THE MONAD INDUCED BY THE HOM-FUNCTOR IN THE CATEGORY OF TOPOLOGICAL SPACES AND ITS ASSOCIATED EILENBERG-MOORE ALGEBRAS

KOENA R. NAILANA

Received 15 September 2001

We discuss the monad associated with the topology of pointwise convergence. We also study examples of the Eilenberg-Moore algebras for this monad.

2000 Mathematics Subject Classification: 18C15, 54B30, 54F05.

1. Introduction. Let Top denote the category of topological spaces and continuous functions. Let \( \mathbb{R} \) denote the real line with the usual topology, and for each topological space \( X \), let \( C(X, \mathbb{R}) \) be the set of continuous real-valued functions from \( X \) to \( \mathbb{R} \). Consider the contravariant hom-functor \( C_p : \text{Top} \to \text{Top}^{\text{op}} \) defined by assigning to each space \( X \) the space of continuous real-valued functions with the topology of pointwise convergence. We denote this space by \( C_pX \). The space \( C_pX \) has been extensively studied. A fundamental reference on \( C_pX \) is Arkhangel’skii [2]. We recall that the sub-basic open sets of \( C_pX \) are sets of the form \([f, V]\), where \([x, V] = \{f \in C_pX : f(x) \in V, V \text{ open in } \mathbb{R}\}\).

2. The monad induced by the hom-functor in \( \text{Top} \) and the associated \( M \)-algebras. We now consider the composite functor \( C_{\text{op}}^p C_p : \text{Top} \to \text{Top}^{\text{op}} \to \text{Top} \) where \( C_{\text{op}}^p \) is the dual functor. Let \( M = C_{\text{op}}^p C_p \). If \( x \in X \), then the function \( \hat{x} : C_pX \to \mathbb{R} \) defined by \( \hat{x}(f) = f(x) \) is called the evaluation map at \( x \). The following propositions are important since they ensure that our morphisms are continuous. The proofs are straightforward and will be omitted.

**Proposition 2.1.** (i) For all \( x \in X \), \( \hat{x} : C_pX \to \mathbb{R} \) is continuous.

(ii) For all \( g \in C_pX \), \( \hat{g} : MC_pX \to \mathbb{R} \) is continuous.

**Proposition 2.2.** Let \( X \) be any topological space. Then

(i) \( \eta_X : X \to MX \), where \( \eta_X(x) = \hat{x} \) is continuous.

(ii) \( \mu_X : MMX \to MX \), where \( \mu_X(\gamma)[g] = \gamma(\hat{g}) \) is continuous.

We recall from [1] that a monad on a category \( A \) is a triplet \( M = (M, \eta, \mu) \) consisting of a functor \( M : A \to A \) and natural transformations \( \eta : \text{id}_A \to M \) and \( \mu : MM \to M \) such that \( \mu \circ M\mu = \mu \circ \mu M \) and \( \mu \circ \eta M = \text{id} \).

**Proposition 2.3.** The triplet \( (M, \eta, \mu) \), where \( \eta : \text{id}_{\text{Top}} \to M \) and \( \mu : MM \to M \) are defined by \( \eta_X(x) = \hat{x} \) and \( \mu_X(\gamma)[g] = \gamma(\hat{g}) \), respectively, where \( x \in X \), \( g \in C_pX \), is a monad.
We first check that $\eta : \text{id}_{\Gamma \text{Top}} \to M$ and $\mu : MM \to M$ are natural transformations. Let $f : X \to Y$ be a continuous function. We show that $M(f) \circ \eta_X = \eta_Y \circ f$. We define $\eta_X$ and $M(f)$ by $\eta_X(x) = \hat{x}$ and $M(f)(y)[g] = y(g \circ f)$ where $g : Y \to \mathbb{R}$ is continuous, $y \in M(X)$, and $\hat{\cdot}$ denotes evaluation, for example, $\hat{x}(g) = g(x)$. Then $M(f) \circ \eta_X(x) = M(f)(\hat{x})$. Let $g \in C_pX$. Then $M(f)(\hat{x})(g) = \hat{x}(g \circ f) = g \circ f(x) = g(f(x)) = \hat{f}(x)[g]$. Hence $M(f)(\hat{x}) = \hat{f}(x)[g]$. Now $\eta_Y \circ f(x) = \eta_Y(f(x)) = \hat{f}(x)$. Let $g \in C_pX$. Then $\hat{f}(x)[g] = g(\hat{f}(x))$. Hence $M(f) \circ \eta_X = \eta_Y \circ f$. We define $\mu_X$ by $\mu_X(y)[g] = \gamma(\hat{g})$ where $\gamma \in MM(X)$, $g \in C_pX$, and $\hat{\cdot}$ denotes the evaluation function at $g$, that is, $\hat{\cdot} : M(X) \to \mathbb{R}$. We now show that $\mu : MM \to M$ is a natural transformation, that is, $M(f) \circ \mu_X = \mu_Y \circ M^2(f)$. Let $h \in C_pY$. Then

