GRADED RADICAL $W$ TYPE LIE ALGEBRAS I

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We get a new $\mathbb{Z}$-graded Witt type simple Lie algebra using a generalized polynomial ring which is the radical extension of the polynomial ring $F[x]$ with the exponential function $e^x$.

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1. Introduction. Let $F$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbb{Z}_+$ and $\mathbb{Z}$ denote the nonnegative integers and the integers, respectively. Let $F[x]$ be the polynomial ring in indeterminate $x$. Let $F(x) = \{ f(x)/g(x) \mid f(x), g(x) \in F[x], g(x) \neq 0 \}$ be the field of rational functions in one variable. We define the $F$-algebra $V^{\infty}_{m,e}$ spanned by

\[ \left\{ e^x f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \mid d, a_1, \ldots, a_m, t \in \mathbb{Z}, f_i \neq x, \right. \]
\[ \left. (a_1, b_1) = 1, \ldots, (a_m, b_m) = 1, 1 \leq i \leq m \right\}, \tag{1.1} \]

where $b_1, \ldots, b_m$ are fixed nonnegative integers, and $(a_i, b_i) = 1, 1 \leq i \leq m$, means that $a_i$ and $b_i$ are relatively primes, and $f_1, \ldots, f_n$ are the fixed relatively prime polynomials in $F[x]$. The $F$-subalgebra $V^+_{m,e}$ of $V^{\infty}_{m,e}$ is spanned by

\[ \left\{ e^x f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \mid d, a_1, \ldots, a_m \in \mathbb{Z}, t \in \mathbb{Z}_+, f_i \neq x, \right. \]
\[ \left. (a_1, b_1) = 1, \ldots, (a_m, b_m) = 1, 1 \leq i \leq m \right\}. \tag{1.2} \]

Let $W_{\infty_{m,e}}(\partial)$ be the vector space over $F$ with elements $\{ f \partial \mid f \in V^{\infty}_{m,e} \}$ and the standard basis $\{ e^x f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial \mid e^x f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial \in W_{\infty_{m,e}} \}$. Define a Lie bracket on $W_{\infty_{m,e}}(\partial)$ as follows:

\[ [f \partial, g \partial] = f(\partial(g)) \partial - g(\partial(f)) \partial, \quad f, g \in V_{\infty_{m,e}}. \tag{1.3} \]

It is easy to check that (1.3) defines a Lie algebra $W_{\infty_{m,e}}(\partial)$ with the underlying vector space $W_{\infty_{m,e}}(\partial)$ (see also [1, 3, 5]). Similarly, we define the Lie subalgebra $W^+_{\infty_{m,e}}(\partial)$ of $W_{\infty_{m,e}}(\partial)$ using the F-algebra $V^+_{m,e}$ instead of $V_{\infty_{m,e}}$.

The Lie algebra $W_{\infty_{m,e}}(\partial)$ has a natural $\mathbb{Z}$-gradation as follows:

\[ W_{\infty_{m,e}}(\partial) = \bigoplus_{d \in \mathbb{Z}} W^d_{\infty_{m,e}}, \tag{1.4} \]

where $W^d_{\infty_{m,e}}$ is the subspace of the Lie algebra $W_{\infty_{m,e}}(\partial)$ generated by elements of the form $e^x f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial \mid f_1, \ldots, f_n \in F[x], a_1, \ldots, a_m, t \in \mathbb{Z}, m \in \mathbb{Z}_+$}. We call the subspace $W^d_{\infty_{m,e}}$ the $d$-homogeneous component of $W_{\infty_{m,e}}(\partial)$. 
We decompose the \(d\)-homogeneous component \(W^d_{\sqrt{m},e}\) as follows:

\[
W^d_{\sqrt{m},e} = \bigoplus_{s_1, \ldots, s_m \in \mathbb{Z}} W_{(d,s_1, \ldots, s_m)},
\]

where \(W_{(d,s_1, \ldots, s_m)}\) is the subspace of \(W^d_{\sqrt{m},e}\) spanned by

\[
\{ e^{dx} f_1^{s_1/b_1} \cdots f_m^{s_m/b_m} x^q \partial \mid q \in \mathbb{Z} \}.
\]

Note that \(W_{(0,0, \ldots, 0)}\) is the Witt algebra \(W(1)\) as defined in [3].

