ON A CLASS OF EVEN-DIMENSIONAL MANIFOLDS STRUCTURED BY AN AFFINE CONNECTION

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We deal with a $2m$ -dimensional Riemannian manifold (M, g) structured by an affine connection and a vector field $\mathcal{T}$, defining a $\mathcal{T}$ -parallel connection. It is proved that $\mathcal{T}$ is both a torse forming vector field and an exterior concurrent vector field. Properties of the curvature 2-forms are established. It is shown that M is endowed with a conformal symplectic structure Ω and $\mathcal{T}$ defines a relative conformal transformation of Ω .

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1. Introduction. In [5], a class of odd-dimensional manifolds endowed with a $\mathcal{T}$ -parallel connection was investigated.

In the present paper, we consider a $2m$ -dimensional Riemannian manifold (M, g) , structured by an affine connection defined by the torsion 2-forms S^A , $A \in \{1, 2, \dots, 2m\}$. If $\{e_A\}$ and $\{\omega^A\}$ are a vector and a covector basis, respectively, and $\mathcal{T}(T^A)$ a vector field (called the *structure* vector field of M), we assume that $\mathcal{T}$ defines a $\mathcal{T}$ -parallel connection, in the sense of [9] (see also [2, 4]), that is, the connection forms associated with $\{e_A\}$ and $\{\omega^A\}$ satisfy

$$\theta_B^A = \langle \mathcal{T}, e_B \wedge e_A \rangle = T^B \omega^A - T^A \omega^B, \quad (1.1)$$

where $\wedge$ means the wedge product of vector fields, which implies $\nabla_{\mathcal{T}} e_A = 0$.

Next, we assume that the torsion forms S^A are *exterior recurrent* (abbreviated ER) [1] with $\alpha = \mathcal{T}^\flat$ as recurrence form, that is, $dS^A = \alpha \wedge S^A$.

Assuming that T^A are also ER with a certain Pfaffian u as recurrence form, that is, $dT^A = T^A u$, and denoting $2t = \|\mathcal{T}\|^2$, we have

$$\nabla \mathcal{T} = 2t dp + (u - \alpha) \otimes \mathcal{T}, \quad (1.2)$$

where dp is the soldering form of M [3], which says that $\mathcal{T}$ is a *torse forming* vector field [8, 11, 12].

We derive

$$\nabla^2 \mathcal{T} = 2t(u + \alpha) \wedge dp, \quad (1.3)$$

that is, $\mathcal{T}$ is an *exterior concurrent* vector field [10] (see also [4]).

Setting $S = S^1 \wedge S^2 \wedge \dots \wedge S^{2m}$, we find that the $4m$ -form S associated with M is ER with $4m\alpha$ as recurrence form.

It is shown that the curvature 2-forms Θ_B^A are ER having the closed 1-form $2(u + \alpha)$ as recurrence form. We agree to define such a manifold as an *exterior recurrent curvature 2-form* manifold.

Finally, assuming that M carries an *almost symplectic form* Ω , that is, a nondegenerate differential 2-form, we prove that Ω is a *conformal* symplectic form.

It is shown that $\mathcal{T}$ defines a *relative conformal transformation* of the conformal symplectic form Ω (see [5]).

The above results are stated in [Theorem 3.1](#).

2. Preliminaries. Let (M, g) be a $2m$ -dimensional oriented Riemannian manifold structured by an affine differential operator ∇ .

Let $\Gamma(TM)$ be the set of sections of the tangent bundle and $\flat : TM \rightarrow T^*M$ and $\sharp : T^*M \rightarrow TM$ the classical musical isomorphisms defined by g (i.e., $\flat$ is the index lowering operator and $\sharp$ is the index raising operator).

Following [7], we denote by

$$A^q(M, TM) = \Gamma \text{Hom}(\wedge^q TM, TM) \quad (2.1)$$

the set of vector-valued q -forms ($q \leq \dim M$) and we write for the affine operator ∇

$$d^\nabla : A^q(M, TM) \longrightarrow A^{q+1}(M, TM). \quad (2.2)$$

If $dp \in A^1(M, TM)$ is the canonical vector-valued 1-form of M , then as an extension of the Levi-Civita operator and by [3], we agree to call dp the *soldering form* of M .