\[ (M(f) \circ \mu_X)(y)[h] = M(f)(\mu_X(y))[h] = \mu_X(y)(h \circ f) = \gamma(h \circ f). \tag{2.1} \]

On the other hand,

\[ \mu_Y \circ M^2(f)(y)[h] = \mu_Y(M^2(f)(y))[h] \]

\[ = M^2(f)(y)(h) \]

\[ = M(M(f))(y)(h) \]

\[ = \gamma(\hat{h} \circ M(f)). \tag{2.2} \]

Let $\lambda : C_pX \to \mathbb{R}$. Then $\hat{h} \circ M(f)(\lambda) = \hat{h}(M(f)(\lambda)) = M(f)(\lambda)[h] = \lambda(h \circ f) = \hat{h} \circ f(\lambda)$. Therefore, $\hat{h} \circ M(f) = h \circ f$. From the equations

\[ M(f) \circ \mu_X(y)[h] = \gamma(h \circ f), \]

\[ \mu_Y \circ M^2(f)(y)[h] = \gamma(\hat{h} \circ M(f)), \tag{2.3} \]

\[ (\hat{h} \circ M(f))(\lambda) = \hat{h} \circ f(\lambda), \]

we get $M(f) \circ \mu_X = \mu_Y \circ M^2(f)$. Therefore $\mu : MM \to M$ is a natural transformation. We now show that the other monad conditions are satisfied. First, we show that $\mu_X \circ M\eta = \text{id}$. We prove that $\mu_X \circ M\eta = \text{id}$. Let $y \in M(X)$ and $f \in C_pX$. Then $\hat{f} : M(X) \to \mathbb{R}$ and $(\mu_X \circ M\eta)(y)[f] = M(\eta_M)(y)(\hat{f}) = \gamma(\hat{f} \circ \eta) = \gamma(f)$. Therefore $\mu_X \circ M\eta = \text{id}$. On the other hand, $(\mu_X \circ \eta_M)(y)[f] = \mu_X(\eta_M)(y)(\hat{f}) = \eta_M(y)(\hat{f}) = \hat{\gamma}(\hat{f}) = \hat{\gamma}(y) = \gamma(f)$. Therefore $\mu_X \circ \eta_M = \text{id}$. Second, we show that $\mu_X \circ \mu_M = \mu_X \circ M\mu_X$. We prove that $\mu_X \circ \mu_M = \mu_X \circ M\mu_X$. Let $y \in MMM(X)$. Then $\mu_X \circ M\mu_X(y)[f] = \mu_X(M\mu_X(y))[f] = \mu_X(y)(\hat{\gamma}) = \gamma(\hat{\gamma} \circ \mu_X) = \gamma(\hat{\gamma})$. Therefore $\mu_X \circ \mu_M = \mu_X \circ M\mu_X$. Therefore $(M, \eta, \mu)$ is a monad.

We now look at examples of the $M$-algebras of the monad $(M, \eta, \mu)$. 

\[ \text{If } \mathbb{M} = (M, \eta, \mu) \text{ is a monad on } \mathbb{A}, \text{ then } (A, h_A) \text{ is called an Eilenberg-Moore algebra or simply an } M\text{-algebra if the algebra map } h_A : MA \to A \text{ satisfies } h_A \circ \eta_A = \text{id}_A \text{ and } h_A \circ Mh_A = h_A \circ \mu_A. \]
**Proposition 2.4.** The real line \( \mathbb{R} \) is an \( M \)-algebra.

