We study the principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem:

\[-\Delta u(x) = \lambda g(x)u(x), \quad x \in D; \quad (\partial u/\partial n)(x) + \alpha u(x) = 0, \quad x \in \partial D,\]

where \(\Delta\) is the standard Laplace operator, \(D\) is a bounded domain with smooth boundary, \(g : D \to \mathbb{R}\) is a smooth function which changes sign on \(D\) and \(\alpha \in \mathbb{R}\). We discuss the relation between \(\alpha\) and the principal eigenvalues.

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1. Introduction. We investigate the property of principal eigenvalues for the boundary value problem

\[-\Delta u(x) = \lambda g(x)u(x), \quad x \in D,\]
\[
\frac{\partial u}{\partial n}(x) + \alpha u(x) = 0, \quad x \in \partial D,
\]

where \(D\) is a bounded region in \(\mathbb{R}^N\) with smooth boundary, \(g : D \to \mathbb{R}\) is a smooth function which changes sign on \(D\) and \(\alpha \in \mathbb{R}\).

Such problems have been studied in recent years since Fleming [4] studied the following associated nonlinear problems arising in the study of population genetics:

\[u_t(x,t) = \Delta u + \lambda g(x)f(u), \quad x \in D,\]

where \(f\) is some function of class \(C^1\) such that \(f(0) = 0 = f(1)\).

Fleming's results suggested that nontrivial steady-state solutions were bifurcating the trivial solutions \(u \equiv 0\) and \(u \equiv 1\). In order to investigate these bifurcation phenomena, it was necessary to understand the eigenvalues and eigenfunctions of the corresponding linearized problem

\[-\Delta u(x) = \lambda g(x)u(x), \quad x \in D.\]

The study of the linear ordinary differential equation case, however, goes back to Bocher [3]. Attention has been confined mainly to the cases of Dirichlet \((\alpha = \infty)\) and Neumann boundary conditions.

In the case of Dirichlet boundary conditions, it is well known (see [5]) that there exists a double sequence of eigenvalues for (1.1)

\[
\cdots < \lambda_2^- < \lambda_1^- < 0 < \lambda_1^+ < \lambda_2^+ < \cdots,
\]
\( \lambda_+^* (\lambda_-^*) \) being the unique positive (negative) principal eigenvalue, that is, (1.1) has solution \( u(v) \) which is positive in \( D \). It is also well known that the case where \( 0 < \alpha < \infty \) is similar to the Dirichlet case.

In the case of Neumann boundary conditions, 0 is clearly a principal eigenvalue and there is a positive (negative) principal eigenvalue if and only if \( \int_D g(x)dx < 0 \) (\( > 0 \)); in the case where \( \int_D g(x)dx = 0 \) there are no positive and no negative principal eigenvalues.

We show that the set of \( \lambda \)'s such that \( \lambda \) is a principal eigenvalue of (1.1) is a bounded set and its bounds are independent of \( \alpha \), and also the positive principal eigenvalue \( \lambda \) of (1.1) is strictly an increasing function of \( \alpha \).

Our analysis is based on a method used by Hess and Kato [5]. Consider, for fixed \( \lambda \), the eigenvalue problem

\[-\Delta u(x) - \lambda g(x)u(x) = \mu u(x), \quad x \in D,\]
\[\frac{\partial u}{\partial n}(x) + \alpha u(x) = 0, \quad x \in \partial D.\]

(1.5)

We denote the lowest eigenvalue of (1.5) by \( \mu(\alpha, \lambda) \). Let

\[A_{\alpha, \lambda} = \left\{ \int_D [|\nabla \phi|^2 + q\phi^2]dx + \alpha \int_{\partial D} \phi^2 ds_x - \lambda \int_D g\phi^2 dx : \phi \in W^{1,2}(D), \int_D \phi^2 dx = 1 \right\}\]

(1.6)

when \( \alpha \geq 0 \), it is clear that \( A_{\alpha, \lambda} \) is bounded below. It is shown in [6], by using variational arguments, that \( \mu(\alpha, \lambda) = \inf A_{\alpha, \lambda} \) and that an eigenfunction corresponding to \( \mu(\alpha, \lambda) \) does not change sign on \( D \). Thus, clearly, \( \lambda \) is a principal eigenvalue of (1.1) if and only if \( \mu(\alpha, \lambda) = 0 \).