Let the unit vector fields $\{e_A\}$ be an orthonormal vector basis and $\{\omega^A\}$ its corresponding cobasis on M , $A = 1, \dots, 2m$. Then, if θ_B^A , S^A , and Θ_B^A denote the connection forms, the torsion 2-forms and the curvature 2-forms, respectively, Cartan's structure equations are expressed by

$$\nabla e_A = \theta_B^A \otimes e_B, \quad (2.3)$$

$$d\omega^A = \omega^B \wedge \theta_B^A + S^A, \quad (2.4)$$

$$d\theta_B^A = \theta_B^C \wedge \theta_C^A + \Theta_B^A. \quad (2.5)$$

We recall the following definitions (cf. [4]).

A vector field $\mathcal{T}$ is said to be a *torse forming* vector field [12] if it satisfies

$$\nabla \mathcal{T} = f\mathcal{T} + v \otimes \mathcal{T}, \quad f \in C^\infty M, \quad v \in \wedge^1 M. \quad (2.6)$$

Also, the vector field $\mathcal{T}$ is called *exterior concurrent* [10] if

$$\nabla^2 \mathcal{T} = \pi \wedge dp, \quad \pi \in \wedge^1 M. \quad (2.7)$$

If $Z, Z' \in \Gamma(TM)$, we also have the following formula:

$$d\omega(Z, Z') = \mathcal{L}_{Z'} \omega(Z) - \mathcal{L}_Z \omega(Z') + \omega([Z, Z']), \quad (2.8)$$

where $\mathcal{L}$ is the Lie derivative.

Since $dp = \omega^A \wedge e_A$, then it follows that

$$d^\nabla(dp) = S^A \otimes e_A. \quad (2.9)$$

3. Manifolds with affine connection. In the present paper, we assume first that the $2m$ -dimensional Riemannian manifold (M, g) carries a structure vector field $\mathcal{T}(T^A)$ which defines a $\mathcal{T}$ -parallel connection, in the sense of [9] (see also [2, 4]). Such a connection is expressed by

$$\theta_B^A = \langle \mathcal{T}, e_B \wedge e_A \rangle = T^B \omega^A - T^A \omega^B. \quad (3.1)$$

Since we quickly find from (3.1) that

$$\nabla_{\mathcal{T}} e_A = 0, \quad (3.2)$$

this agrees with the definition of $\mathcal{T}$ -parallel connection.

Setting $2t = \|T\|^2$, we derive

$$\nabla \mathcal{T} = 2t dp - \alpha \otimes \mathcal{T} + \sum_A dT^A \otimes e_A, \quad (3.3)$$

where $\alpha = \mathcal{T}^\flat$ is the dual 1-form of $\mathcal{T}$. Also, we find by (3.1) and (2.4) that

$$d\omega^A = \alpha \wedge \omega^A + S^A. \quad (3.4)$$

Second, we assume that the torsion forms S^A are exterior recurrent [1] having α as recurrence form, that is,

$$dS^A = \alpha \wedge S^A, \quad (3.5)$$

and T^A are ER with the Pfaffian u as recurrence form, that is,

$$dT^A = T^A u. \quad (3.6)$$

We obtain $d\alpha = 0$, that is, $\alpha^\sharp = \mathcal{T}$ is a closed vector field.

Under these conditions, it follows from (3.3) and (3.6) that

$$\nabla \mathcal{T} = 2t dp + (u - \alpha) \otimes \mathcal{T}; \quad (3.7)$$

this proves that $\mathcal{T}$ is a torse forming vector field [4, 8, 11, 12]. Since the operator ∇ acts inductively and clearly by (3.6), then

$$dt = 2tu, \quad (3.8)$$

we infer

$$d^\nabla(\nabla \mathcal{T}) = \nabla^2 \mathcal{T} = 2t(u + \alpha) \wedge dp. \quad (3.9)$$

This means that the vector field $\mathcal{T}$ is an exterior concurrent vector field [6, 10].

By [6], (3.9) implies that

$$\mathcal{R}(\mathcal{T}, Z) = -(2m-1)2tg(\mathcal{T}, Z), \quad Z \in \Gamma(TM), \quad (3.10)$$

where $\mathcal{R}$ denotes the Ricci tensor field on M .