**Proof.** We define \( h_\mathbb{R} : M\mathbb{R} \to \mathbb{R} \) as \( \hat{1}_\mathbb{R} \), that is, the identity map with respect to \( \mathbb{R} \), and show that the \( M \)-algebra conditions are satisfied. It is obvious that the map \( h_\mathbb{R} \) is continuous. Let \( x \in \mathbb{R} \). Then \( h_\mathbb{R} \circ \eta_\mathbb{R}(x) = h_\mathbb{R}(\hat{x}) = \hat{x}(\hat{1}_\mathbb{R}) = 1_\mathbb{R}(x) \). Therefore \( h_\mathbb{R} \circ \eta_\mathbb{R} = 1_\mathbb{R} \). Now let \( y \in MM(\mathbb{R}) \). Then \( h_\mathbb{R} \circ \mu_\mathbb{R}(y) = \hat{1}_\mathbb{R}(\mu_\mathbb{R}(y)) = \mu_\mathbb{R}(y)(1_\mathbb{R}) = y(1_\mathbb{R}) \). On the other hand, \( h_\mathbb{R} \circ MH_\mathbb{R}(y) = \hat{1}_\mathbb{R}(M\hat{1}_\mathbb{R}(y)) = \hat{1}_\mathbb{R}(y \circ C_p(\hat{1}_\mathbb{R})) = y \circ C_p(\hat{1}_\mathbb{R})(1_\mathbb{R}) = y(C_p(\hat{1}_\mathbb{R})(1_\mathbb{R})) = y(1_\mathbb{R} \circ \hat{1}_\mathbb{R}) = y(\hat{1}_\mathbb{R}) \). Therefore \( h_\mathbb{R} \circ \mu_\mathbb{R} = h_\mathbb{R} \circ MH_\mathbb{R} \).

**Proposition 2.5.** For each \( X \in \text{Top} \), \( C_pX \) is an \( M \)-algebra with \( h_{C_pX} = C_p(\eta_X) \).

**Proof.** We first define \( h_{C_pX} : MC_pX \to C_pX \). Let \( \varphi \in MC_pX \). We define \( h_{C_pX} \) by \( h_{C_pX}(\varphi) = \varphi \circ \eta_X = C_p\eta_X(\varphi) \). Then the map \( h_{C_pX} \) is continuous, since it is the composite of continuous functions \( \varphi \) and \( \eta_X \). We now show that the conditions for an \( M \)-algebra are satisfied. Thus, we must show that \( h_{C_pX} \circ \eta_{C_pX} = \text{id}_{C_pX} \). Let \( f \in C_pX \).

Then \( h_{C_pX} \circ \eta_{C_pX}(f) = h_{C_pX}(\eta_{C_pX}(f)) = C_p\eta_X(^\hat{f}) = \hat{f} \circ \eta_X = f = \text{id}_{C_pX}(f) \), since \( \hat{f} \circ \eta_X(x) = \hat{f}(\eta_X(x)) = \hat{x}(f) = f(x) \). Therefore \( h_{C_pX} \circ \eta_{C_pX} = \text{id}_{C_pX} \). We must now show that \( h_{C_pX} \circ \mu_{C_pX} = h_{C_pX} \circ MH_{C_pX} \). Let \( \gamma \in MMC_pX \). Then \( h_{C_pX} \circ \mu_{C_pX}(\gamma) = h_{C_pX}(\mu_{C_pX}(\gamma)) = C_p\eta_X(\mu_{C_pX}(\gamma)) = \mu_{C_pX}(\gamma) \circ \eta_X \). Now let \( x \in X \). Then \( \mu_{C_pX}(\gamma)(\eta_X(x)) = \mu_{C_pX}(\gamma)(\hat{x}) = \gamma(\hat{x}) \). On the other hand, \( h_{C_pX} \circ MH_{C_pX}(\gamma) = C_p\eta_X(\gamma) = C_p(M\eta_X \circ \eta_X)(\gamma) = \gamma \circ M\eta_X \circ \eta_X \). Let \( x \in X \). Then \( M\eta_X \circ \eta_X(x) = M\eta_X(\hat{x}) = \hat{x} \circ C_p\eta_X(\hat{x}) = \hat{x} \). Therefore \( h_{C_pX} \circ \mu_{C_pX} = h_{C_pX} \circ MH_{C_pX} \). Hence \( C_pX \) is an \( M \)-algebra.

