

ADJOINT REGULAR RINGS

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Let R be a ring. The *circle operation* is the operation $a \circ b = a + b - ab$, for all $a, b \in R$. This operation gives rise to a semigroup called the *adjoint semigroup* or *circle semigroup* of R . We investigate rings in which the adjoint semigroup is regular. Examples are given which illustrate and delimit the theory developed.

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1. Introduction. This paper continues the authors' investigation of adjoint semigroups of rings [13, 14]. Here R will always denote a ring (not necessarily commutative and not necessarily with unity). The Jacobson *circle* or *adjoint* operation, $a \circ b = a + b - ab$, for each $a, b \in R$, yields a monoid (R, \circ) , the *adjoint semigroup* of R . Here we primarily consider the situation where R is a (von Neumann) regular ring or where (R, \circ) is (von Neumann) regular. Previous work along these lines has been done by Du [7] and Clark [3]. Our viewpoint is that of the interplay between semigroup properties of (R, \circ) or (R, \cdot) and ring properties of R .

For $x \in R$, we use $\mathbf{l}_R(x)$ and $\mathbf{r}_R(x)$ for the left and right annihilator sets of x in R , respectively. When no ambiguity will arise, we use simply $\mathbf{l}(x)$ and $\mathbf{r}(x)$. The Jacobson radical of R is denoted by $\mathcal{J}(R)$.

Let S be a semigroup. We use $E(S)$ for the set of idempotents in S and $Z(S)$ for the center of S . Frequent use will be made of the fact that $E(R, \cdot) = E(R, \circ)$ and $Z(R, \cdot) = Z(R, \circ)$. Consequently we use $E(R)$ and $Z(R)$ for these sets, respectively.

Of particular interest here are the following types of regular semigroups. Let T be a regular semigroup. If the idempotents of T commute among one another, T is said to be *inverse*; this is equivalent to the condition that the von Neumann inverse of each element in T is unique, [5, Theorem 4.11]. If the idempotents of T are central, then T is said to be *Clifford*. It is well known that a Clifford semigroup is a union of groups, [5, Theorem 1.17]. If the idempotents form a subsemigroup, then T is called *orthodox*. If each element commutes with one of its von Neumann pseudoinverses, then T is *completely regular*. This last condition is equivalent to the condition that T is a union of groups, [19, Theorem II.1.4].

Let \mathcal{P} be a semigroup property (or from another vantage point, \mathcal{P} could be thought of as a class of semigroups closed under isomorphism). If (R, \cdot) has property \mathcal{P} we say R is a \mathcal{P} -ring, and if (R, \circ) has property \mathcal{P} we say R is an *adjoint \mathcal{P} -ring*. Exemplary of such properties are regular, completely regular, or Clifford. It is worth noting that what here is called an adjoint completely regular ring is called a generalized radical ring in [3, 8]. (For terminology and basic facts on semigroups, see [5] or [19].)

2. Preliminaries. In this section, we present some preliminary results. Let R^1 be the standard Dorroh extension of a ring R to the ring R^1 , which has unity. Recall that this embeds R as an ideal of R^1 , so we can identify R as the ideal \bar{R} in R^1 . We will at times use R^1 in conjunction with R in order to make use of the simplifying attributes of having an identity for calculation. Recall that the mapping $\phi : x \rightarrow 1 - x$ is an isomorphism from (R^1, \circ) onto (R^1, \cdot) , and ϕ restricted to R yields an injective homomorphism. Observe that if R has identity, then this same mapping, $x \rightarrow 1 - x$, yields an isomorphism from (R, \circ) onto (R, \cdot) . This observation makes the next result immediate. (See also [9, Lemma 20].)

PROPOSITION 2.1. *Let $a, b \in R$. Then*

- (a) $a \circ b \in E(R)$ if and only if $(1 - a)(1 - b) = 1 - (a \circ b)$;
- (b) $a \circ b \circ a = a$ if and only if $(1 - a)(1 - b)(1 - a) = (1 - a)$;
- (c) $a \circ R = b \circ R$ if and only if $(1 - a)R^1 = (1 - b)R^1$;
- (d) a is adjoint regular in R if and only if $1 - a$ is a regular element in R^1 .

The next result is immediate, but it is useful enough to warrant stating.

PROPOSITION 2.2. *An element $a \in R$ is adjoint regular if and only if there exist $b \in R$, $e = e^2 \in R$ such that $a + b - ab = e$ and $ea = e$.*

Note that a quasi-regular element satisfies the conditions of Proposition 2.2 with $e = 0$. Hence, the conditions for adjoint regularity are a natural generalization of the condition for quasi-regularity.