When \( \alpha < 0 \), the boundedness below of \( A_{\alpha, \lambda} \) is no longer obvious a priori, and it is shown by Afrouzi and Brown [2].

2. Boundedness and monotonicity of principal eigenvalues. The following theorem is proved in [1, Theorem 1.8].

**Theorem 2.1.** If

\[ \lambda_1 = \inf \left\{ \int_D [|\nabla \phi|^2 + q\phi^2]dx + \alpha \int_{\partial D} \phi^2 ds_x : \phi \in W^{1,2}(D), \int_D \phi^2 dx = 1 \right\}, \]

(2.1)

where \( q \in L^\infty(D) \), then there exists \( \phi_1 \in W^{1,2}(D), \int_D \phi_1^2 dx = 1 \), such that

\[ \lambda_1 = \int_D [|\nabla \phi_1|^2 + q\phi_1^2]dx + \alpha \int_{\partial D} \phi_1^2 ds_x. \]

(2.2)

Moreover, \( \lambda_1 \) is the principal eigenvalue and \( \phi_1 > 0 \) is a principal eigenfunction of

\[-\Delta u(x) + q(x)u(x) = \lambda u(x), \quad x \in D,\]
\[\frac{\partial u}{\partial n}(x) + \alpha u(x) = 0, \quad x \in \partial D.\]

(2.3)
It is obvious that $\lambda_1$ is the principal eigenvalue of (1.1) if and only if 0 is the principal eigenvalue of

$$-\Delta u(x) - \lambda_1 g(x)u(x) = \mu u(x), \quad x \in D,$$

$$\frac{\partial u}{\partial n}(x) + \alpha u(x) = 0, \quad x \in \partial D.$$  (2.4)

Here we are ready to prove one of the main results of this section about the uniformly boundedness of principal eigenvalues of (1.1) with respect to $\alpha$.

**Theorem 2.2.** There exist $m < 0$ and $M > 0$ such that if $\lambda$ is a principal eigenvalue of (1.1), then $\lambda \in [m, M]$ and also $m, M$ are independent of $\alpha$.

**Proof.** Suppose that $\lambda_1$ is a principal eigenvalue of (1.1). Then 0 is a principal eigenvalue of (2.4) and so by Theorem 2.1, we have

$$0 = \inf \left\{ \int_D |\nabla \phi|^2 + \alpha \int_{\partial D} \phi^2 ds - \lambda_1 \int_D g\phi^2 dx : \phi \in W^{1,2}(D), \int_D \phi^2 dx = 1 \right\}. \quad (2.5)$$

Now, by considering test functions $\phi_1, \phi_2 \in C_0^\infty(D)$ such that $\int_D \phi_1^2 dx = 1$ and $\int_D g\phi_1^2 dx > 0$ also $\int_D \phi_2^2 dx = 1$ and $\int_D g\phi_2^2 dx < 0$ we have

$$0 \leq \int_D |\nabla \phi_1|^2 + \alpha \int_{\partial D} \phi_1^2 ds - \lambda_1 \int_D g\phi_1^2 dx \quad (2.6)$$

and also

$$0 \leq \int_D |\nabla \phi_2|^2 + \alpha \int_{\partial D} \phi_2^2 ds - \lambda_1 \int_D g\phi_2^2 dx. \quad (2.7)$$

Hence from (2.6) and (2.7) we obtain, respectively,

$$\lambda_1 \leq \frac{\int_D |\nabla \phi_1|^2 dx}{\int_D g\phi_1^2 dx}, \quad \frac{\int_D |\nabla \phi_2|^2 dx}{\int_D g\phi_2^2 dx} \leq \lambda_1. \quad (2.8)$$

So by assuming $M = \int_D |\nabla \phi_1|^2 dx / \int_D g\phi_1^2 dx$ and $m = \int_D |\nabla \phi_2|^2 dx / \int_D g\phi_2^2 dx$, we have obtained $\lambda \in [m, M]$, and also we see that $m, M$ are independent of $\alpha$. \qed