**Proposition 2.6.** Retracts of \( C_pX \) are \( M \)-algebras.

**Proof.** Let \( g : C_pY \to X \) be a retraction. Then there is a continuous function \( f : X \to C_pY \) such that \( g \circ f = \text{id}_X \). The following diagram will help us define the algebra map \( h_X : MX \to X \):

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & MX \xrightarrow{\text{id}_M} MC_pY \xrightarrow{C_p}\eta_Y \xrightarrow{C_p \eta_Y} C_pY \\
\downarrow{\text{id}_X} & & \downarrow{M} \\
X & \xrightarrow{g \circ \eta_X} & C_pX \xrightarrow{g \circ \mu_Y} C_pY \\
X & \xrightarrow{h_X} & X
\end{array}
\]

(2.4)

Define

\[
h_X = g \circ C_p \eta \circ Mf = g \circ C_p(f \circ \eta_Y).
\]

(2.5)

Since \( h_X \) is the composite of continuous functions, then it is continuous.
Now,
\[ h_X \circ \eta_X(x) = h_X(\hat{x}) = g(C(f \circ \eta_Y)(\hat{x})) = g(\hat{x} \circ C \circ \eta_Y) = g(f(\hat{x}) \circ \eta_Y) = g(f(x)) = \text{id}_X(x), \] (2.6)
since \( g \) is a retraction.

We now show that \( h_X \circ \mu_X = h_X \circ Mh_X \).

Let \( \gamma \in MMX \). Then
\[ h_X \circ \mu_X(\gamma) = h_X(\mu_X(\gamma)) = g(C_p(C_p f \circ \eta_Y)(\mu_X(\gamma))) = g(C_p(C_p f \circ \eta_Y)(\mu_X(y))) = g(\mu_X(y) \circ C_p f \circ \eta_Y). \] (2.7)

If \( k \in C_pX \), then
\[ \mu_X(y)(k) = \gamma(\hat{k}). \] (2.8)

On the other hand,
\[ h_X \circ Mh_X = g(C_p(C_p f \circ \eta_Y) \circ M(g(C_p(C_p f \circ \eta_Y))) = g(C_p f \circ \eta_Y) \circ M(g(C_p f \circ \eta_Y)) = g(C_p f \circ \eta_Y) \circ M(id_X) \circ M(C_p f \circ \eta_Y) = g(C_p f \circ \eta_Y) \circ M(C_p f \circ \eta_Y) = g(C_p f \circ \eta_Y). \] (2.9)

Now,
\[ h_X \circ Mh_X(y) = g(C_p(\eta_{C_p X} \circ C_p f \circ \eta_Y)(y)) = g(y \circ \eta_{C_p X} \circ C_p f \circ \eta_Y). \] (2.10)

We only need to show that \( y \circ \eta_{C_p X} = \mu_X \). Let \( k \in C_pX \). Then \( y \circ \eta_{C_p X}(k) = y(\eta_{C_p X}(k)) = y(\hat{k}) \). From (2.8), we have \( y \circ \eta_{C_p X} = \mu_X \) and therefore \( h_X \circ \mu_X = h_X \circ Mh_X \). Hence retracts of \( C_pX \) are \( M \)-algebras.

3. The algebra morphisms and the transfer of ring structure from \( MX \) to \( X \) for an \( M \)-algebra \( (X, h_X) \). For an \( M \)-algebra \( (X, h_X) \) the ring structure on \( MX \) can be transferred to \( X \), via \( h_X \), in such a way that \( X \) becomes a ring with respect to the induced operations.

**Definition 3.1.** On an \( M \)-algebra \( (X, h_X) \) define
(i) \( x_1 + x_2 \) to be \( h_X(\eta_X(x_1) + \eta_X(x_2)) \),
(ii) \( x_1 \cdot x_2 \) to be \( h_X(\eta_X(x_1) \cdot \eta_X(x_2)) \).

In addition to the ring structure defined above we also define the scalar multiplication in the following way: define \( tX \) to be \( h_X(t\eta_X(x)) \), where \( t \) is a scalar.

According to **Definition 3.1**, \( C_pX \) (being an \( M \)-algebra, **Proposition 2.5**) has now two concepts of the operations “+” and “.”, the natural one defined pointwise
\[ (h_X(x + y) = h_X(x) + h_X(y), h_X(xy) = h_X(x)h_X(y)) \] (3.1)
and Definition 3.1. The same applies to $MX$. We omit the straightforward proof of the following proposition.