Du [7, Theorem 1] has shown that if R is a regular ring, then (R, \circ) is a regular monoid. We next give a different proof of that result.

PROPOSITION 2.3. *If R is a regular ring, then (R, \circ) is a regular monoid.*

PROOF. There is an injective ring homomorphism, $\phi : R \rightarrow R^*$, that embeds R as an ideal, \hat{R} , in the regular ring R^* which has identity [10, Theorem 1]. Consequently, $\phi : (R, \circ) \rightarrow (R^*, \circ)$ is an isomorphism. Since R^* has identity we have that $(R^*, \cdot) \cong (R^*, \circ)$ and that each ideal of (R^*, \cdot) is regular. So each ideal of (R^*, \circ) is regular. Hence (\hat{R}, \circ) is regular and so is its isomorphic image (R, \circ) . \square

Since every ideal of a regular semigroup is regular, we immediately have that if R is a regular ring and X is an ideal of (R, \circ) , then (X, \circ) is regular.

By using the powerful Fuchs-Halperin result, this proof completely bypasses the calculations used in Du's proof [7]. Also, using similar methods, we can obtain analogous results by replacing the term *regular* in Proposition 2.3 by *orthodox* or *inverse*. Finally, note that the converse of Proposition 2.3 does not hold, as any example of a Jacobson radical ring will illustrate.

3. Equivalent conditions under regularity. In this section, we discuss various equivalent conditions to R strongly regular and to R adjoint completely regular. Examples are given which illustrate these results and show limitations to extending them.

It is well known that the following conditions are equivalent for a ring R :

- (1) for each $a \in R$ there exists $b \in R$ such that $a = a^2b$;

- (2) for each $a \in R$ there exists $c \in R$ such that $a = c^2 a$;
- (3) R is regular and $E(R) \subseteq Z(R)$.

A ring R satisfying any of (and hence all of) these conditions is called a *strongly regular* ring, [6]. Note that these conditions are not equivalent for semigroups, [19, Theorem II.1.4]. The next proposition ties strongly regular with several other conditions on the multiplicative semigroup of a ring. The proposition is a compilation of results from various sources. First some terminology is needed.

Following [17] we say that a semigroup S is *E-solid* if, whenever $e, f, g \in E(S)$ such that $e\mathcal{L}f\mathcal{R}g$, there exists $h \in E(S)$ such that $e\mathcal{R}h\mathcal{L}g$. (Here \mathcal{L} and \mathcal{R} are the standard Green's relations, [5, page 47].) Recall that the *core* of S , denoted by $C(S)$, is the subsemigroup of S generated by $E(S)$, [19, page 89]. So S is orthodox if and only if S is regular and $E(S) = C(S)$. Let \mathcal{P} be a semigroup property. Then S is *locally- \mathcal{P}* if eSe has property \mathcal{P} for every $e \in E(S)$.

PROPOSITION 3.1. *Let R be a regular ring. The following are equivalent:*

- (a) R is strongly regular;
- (b) R is orthodox;
- (c) R is completely regular;
- (d) R is inverse;
- (e) R is Clifford;
- (f) R is locally inverse;
- (g) R is E-solid;
- (h) R is locally E-solid;
- (i) $C(R)$ is completely regular.

PROOF. As mentioned above, (a) \Leftrightarrow (e). So (e) \Rightarrow (b) is trivial, while (b) \Rightarrow (e) follows from [22, Remark 15]. It is known that (a) \Leftrightarrow (c), even for semigroups [19, page 58]. Next, (d) \Leftrightarrow (e) comes from [13], while (d) \Leftrightarrow (f) \Leftrightarrow (g) \Leftrightarrow (h) comes from [17], and (g) \Leftrightarrow (i) comes from [12, Theorem 3]. This completes the logical circuit.

This by no means exhausts the vast number of known equivalent conditions to R strongly regular, but it suffices for our purposes and gives a good sample of what is known. For more equivalent conditions to R strongly regular see [5, 6, 11, 16, 18, 19, 21, 22].

We next consider various equivalent conditions on the adjoint semigroup of a regular ring. □

PROPOSITION 3.2. *Let R be regular. The following are equivalent:*

- (a) R is adjoint completely regular;
- (b) R is adjoint Clifford;
- (c) R is strongly regular.