In the case $0 < \alpha < \infty$, it is known [1, Lemmas 1.18 and 1.19] that problem (1.1) has the unique positive (negative) principal eigenvalue, that is, $\lambda^+_1 (\lambda^-_1)$, such that if $u$ and $v$ are being eigenfunctions corresponding to $\lambda^+_1$ and $\lambda^-_1$, respectively, then $\int_D g\mu^2 dx > 0$ and $\int_D g\nu^2 dx < 0$. Also in the case $\alpha < 0$, the following theorem [2, Theorem 5] is proved.

**Theorem 2.3.** There exists $\alpha_0 \leq 0$ such that

(i) if $\alpha < \alpha_0$, then (1.1) does not have a principal eigenvalue;

(ii) if $\alpha = \alpha_0$, then (1.1) has a unique principal eigenvalue with the corresponding eigenfunction $u_0$ such that $\int_D g(x)u_0^2(x)dx = 0$;

(iii) if $\alpha > \alpha_0$, then (1.1) has exactly two principal eigenvalues $\lambda$ and $\mu, \lambda < \mu$; if $u_0$ and $v_0$ are eigenfunctions corresponding to $\lambda < \mu$, respectively, then $\int_D g(x)u_0^2(x)dx < 0$ and $\int_D g(x)v_0^2(x)dx > 0$;
(iv) \( \alpha_0 = 0 \) if and only if \( \int_D g(x) \, dx = 0 \).

Now we prove the monotonicity of principal eigenvalues of (1.1) with respect to \( \alpha \).

**Theorem 2.4.** Suppose that \( \lambda_1 \) is a principal eigenvalue of

\[
- \Delta u(x) = \lambda g(x) u(x), \quad x \in D,
\]

\[
\frac{\partial u}{\partial n}(x) + \alpha_1 u(x) = 0, \quad x \in \partial D
\]

such that the corresponding principal eigenvalue, say \( u_1 \), satisfies

\[
\int_D g(x) u_1^2 \, dx > 0.
\]

If \( \alpha_2 > \alpha_1 \) and \( \lambda_2, u_2 \) are, respectively, principal eigenvalue and eigenfunction of

\[
- \Delta u(x) = \lambda g(x) u(x), \quad x \in D, \\
\frac{\partial u}{\partial n}(x) + \alpha_2 u(x) = 0, \quad x \in \partial D
\]

such that \( \int_D g(x) u_2^2 \, dx > 0 \), then \( \lambda_2 < \lambda_1 \).

**Proof.** Since \( \lambda_1 \) is a principal eigenvalue of (2.9), so 0 is a principal eigenvalue of

\[
- \Delta u(x) - \lambda_1 g(x) u(x) = \mu u(x), \quad x \in D, \\
\frac{\partial u}{\partial n}(x) + \alpha_1 u(x) = 0, \quad x \in \partial D
\]

and so we have

\[
0 = \int_D |\nabla u_1|^2 \, dx + \alpha_1 \int_{2D} u_1^2 \, dx - \lambda_1 \int_D g u_1^2 \, dx
\]

(2.12)

and also

\[
0 = \inf \left\{ \int_D |\nabla u|^2 \, dx + \alpha_2 \int_{2D} u^2 \, dx - \lambda_2 \int_D g u^2 \, dx : u \in W^{1,2}(D), \int_D u^2 \, dx = 1 \right\}. 
\]

(2.13)

If \( \lambda_2 \geq \lambda_1 \), then

\[
0 = \int_D |\nabla u_1|^2 \, dx + \alpha_1 \int_{2D} u_1^2 \, dx - \lambda_1 \int_D g u_1^2 \, dx \\
> \int_D |\nabla u_1|^2 \, dx + \alpha_2 \int_{2D} u_1^2 \, dx - \lambda_2 \int_D g u_1^2 \, dx \\
\geq 0
\]

(2.14)

which is impossible. Hence \( \lambda_2 < \lambda_1 \) and the proof is complete.

**References**


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