**Proposition 3.2.** The natural operations on $MX$ defined pointwise coincide with the corresponding ones defined above.

**Lemma 3.3.** The topology on $X$ is initial with respect to $\eta_X$, that is, $X$ has the weak topology induced by $\eta_X$ into $C_pC_pX = MX$.

**Proof.** Basic neighborhoods of $\eta_X(x)$ have inverse images of $\eta_X$ of the form $\cap_{i=1}^{n}f_i^{-1}[W_i]$. \hfill $\Box$

**Lemma 3.4** [2]. Let $\varphi \in C_pC_pX$ such that $\varphi : (C_pX,h_{C_pX}) \to (\mathbb{R},h_{\mathbb{R}})$ is a linear functional. Then there are $x_1,\ldots,x_n \in X$, $1 \leq f_i \leq 1$. Let $\lambda_i = \varphi(g_i)$, $g_i \in C_pX$ being such that $g_i(x_i) = 1$, $g_i(x_j) = 0$ for $i \neq j$, $0 \leq g_i \leq 1$. Now $\varphi(g_i^2) = \varphi(g_i)^2 = \lambda_i$. Also $\varphi(g_i^2) = \sum_{i=1}^{n} \lambda_i \hat{x}_i(g_i^2) = \sum_{i=1}^{n} \lambda_i g_i^2(x_i) = \lambda_k$. Thus $\lambda_k = \lambda_k^2$, so that $\lambda_k = 0$ or $\lambda_k = 1$ for $k = 1,2,\ldots,n$. Moreover, $\lambda_k = g_k(x_k) \geq 0$. Furthermore, $\varphi(1) = 1$ gives $1 = \varphi(1) = \sum_{i=1}^{n} \lambda_i \hat{x}_i(1) = \sum_{i=1}^{n} \lambda_i$. Consequently, all $\lambda_i$’s except one are zero, the exceptional one being one 1. Let $x = x_l$, where $\lambda_l = 1$. Then $\lambda_i = 0$ for $i \neq l$, so that $\varphi = \lambda_l \hat{x}_l = \hat{x}_l$. \hfill $\Box$

**Proposition 3.5.** If $\varphi : (C_pX,h_{C_pX}) \to (\mathbb{R},h_{\mathbb{R}})$ is a nontrivial continuous multiplicative linear functional, then there is $x \in X$ such that $\varphi = \hat{x}$, that is, $\varphi$ is a point evaluation.

**Proof.** By Lemma 3.4, there are points $x_1,\ldots,x_n \in X$, and scalars $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$ such that $\varphi = \sum_{i=1}^{n} \lambda_i \hat{x}_i$ where $\lambda_i = \varphi(g_i)$, $g_i \in C_pX$ being such that $g_i(x_i) = 1$, $g_i(x_j) = 0$ for $i \neq j$, $0 \leq g_i \leq 1$. Now $\varphi(g_i^2) = \varphi(g_i)^2 = \lambda_i$. Also $\varphi(g_i^2) = \sum_{i=1}^{n} \lambda_i \hat{x}_i(g_i^2) = \sum_{i=1}^{n} \lambda_i g_i^2(x_i) = \lambda_k$. Thus $\lambda_k = \lambda_k^2$, so that $\lambda_k = 0$ or $\lambda_k = 1$ for $k = 1,2,\ldots,n$. Moreover, $\lambda_k = g_k(x_k) \geq 0$. Furthermore, $\varphi(1) = 1$ gives $1 = \varphi(1) = \sum_{i=1}^{n} \lambda_i \hat{x}_i(1) = \sum_{i=1}^{n} \lambda_i$. Consequently, all $\lambda_i$’s except one are zero, the exceptional one being one 1. Let $x = x_l$, where $\lambda_l = 1$. Then $\lambda_i = 0$ for $i \neq l$, so that $\varphi = \lambda_l \hat{x}_l = \hat{x}_l$. \hfill $\Box$

**Proposition 3.6.** Let $\varphi : (C_pX,h_{C_pX}) \to (\mathbb{R},h_{\mathbb{R}})$ be an algebra map. Then $\varphi$ is a continuous ring homomorphism.