PROOF. Assume (a). Du [8, Lemma 11] has shown that (R, \circ) completely regular implies that $E(R)$ is closed under the ring multiplication. Since R is regular, this yields that R is orthodox. Thus R is Clifford and by Proposition 3.1 R is strongly regular. Consequently, R is adjoint regular. Conversely, R regular and adjoint Clifford implies that R is Clifford, and hence R is strongly regular. We have established (a) \Rightarrow (c) \Leftrightarrow (b). Since

a Clifford semigroup is a union of groups, (R, \circ) Clifford implies (R, \circ) completely regular, yielding $(b) \Rightarrow (a)$ and finishing the logical chain. \square

It is known [13] that R is adjoint inverse if and only if R is adjoint Clifford. Recently Du [9] has shown that adjoint orthodox implies adjoint completely regular. However, an adjoint completely regular ring need not be adjoint Clifford, as the next example shows.

EXAMPLE 3.3. Let S be a right zero semigroup with more than one element and let R be the semigroup ring $\mathbb{Z}_2[S]$. So every nonzero element of R has the form $x = e_1 + \dots + e_n$, for some distinct $e_1, \dots, e_n \in S$. Let $y = f_1 + \dots + f_m$, where f_1, \dots, f_m are distinct terms from S . Then $xy = ny$. So $xy = 0$, if n is even, and $xy = y$, if n is odd. Then x is an idempotent if and only if n is odd, and if y is also an idempotent, then $xy = y$. Consequently $E(R)$ is closed under ring multiplication. Observe that $R = N(R) \cup E(R)$; hence R is adjoint orthodox. Since elements in $E(R)$ are completely regular in (R, \circ) , and since elements in $N(R)$ are quasi-regular in R , and hence are completely regular in R , we have that R is adjoint completely regular. However, since $(e_1 + e_2)y(e_1 + e_2) = 0$, for each $y \in R$, we see that R is not regular. Also, R is not a Jacobson radical ring. Since $e_1e_2 \neq e_2e_1$, for distinct $e_1, e_2 \in S$, we see that $e_1 \circ e_2 \neq e_2 \circ e_1$, and hence R is not adjoint inverse.

Having R regular implies that R is adjoint regular, as we have seen. However, R being regular does not imply that R is adjoint inverse, adjoint completely regular, nor adjoint orthodox, as the next example illustrates.

EXAMPLE 3.4. Let A be a regular ring with identity and let $R = M_2(A)$, the ring of 2×2 matrices over A . So R is regular and hence R is adjoint regular. But there are noncommuting idempotents in R ; so R is not adjoint inverse, and hence not adjoint completely regular by Proposition 3.2. By [9, Theorem 14], R is not adjoint orthodox.

4. Decomposition. Let R be a ring, and let S_1, \dots, S_n be subrings of R . If $R = S_1 + \dots + S_n$ and whenever $s_i \in S_i$, $i = 1, \dots, n$ such that $s_1 + \dots + s_n = 0$, then $s_i = 0$, $i = 1, \dots, n$, then we say that R is a *supplementary sum* of the S_i , $i = 1, \dots, n$, [1]. This is equivalent to $R^+ = \sum_{i=1}^n S_i^+$, as a direct sum of abelian groups. We write $R = S_1 \dot{+} \dots \dot{+} S_n$ for such a supplementary sum. We next state as a lemma the well-known two-sided Peirce decomposition, given here without using an identity in the ring. (See [1, 15].)

LEMMA 4.1. *Let $e \in E(R)$. Then $R = eRe \dot{+} e \cdot \mathbf{l}_R(e) \dot{+} \mathbf{r}_R(e) \cdot e \dot{+} \mathbf{r}_R(e) \cap \mathbf{l}_R(e)$.*

Recall [2] that $e \in E(R)$ is said to be a *left semicentral idempotent* in R if $eRe = Re$, (equivalently, $ere = re$, for each $r \in R$). It is well known that in this case $\mathbf{l}_R(e)$ is an ideal of R and $R/\mathbf{l}_R(e) \cong eRe$.

PROPOSITION 4.2. *Let e be a left semicentral idempotent in R .*

- (a) *$eRe = Re$ is a left ideal of R ; $\mathbf{l}_R(e) = \mathbf{l}_R(Re)$ is an ideal of R ; and $e \cdot \mathbf{l}_R(e)$ is a right ideal of R ;*

- (b) $R = Re \oplus_l l_R(Re)$ as a direct sum of left ideals of R and $R/l_R(e) \cong eRe$, a ring with unity e ;
- (c) $eR = Re \dot{+} e \cdot l_R(Re)$.