The two radical-homogeneous components \(W_{(d,a_1, \ldots, a_m)}\) and \(W_{(d,r_1, \ldots, r_m)}\) are equivalent if \(a_1 - r_1, \ldots, a_m - r_m \in \mathbb{Z}\). This defines an equivalence relation on \(W^d_{\sqrt{m},e}\). Thus we note that the equivalent class of \(W_{(d,a_1, \ldots, a_m)}\) without ambiguity. It is possible to choose the minimal positive integers \(a_1, \ldots, a_m\) for the radical homogeneous equivalent component \(W_{(d,a_1, \ldots, a_m)}\).

We give the lexicographic order on all the radical homogeneous equivalent components \(W_{(d,a_1, \ldots, a_m)}\) using \(\mathbb{Z} \times \mathbb{Z}^m_+\).

The radical equivalent homogeneous component \(W^d_{\sqrt{m},e}\) can be written as follows:

\[
W^d_{\sqrt{m},e} = \sum_{(d,a_1, \ldots, a_m) \in \mathbb{Z}^m_+} W_{(d,a_1, \ldots, a_m)}.
\]

Thus for any element \(l \in W_{\sqrt{m},e}(\partial)\), \(l\) can be written uniquely as follows:

\[
l = \sum_{(d,a_1, \ldots, a_m) \in \mathbb{Z} \times \mathbb{Z}^m_+} l_{(d,a_1, \ldots, a_m)}.
\]

For any such element \(l \in W_{\sqrt{m},e}(\partial)\), \(H(l)\) is defined as the number of different homogeneous components of \(l\) as in (1.4), and \(L_d(l)\) as the number of nonequivalent radical \(d\)-homogeneous components of \(l\) in (1.8). For each basis element \(e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial\) of \(W_{\sqrt{m},e}(\partial)\) (or \(W^+_{\sqrt{m},e}(\partial)\)), define \(\deg_{Lie}(e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial) = t\). Since every element \(l\) of \(W_{\sqrt{m},e}(\partial)\) is the sum of the standard basis element, we may define \(\deg_{Lie}(l)\) as the highest power of each basis element of \(l\). Note that the Lie algebra \(W_{\sqrt{m},e}(\partial)\) is self-centralized, that is, the centralizer \(C_l(W_{\sqrt{m},e}(\partial))\) of every element \(l\) in \(W_{\sqrt{m},e}(\partial)\) is one dimensional [1]. We find the solution of

\[
1^{1/3} = y
\]

in \(\mathbb{Z}_7\). Equation (1.9) implies that

\[
1 \equiv y^3 \mod 7.
\]

The solutions of (1.10) are 1, 2, or 4. Thus \(1^{1/3} = 1, 2, \) or \(4 \mod 7\). Thus the radical number in \(\mathbb{Z}_p\) is not uniquely determined generally. So we may not consider the Lie algebras in this paper over a field of characteristic \(p\) differently from the Lie algebras in [2, 3, 4]. It is easy to prove that the Lie algebra \(W_{(0, 0, \ldots, 0)}\) is simple [3].
2. Main results. We need several lemmas for Theorem 2.5.

**Lemma 2.1.** For any element \( l \) in the \((d,a_1,\ldots,a_m)\)-radical-homogeneous component of \( W_{\sqrt{n},e}(\partial) \), and for any element \( l_1 \in W_{(0,0,\ldots,0)} \), \([l,l_1]\) is an element in the \((d,a_1,\ldots,a_m)\)-radical homogeneous equivalent component.

The proof of **Lemma 2.1** is straightforward.

**Lemma 2.2.** A Lie ideal \( I \) of \( W_{\sqrt{n},e}(\partial) \) which contains \( \partial \) is \( W_{\sqrt{n},e}(\partial) \).

**Proof.** Let \( I \) be the ideal in the lemma. The Lie subalgebra which has the standard basis \( \{x_i\partial\mid i \in \mathbb{Z}_+\} \) is simple. Let \( I \) be any ideal of \( W_{\sqrt{n},e}(\partial) \) which contains \( \partial \). Then for any \( f\partial \in W_{\sqrt{n},e}(\partial) \),
\[
[x\partial,f\partial] = x\partial(f)\partial - f\partial \in I. \quad (2.1)
\]
On the other hand,
\[
[\partial,xf\partial] = f\partial + x\partial(f)\partial \in I. \quad (2.2)
\]
Thus by subtracting (2.2) from (2.1) we get \( 2f\partial \in I \). Therefore, we have proven the lemma, since \( I \cap W_{(0,0,\ldots,0)} \) contains nonzero elements and so \( I \supset W_{(0,0,\ldots,0)} \).