**Proof.** Given $f,g \in C_pX$, consider $\eta_{C_pX}(f) + \eta_{C_pX}(g)$ in $MC_pX$. We have

$$h_{\mathbb{R}} \circ C^2 \varphi(\eta_{C_pX}(f) + \eta_{C_pX}(g)) = h_{\mathbb{R}} \circ M \varphi(\eta_{C_pX}(f)) + h_{\mathbb{R}} \circ M \varphi(\eta_{C_pX}(g))$$

(3.2)

by Lemma 3.4.

Hence $\varphi \circ h_X(\eta_{C_pX}(f) + \eta_{C_pX}(g)) = \varphi \circ h_X(\eta_{C_pX}(f)) + \varphi \circ h_X(\eta_{C_pX}(g))$, so that $\varphi(f + g) = \varphi(f) + \varphi(g)$, since $h_X$ preserves the ring structure. Similarly, $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$. We also have $\varphi(t \cdot f) = t \cdot \varphi(f)$, $t \in \mathbb{R}$. Moreover $\varphi(1) = 1$, where 1 denotes the constant function with value equal to 1. \hfill $\Box$

**Proposition 3.7.** Every algebra map $\varphi : (C_pX,h_{C_pX}) \to (\mathbb{R},h_{\mathbb{R}})$ is a point evaluation map.

**Proof.** By the above proposition, $\varphi$ is a continuous ring homomorphism, that is, a continuous multiplicative linear functional. Thus, there is some $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in C_pX$, by the above proposition. \hfill $\Box$

**Theorem 3.8.** The algebra morphisms $\varphi : (C_pX,h_{C_pX}) \to (\mathbb{R},h_{\mathbb{R}})$ are precisely the morphisms $\hat{x}$, where $x \in X$, that is, the point evaluation map.
PROOF. Suppose \( \varphi = \hat{x} \), for some \( x \in X \). Let \( y \in MC_pX \). Take \( h_X = C_p\eta_{CPX} \) and \( h_\mathbb{R} = 1_\mathbb{R} \). Then \( \varphi \circ h_X(y) = \hat{x} \circ C_p\eta_{CPX}(y) = \hat{x}(C_p\eta_{CPX}(y)) = y \circ \eta_X(x) = y(\hat{x}) \). On the other hand, \( h_\mathbb{R} \circ M(\varphi)(y) = h_\mathbb{R}(M(\varphi)(y)) = \hat{1}_\mathbb{R}(M(\varphi)(y)) = M(\varphi)(y)(1_\mathbb{R}) = y(1_\mathbb{R} \circ \varphi) = y(\varphi) = y(\hat{x}) \). Therefore, \( \varphi \circ h_X = h_\mathbb{R} \circ M(\varphi) \) and thus \( \varphi = \hat{x} \) is an algebra morphism. The converse follows from Proposition 3.7.

PROPOSITION 3.9. The algebra morphisms \( \varphi : (C_pX, h_{CPX}) \rightarrow (C_pY, h_{CPY}) \) are the maps \( C_p(f) \), where \( f : Y \rightarrow X \) is continuous.

PROOF. Suppose that \( \varphi : (C_pX, h_{CPX}) \rightarrow (C_pY, h_{CPY}) \) is an algebra map. Given \( y \in Y \), \( \hat{y} \circ \varphi : (C_pX, h_{CPX}) \rightarrow (\mathbb{R}, h_\mathbb{R}) \) is an algebra map, since the composition of two algebra maps is an algebra map. Thus the following diagram is commutative:

\[
\begin{array}{ccc}
MC_pX & \xrightarrow{M\varphi} & MC_pY \\
\downarrow h_{CPX} & & \downarrow h_{CPY} \\
C_pX & \xrightarrow{\varphi} & C_pY \\
\downarrow y & & \downarrow y \\
& \mathbb{R} & \\
\end{array}
\] (3.3)