PROOF. By definition, $Re = eRe$. The rest is routine. □

In a strictly analogous fashion, we define right semicentral idempotent and obtain a dual result. For a central idempotent we naturally obtain a stronger result.

COROLLARY 4.3. *If e is a central idempotent in R , then $R = eRe \oplus \text{Ann}_R(eRe)$, as a direct sum of ideals of R .*

Following Clark and Lewin [4], $e \in E(R)$ is said to be a *principal idempotent* in R if the homomorphic image of e in $\bar{R} = R/J(R)$ is the identity in \bar{R} . This implies that if $u \in E(R)$ such that $eu = 0 = ue$, then $u = 0$. (The latter condition, together with $e \neq 0$, is what Albert used in defining the *principal idempotent* [1, page 25].)

Principal idempotents play a key role in our next decomposition, whose proof makes use of the following result due to Du [7, Corollary 2].

LEMMA 4.4. *If R is adjoint regular and e is a principal idempotent in R , then $R = eRe \dot{+} J(R)$, as a supplementary sum of subrings.*

We are now ready to give a much shorter proof of the main result in [3, Theorem B].

PROPOSITION 4.5. *Let e be an idempotent in a ring R . The following are equivalent:*

- (a) R is adjoint completely regular and e is a principal idempotent in R ;
- (b) $R = eRe \dot{+} J(R)$ and eRe is a strongly regular ring.

PROOF. Assume (a). By Lemma 4.4, $R = eRe \dot{+} J(R)$; so $R/J(R) \cong eRe$. Since the ring eRe inherits the adjoint completely regular condition from R and eRe is regular, by Proposition 3.2 we have that ring eRe is strongly regular.

Assume (b). Then eRe is adjoint regular, so by [7] the ring R is adjoint regular. Since all the idempotents of R are central, R is adjoint completely regular. Clearly e is a principal idempotent of R . □

In view of Proposition 4.5 and the results of Section 3, we immediately have a plethora of conditions equivalent to part (a) of Proposition 4.5.

PROPOSITION 4.6. *Let $R = A \oplus B$ as a direct sum of left (right) ideals. If R is adjoint regular, then A and B are adjoint regular rings.*

PROOF. Let $a \in A$. Then there exist $a_1 \in A, b_1 \in B$ such that $a = a \circ (a_1 + b_1) \circ a = a \circ (a_1 + b_1) + a - [a \circ (a_1 + b_1)]a$. So $a \circ (a_1 + b_1) = [a \circ (a_1 + b_1)]a \in A$. Then $a \circ (a_1 + b_1) = a + a_1 + b_1 - a(a_1 + b_1)$, or $b_1 - ab_1 = a \circ (a_1 + b_1) - a - a_1 \in A$. But $b_1 - ab_1 \in B$; so $b_1 - ab_1 = 0$. Then $a = a \circ (a_1 + b_1) \circ a = [a + a(a_1 + b_1) - a(a_1 + b_1)] \circ a$, or $a = (a + a_1 - aa_1) \circ a = a \circ a_1 \circ a$. Proceed similarly for right ideals. □

LEMMA 4.7. *Let R be adjoint regular.*

- (a) *Either $J(R) = R$ or R contains a nonzero idempotent.*
- (b) *If the module ${}_R R$ is indecomposable and $R \neq J(R)$, then $R = eRe \oplus \mathfrak{r}(e)$, as a direct sum of right ideals, with eRe a regular ring with identity e , and $\mathfrak{r}(e)$ is a square zero ideal of R .*

(c) *If the modules ${}_R R$ and R_R are indecomposable, then either $R = J(R)$ or R is a division ring.*

PROOF. (a) Let $r \in R, r \neq 0$. Then there exists $\bar{r} \in R$ such that $r \circ \bar{r} \circ r = r$, and $r \circ \bar{r}$ and $\bar{r} \circ r$ are nonzero idempotents in (R, \circ) . So $r \circ \bar{r}$ and $\bar{r} \circ r$ are also idempotents in R . If R has no nonzero idempotent, then $r \circ \bar{r} = \bar{r} \circ r$. So in this case each element of R is quasi-regular, that is, $R = J(R)$.