**Lemma 2.3.** A Lie ideal \( I \) of \( W_{\sqrt{n},e}(\partial) \) which contains a nonzero element in \( W_{(d,a_1,\ldots,a_m)} \) is \( W_{\sqrt{n},e}(\partial) \), for a fixed \((d,a_1,\ldots,a_m) \in \mathbb{Z} \times \mathbb{Z}_+ \).

**Proof.** Let \( I \) be a Lie ideal of \( W_{\sqrt{n},e}(\partial) \) and \( l \) a nonzero element in the ideal \( I \). Then we take an element \( l_1 = e^{-dx}f_1^{-a_1/b_1} \cdots f_m^{-a_m/b_m}x^p\partial \) with \( p \) a sufficiently large positive integer such that \([l,l_1] \neq 0\). Then \([f\partial,[l,l_1]]\) is a nonzero element in \( W_{(0,0,\ldots,0)} \) by taking an element \( f_{1}^{t_1} \cdots f_{m}^{t_m} \in F[x] \), where \( t_1,\ldots,t_m \) are sufficiently large integers. Thus \( I \cap W_{(0,0,\ldots,0)} \) contains nonzero elements, and hence, \( \partial \in I \cap W_{(0,0,\ldots,0)} \) by simplicity of \( W_{(0,0,\ldots,0)} \). Then the lemma follows from **Lemma 2.2**.

Throughout this paper, \( a \gg b \) means that \( a \) is a number sufficiently larger than \( b \).

**Lemma 2.4.** Let \( I \) be any nonzero Lie ideal of \( W_{\sqrt{n},e}(\partial) \). For any nonzero element \( l \in I \), there is an element \( x^s\partial, s \gg 0 \), such that \([x^s\partial,l] \) is the sum of elements in \( W_{\sqrt{n},e}(\partial) \) with \( \deg_{\text{Lie}}([x^s\partial,l]) > 0 \).

**Proof.** It is straightforward by choosing a sufficiently large positive integer \( s \).

**Theorem 2.5.** The Lie algebra \( W_{\sqrt{n},e}(\partial) \) is simple.

**Proof.** Let \( I \) be a nonzero Lie ideal of \( W_{\sqrt{n},e}(\partial) \). Let \( l \) be a nonzero element of \( I \). By **Lemma 2.4**, we may assume that \( l \) has polynomial terms with positive powers for each basis element of \( l \). We prove this theorem in several steps.

**Step 1.** If \( l \) is in the 0-homogeneous component, then the theorem holds. We prove this step, by induction on the number \( L_0(l) \) of nonequivalent radical-homogeneous components of the element \( l \) of \( I \). If \( L_0(l) \) is 1 and \( l \in W_{(0,0,\ldots,0)} \), then the theorem holds by Lemmas 2.2, 2.3, and the fact that \( W_{(0,0,\ldots,0)} \) is simple.
Assume that \( l \in \mathcal{W}^{0,0,...,0}(0) \) with \( a_r \neq 0 \). If we take an element \( f_{1}^{h_{r}/k_{r}} \cdots f_{n}^{h_{n}/k_{n}}x^{h_{n+1}} \partial \) such that \( h_{r} \gg k_{r}, \ldots, h_{n} \gg k_{r} \) and \((h_{r} + k_{r})/k_{r} \in \mathbb{Z}_{+}, \ldots, (h_{m} + k_{m})/k_{m} \in \mathbb{Z}_{+}\), then we have \( l_{1} = [f_{1}^{h_{r}/k_{r}} \cdots f_{n}^{h_{n}/k_{n}}x^{h_{n+1}} \partial, I] \neq 0 \). This implies that \( l_{1} \) is in \( W(0,0,...,0) \). Thus we have proven the theorem by Lemma 2.2.

