(\lambda x) \odot \cos(\mu x) &= \frac{1}{4} \pi \mu^{-1} [\operatorname{sgn}(\lambda + \mu) + \operatorname{sgn}(\lambda - \mu)] \sin(\mu x) \\ &\quad + \frac{1}{2} \mu^{-1} \ln \left| \frac{\lambda + \mu}{\lambda - \mu} \right| \cos(\mu x) \end{aligned} \quad (55)$$

if $\lambda \neq \pm\mu$; and

$$\begin{aligned} \text{Si}_-(\lambda x) \odot \sin(\lambda x) &= \frac{1}{4} \lambda^{-1} \operatorname{sgn} \lambda \cdot \pi \cos(\lambda x) - \frac{1}{2} \lambda^{-1} [c - \ln |2\lambda|] \sin(\lambda x), \\ \text{Si}_-(\lambda x) \odot \cos(\lambda x) &= \frac{1}{4} \lambda^{-1} \operatorname{sgn} \lambda \cdot \pi \sin(\lambda x) - \frac{1}{2} \lambda^{-1} [c - \ln |2\lambda|] \cos(\lambda x) \end{aligned} \quad (56)$$

if $\lambda = \pm\mu$.

PROOF. Equations (54) and (55) follow on replacing x by $-x$ in (40) and (50), respectively; and (56) follow on replacing x by $-x$ in (41) and (51), respectively. $\square$

The final neutrix convolutions follow easily from the above results:

$$\text{Si}(\lambda x) \odot \sin(\mu x) = 0, \quad \text{Si}(\lambda x) \odot \cos(\mu x) = 0 \quad (57)$$

if $\lambda \neq \pm\mu$; and

$$\text{Si}(\lambda x) \odot \sin(\lambda x) = 0, \quad \text{Si}(\lambda x) \odot \cos(\lambda x) = 0 \quad (58)$$

if $\lambda = \pm\mu$.

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