By (3.9) and by standard calculation, we derive

$$\nabla^4 \mathcal{T} = 0 \quad (3.11)$$

and therefore we may say that the vector field $\mathcal{T}$ is an element of

$$\Gamma \text{Hom}(\wedge^4 TM, TM). \quad (3.12)$$

On the other hand, recall that the Bianchi forms in the sense of Tachibana are defined by

$$\Omega_{\alpha_1, \dots, \alpha_{2p}}^{(p)} = \Omega_{\alpha_1}^{\alpha_2} \wedge \Omega_{\alpha_2}^{\alpha_3} \wedge \dots \wedge \Omega_{\alpha_{2p-1}}^{\alpha_{2p}}, \quad (3.13)$$

where $\Omega_{\alpha_q}^{\alpha_{q+1}}$ are 2-forms. Thus, setting

$$S = S^1 \wedge S^2 \wedge \dots \wedge S^{2m}, \quad (3.14)$$

we find that

$$dS = 4m\alpha \wedge S. \quad (3.15)$$

Therefore, we may say that the $4m$ -form S associated with M is ER with $4m\alpha$ as recurrence form.

By (3.4) we may set

$$S^A = u \wedge \omega^A \quad (3.16)$$

and by (3.1) and the structure equations (2.5) we get after some calculations

$$\Theta_B^A = 2(u + \alpha) \wedge \omega_B^A + 2t\omega^B \wedge \omega^A. \quad (3.17)$$

Next, performing the exterior differentiation of Θ_B^A , we derive, taking account of (3.8)

$$d\Theta_B^A = 2(u + \alpha) \wedge \Theta_B^A. \quad (3.18)$$

This shows that all curvature forms Θ_B^A are ER and have the closed 1-form $2(u + \alpha)$ as recurrence form.

We agree to define such an even-dimensional manifold M as an *exterior recurrent curvature 2-form manifold*.

Finally, assume that M carries an almost symplectic form Ω . Then, we may express Ω as

$$\Omega = \sum_{a=1}^m \omega^a \wedge \omega^{a*}, \quad a^* = a + m. \quad (3.19)$$

Taking the exterior differentiation of Ω , we find by (3.4) and (3.16) that

$$d\Omega = 2(\alpha + u) \wedge \Omega. \quad (3.20)$$

This shows that the manifold under consideration is endowed with a *conformal* symplectic structure having $\alpha + u$ as covector of Lee.

Moreover, taking the Lie differentiation of Ω with respect to the structure vector field $\mathcal{T}$, we infer

$$\mathcal{L}_{\mathcal{T}}\Omega = u\mathcal{T}\Omega + 2(u + \alpha) \wedge \sum_{a=1}^m (T^a \omega^{a*} - T^{a*} \omega^a). \quad (3.21)$$

Using (3.8) and (3.6), the exterior differentiation of (3.21) gives

$$d\mathcal{L}_{\mathcal{T}}\Omega = 8tu \wedge \Omega. \quad (3.22)$$

Hence, by [4], the above equation says that $\mathcal{T}$ defines a *relative conformal transformation* of the conformal symplectic form Ω .

Summing up, we state the following theorem.

THEOREM 3.1. *Let (M, g) be a $2m$ -dimensional Riemannian manifold structured by an affine connection defined by the torsion 2-forms S^A , $A = 1, \dots, 2m$. Let $\mathcal{T}(T^A)$ be a structure vector field, which defines a $\mathcal{T}$ -parallel connection and assume that S^A are exterior recurrent, having $\mathcal{T}^b$ as recurrence form ($\mathcal{T}^b = \alpha$ is a closed Pfaffian).*

Then the following properties hold:

- (i) $\mathcal{T}$ is both a torse forming and an exterior concurrent vector field;
- (ii) the structure curvature 2-forms Θ_B^A are exterior recurrent with the closed Pfaffian $2(u + \alpha)$ as recurrence form;
- (iii) the manifold M is endowed with a conformal symplectic structure Ω having $u + \alpha$ as covector of Lee;
- (iv) the vector field $\mathcal{T}$ defines a relative conformal transformation of Ω , that is, $d\mathcal{L}_{\mathcal{T}}\Omega = 8tu \wedge \Omega$, where $2t = \|\mathcal{T}\|^2$.

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