By induction, we may assume that the theorem holds for \( l \in I \) such that \( L_{0}(l) = k \), for some fixed nonnegative integer \( k \geq 1 \). Assume that \( L_{0}(l) = k + 1 \). If \( l \) has a \( W(0,0,...,0) \) radical-homogeneous equivalent component, we take \( l_{2} \in \mathcal{W}(0,0,...,0) \) such that \([l,l_{2}]\) can be written as follows: \([l,l_{2}] = l_{3} + l_{4} \) where \( l_{3} \) is a sum of nonzero radical-homogeneous components, and \( l_{4} = f \partial \) with \( f \in F[x] \). Thus we have the nonzero element
\[
\partial,[\cdots,[\partial,l] \cdots] = l_{2} \in I
\] (2.3)

which has no terms in the homogeneous equivalent component \( \mathcal{W}(0,0,...,0) \), where we applied Lie brackets until \( l_{2} \) has no terms in the radical homogeneous equivalent component \( \mathcal{W}(0,0,...,0) \). Then \( l_{2} \in I \) such that \( H(l_{2}) \leq k \). Therefore, we have proven the theorem by Lemmas 2.2, 2.3, and induction. If \( l \) has no terms in the radical homogeneous equivalent component \( (0,0,...,0) \), then \( l \) has a term in the radical homogeneous equivalent component \( \mathcal{W}(0,a_{1},...,a_{n}) \). Take an element \( l_{3} = f_{1}^{c_{1}/p_{1}} \cdots f_{m}^{c_{m}/p_{m}}x^{c_{m+1}} \partial \) such that \( c_{1}, \ldots, c_{m+1} \) are sufficiently large positive integers such that \( c_{1} + a_{1} \in \mathbb{Z} \cdots c_{m} + a_{m} \in \mathbb{Z} \), and which is in a radical homogeneous equivalent component \( \mathcal{W}(0,a_{1},...,a_{n}) \). Then \([l_{3},l]\) is nonzero and which has a term in the radical homogeneous equivalent component \( \mathcal{W}(0,0,...,0) \). So in this case we have proven the theorem by induction.

**Step 2.** Assume that \( l \) is in the \( d \)-homogeneous component such that \( 0 \neq d \) and \( L_{0}(l) = 1 \), then the theorem holds. By taking \( e^{-dx}x^{t} \partial \), we have \( 0 \neq [e^{-dx}x^{t} \partial, l] \in \mathcal{W}(0,0,...,0) \) by taking a sufficiently large positive integer \( t \). Thus we have proven the theorem by Step 1.

**Step 3.** If \( l \) is the sum of \( (k-1) \) nonzero homogeneous components and 0-homogeneous component, then the theorem holds. We prove the theorem by induction on the number of distinct homogeneous components by Steps 1 and 2. Assume that we have proven the theorem when \( l \) has \( (k-1) \) radical-homogeneous components. Assume that \( l \) has \( k \) terms in \( \mathcal{W}(0,0,...,0) \). By Step 1, we have an element \( l_{1} \in I \), such that \( l_{1} = l_{2} + f \partial \), where \( l_{2} \) has \( (k-1) \) homogeneous components and \( f \in F[x] \). Then 
\[ 0 \neq \partial,[\cdots,[\partial,l_{1}] \cdots] \in I \] has \( (k-1) \) homogeneous components, where we applied the Lie bracket until it has no terms in \( \mathcal{W}(0,0,...,0) \). Therefore, we have proven the theorem by induction.

Assume that \( l \) has \( k \) homogeneous equivalent components. We may assume \( l \) has the terms which is in \( 0 \neq d \)-homogeneous component. By taking a sufficiently large positive integer \( r \), we have \([e^{-dx}x^{r} \partial, l] \neq 0 \) and it has \( k \) homogeneous components with a term in the radical-homogeneous component \( \mathcal{W}(0,0,...,0) \). Therefore, we have proven the theorem by Step 3. □

**Corollary 2.6.** The Lie algebra \( \mathcal{W}_{\frac{1}{2},e}^{0} (\partial) \) is simple.

**Proof.** It is straightforward from Theorem 2.5 without using Lemma 2.4. □

**Corollary 2.7.** The Lie subalgebra \( \mathcal{W}_{\frac{1}{2},e}^{0} \) of \( \mathcal{W}_{\frac{1}{2},e}^{0} (\partial) \) is simple.
Proof. It is straightforward from Step 1 of Theorem 2.5.

Proposition 2.8. For any nonzero Lie automorphism $\theta$ of $W_{\sqrt{m,e}}^+(\partial)$, $\theta(\partial) = \partial$ holds.

Proof. It is straightforward from the relation $\theta([\partial, x\partial]) = \theta(\partial)$ and the fact that $W_{\sqrt{m,e}}^+(\partial)$ is self-centralized and $\mathbb{Z}$-graded.

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References


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