(b) If $R \neq J(R)$, then there exists a nonzero $r \in R$ such that $e = r \circ \bar{r}$ is a nonzero idempotent in R . Then $R = Re \oplus \mathbf{I}(e)$, as a direct sum of left ideals. Since Re and $\mathbf{I}(e)$ are submodules of ${}_R R$, and since $Re \neq 0$, we have $\mathbf{I}(e) = 0$, and hence e is a right identity of R . So $R = eR \oplus \mathbf{r}(e)$, as a direct sum of right ideals of R . However, $R \cdot \mathbf{r}(e) = (Re)\mathbf{r}(e) = 0$, so $\mathbf{r}(e)$ is an ideal of R and $\mathbf{r}(e) \subseteq \mathbf{r}(R)$. Since R is adjoint regular, we have that eRe is a regular ring [7, Proposition 2].

(c) Continuing from the proof of (b), since R_R is indecomposable and since $eRe \neq 0$, we have that $\mathbf{r}(e) = 0$, and hence e is also a left identity. Thus $R = eRe$, a regular ring with identity. Since the ring is indecomposable, either $R = J(R)$ or R is a division ring. □

Note that from Lemma 4.7, we have that if R is indecomposable in terms of both left and right ideal decompositions, then either R is a Jacobson radical ring and the regular radical (see [20, Chapter VI]) is zero, or R is equal to its regular radical and the Jacobson radical is zero.

PROPOSITION 4.8. *Let R be adjoint regular. If R has a right (left) nonzero semicentral idempotent, then there exist submonoids A and B of (R, \circ) such that*

- (a) $R = A \circ B$ and $A \cap B = 0$;
- (b) $R = A \oplus B$ as a direct sum of adjoint regular right (left) ideals of R ;
- (c) A is a regular ring with identity and B is a two-sided ideal of R .

PROOF. Let e be a nonzero right semicentral idempotent in R . Then $R = eR \oplus r(e)$ as a direct sum of right ideals, where $eR = eRe$ is a ring with unity and $r(e)$ is a two-sided ideal of R . Let $A = eR$ and $B = r(e)$. Observe that $R = A \circ B$ because $AB = 0$. By Proposition 4.6, (A, \circ) is regular. Since A is a ring with unity, A is regular. Proceed similarly for e left semicentral. □

5. Radicals for adjoint Clifford rings. We show the equivalence of several standard radicals for rings which are adjoint Clifford and obtain a characterization for the Jacobson radical of such rings.

PROPOSITION 5.1. *If R is adjoint Clifford, then the Brown-McCoy radical, $\mathcal{G}(R)$, and $\mathcal{J}(R)$ are equal.*

PROOF. For purposes of contradiction suppose that there is an adjoint Clifford ring R with $\mathcal{J}(R) \neq \mathcal{G}(R)$. Then $R/\mathcal{J}(R)$ is also adjoint Clifford and $\mathcal{J}(R/\mathcal{J}(R)) = 0$. So, without loss of generality, take $\mathcal{J}(R) = 0$. Since R is adjoint Clifford it is a subdirect product of division rings by [13, Proposition 3.7]. Because $\mathcal{G}(R) \neq 0$, at least one of these homomorphic image division rings must have nonzero Brown-McCoy radical, a contradiction. □

EXAMPLE 5.2. Proposition 5.1 cannot be extended to all adjoint regular rings, even if the ring is also adjoint simple, as the following example illustrates. Let V be a vector space over a field F with $\dim_F V = \aleph_\omega$, and let ω be the first infinite limit ordinal. Let $R = \{\phi \in \text{End}_F V \mid \text{rank } \phi < \aleph_\omega\}$. It is well known that R is a regular ring with no maximal ideal. So $J(R) = 0$ and $G(R) = R$. It is also known that (R, \circ) is simple, [4, Example 3B].

In the next proposition, when R does not have unity we use the unity element in the Dorroh extension for convenience of expression.

PROPOSITION 5.3. *If R is adjoint Clifford, then $\mathcal{F}(R) = \bigcap_{e \in E} (1 - e)R$.*

PROOF. Let $R = eR \oplus (1 - e)R$ as ideals. Then eR is adjoint Clifford because the map $\phi : R \rightarrow eR$ is a ring homomorphism and so $\phi : (R, \circ) \rightarrow (eR, \circ)$ is a semigroup homomorphism. Therefore $\mathcal{F}(R) \subseteq (1 - e)R$. Since e is arbitrary, we have that $\mathcal{F}(R) \subseteq \bigcap_{e \in E} (1 - e)R$.

Conversely, let $I = \bigcap_{e \in E} (1 - e)R$. Then I is an ideal, hence adjoint Clifford. But I contains no nonzero idempotent. Therefore (I, \circ) is a group, so $I \subseteq \mathcal{F}(R)$. \square

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