By Theorem 3.8, \( \hat{y} \circ \varphi = \hat{x} \) for some \( x \in X \). Put \( x = f(y) \). Thus \( f \) maps \( Y \) into \( X \). Since \( X \) has the initial topology induced by \( \eta_X \), \( f \) will be continuous if \( \eta_X \circ f \) is continuous. Now \( \eta_X \circ f(y) = \hat{x} = \hat{y} \circ \varphi = C_p(\eta_Y(y)) \). Thus \( \eta_X \circ f = C_p(\varphi) \circ \eta_Y \), so that \( \eta_X \circ f \) is continuous, hence \( f \) is continuous, as required. It remains to prove that \( \varphi = C_p(f) \). Since the functions \( \hat{y} \) distinguish the points of \( C_pY \), it suffices to prove that \( \hat{y} \circ \varphi = \hat{y} \circ C_p(f) \) for every \( y \in Y \). Now \( \hat{y}(C_p(f)(y)) = \hat{y}(g \circ f) = g \circ f(y) = g(f(y)) = g(x) \). Also \( \hat{y}(\varphi(g)) = \hat{y} \circ \varphi(g) = \hat{x}(g) = g(x) \). Hence \( \hat{y} \circ \varphi = \hat{y} \circ C_p(f) \) for all \( y \in Y \), so that \( \varphi = C_p(f) \).

Conversely suppose the morphism \( \varphi : (C_pX, h_{CPX}) \rightarrow (C_pY, h_{CPY}) \) is such that \( \varphi = C_p(f) \). Then by Proposition 3.9, \( \varphi \) is an algebra morphism.

PROPOSITION 3.10. The map \( h_{CPX} : MC_pX \rightarrow C_pX \) preserves the ring structure of the function spaces, operations being defined pointwise.

PROOF. Let \( \varphi, \psi \in MC_pX \), so that \( \varphi, \psi : MX \rightarrow \mathbb{R} \). The maps \( \varphi + \psi, \varphi \cdot \psi, \) and \( t \varphi \) (where \( t \in \mathbb{R} \)) are both defined pointwise, so that \( (\varphi + \psi)(\lambda) = \varphi(\lambda) + \psi(\lambda), \varphi \cdot \psi(\lambda) = \varphi(\lambda) \cdot \psi(\lambda), \) and \( (t \varphi)(\lambda) = t \varphi(\lambda) \), for all \( \lambda \in C_pX \). Now \( h_{CPX}(\varphi) = C_{\eta_X}(\varphi) = \varphi \circ \eta_X \), hence \( h_{CPX}(\varphi + \psi) = (\varphi + \psi) \circ \eta_X \). Thus

\[
(\varphi + \psi) \circ \eta_X(x) = (\varphi + \psi)(\eta_X(x)) = \varphi(\eta_X(x)) + \psi(\eta_X(x))
\]

\[
= h_{CPX}(\varphi)(x) + h_{CPX}(\psi)(x) = (h_{CPX}(\varphi) + h_{CPX}(\psi))(x). \tag{3.4}
\]

Since this holds for every \( x \in X \), we have \( h_{CPX}(\varphi + \psi) = h_{CPX}(\varphi) + h_{CPX}(\psi) \).

The proof that \( h_{CPX}(\varphi \cdot \psi) = h_{CPX}(\varphi) \cdot h_{CPX}(\psi) \) is similar. We also have \( h_{CPX}(t \varphi) = th_{CPX}(\varphi) \), where \( t \) is a scalar.

PROPOSITION 3.11. For any \( f : Y \rightarrow X \), the map \( C_p(f) : C_pX \rightarrow C_pY \) preserves the ring structure.
**Proof.** Since $C_p f$ acts by composition on the right, the result is clear. We will verify one case only: $C_p f (\varphi + \psi) = C_p f (\varphi) + C_p f (\psi)$. Then

$$C_p f (\varphi + \psi) (y) = (\varphi + \psi) (f (y)) = \varphi (f (y)) + \psi (f (y)) = C_p f (\varphi) (y) + C_p f (\psi) (y) = (C_p f (\varphi) + C_p f (\psi)) (y).$$

(3.5)

Since the equality holds for every $y \in Y$, $C_p f (\varphi + \psi) = C_p f (\varphi) + C_p f (\psi)$. □

**Problem 3.12.** Characterize fully the Eilenberg-Moore category of $M$-algebras.

**References**


KOENA R. NAILANA: DEPARTMENT OF MATHEMATICS, APPLIED MATHEMATICS AND ASTRONOMY, UNIVERSITY OF SOUTH AFRICA, P.O. BOX 392, PRETORIA 0003, SOUTH AFRICA

E-mail address: nailakr@unisa.ac.za
Submit your manuscripts at http://www.hindawi.com