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ON THE EXISTENCE OF BOUNDED SOLUTIONS OF NONLINEAR ELLIPTIC SYSTEMS

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We study the existence of bounded solutions to the elliptic system $-\Delta_p u = f(u,v)+h_1$ in Ω, $-\Delta_q v = g(u,v)+h_2$ in Ω, $u=v=0$ on $\partial\Omega$, non-necessarily potential systems. The method used is a shooting technique. We are concerned with the existence of a negative subsolution and a nonnegative supersolution in the sense of Hernandez; then we construct some compact operator T and some invariant set K where we can use the Leray Schauder's theorem.

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1. Introduction. The aim of this paper is to study the existence of solutions for the following system:

$$\begin{gathered} -\Delta_p u = f(u,v)+h_1, \quad -\Delta_q v = g(u,v)+h_2 \quad \text{in } \Omega, \\ u=v=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, where $N\geq 1$, $p,q>1$, f,g are continuous functions of $\mathbb{R}^2$ into $\mathbb{R}$ and h_1,h_2 are the functions given in $L^{+\infty}(\Omega)$.

System (1.1) results from the study of the nonlinear phenomena, such as the evolution of population, of chemical reaction, and so forth. A great attention was given to the existence of the solutions for a system of the (1.1) type, by using various approaches (cf. [3, 4, 5, 7, 13]). When the system has a variational structure, the existence of the solutions for (1.1) can be established by means of the variational approaches under adapted conditions (cf. [9, 13]). When (1.1) does not have a variational structure, as in Vélin and de Thélin [13], where the authors obtained some results for the existence of solutions to problem (1.1) with the following growth conditions of nonlinearity f and g:

$$\begin{aligned} |f(u,v)| &\leq a_1|u|^{\alpha_0}|v|^{\beta_0+1}+a_2|u|^{\alpha_1-1}+a_3|v|^{\beta_1-1}, \\ |g(u,v)| &\leq a_4|u|^{\alpha_0+1}|v|^{\beta_0}+a_5|u|^{\alpha_2-1}+a_6|v|^{\beta_2-1}, \end{aligned} \tag{1.2}$$

where a_i $(i=1,\dots,6)$ are positive constants and α_i and β_i $(i=0,1,2)$ satisfy

$$\begin{gathered} \frac{\alpha_0+1}{p}+\frac{\beta_0+1}{q}<1, \\ 1<\alpha_1<p; \qquad 0<\beta_1-1<\frac{q}{p^*}, \\ 1<\alpha_2-1<\frac{p}{q^*}; \qquad 0<\beta_2<q. \end{gathered} \tag{1.3}$$

Always in the case of a system, we can notice the existence results obtained in Baoyao [2], and Brézis and Lieb [4].

The case of a scalar equation has been studied by many authors, see de Figueiredo and Gossez [6], Fernandes et al. [10] and Fonda et al. [11]. More recently, some interesting results have been obtained by Gossez and El Hachimi [12] and Anane and Chakrone [1]. Those authors derived the solvability of the following problem:

$$\begin{aligned} -\Delta_m u &= f(u)+h \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

under the following condition:

$$\liminf_{u\to\infty} \frac{pF(u)}{u^m} < \mu'_m, \tag{1.5}$$

where

$$\mu'_m = (m-1)\left[\frac{2}{R(\Omega)}\int_0^1 \frac{ds}{\sqrt[m]{1-s^m}}\right]^m, \tag{1.6}$$

and $R(\Omega)$ denotes the radius of the smallest open ball $B(0,R)$ containing Ω. The particular cases $N=1$ and $m=2$ were considered in [10]. It was shown there that (1.4) is solvable for any $h\in L^\infty(\Omega)$ if

$$\liminf_{u\to\infty} \frac{2F(u)}{u^2} < \lambda_{1,2}, \tag{1.7}$$

where $\lambda_{1,2}$ is the first eigenvalue of $-\Delta$ and $\Omega =]a,b[$. Observe that for $N>1$, we have $\mu'_2 < \lambda_{1,2}(\Omega)$. Then, the question naturally arises whether μ'_m can be replaced by $\lambda_{1,m}(\Omega)$ in (1.5), where $\lambda_{1,m}$ is the first eigenvalue of $-\Delta_m$. This problem remains open.

The goal of this paper is to show that the same approach in [12] can be applied for some quasilinear elliptic systems with the constants μ_p and μ_q, defined below, associated, respectively, with the operators $-\Delta_p$ and $-\Delta_q$, and where μ_m $(m=p,q)$, better than μ'_m, is presented in (1.5). In this case, we treat the question of the existence of the solutions for system (1.1) without imposing variational structures, which is often the case for system (1.1) and without necessarily the growth conditions for f and g.

2. Main result. We make the following assumptions:

(H_1) (i) The function $f(u,\cdot)$ is a nonincreasing function on $\mathbb{R}$ for all u in $\mathbb{R}$,

(ii) The function $g(\cdot,v)$ is a nonincreasing function on $\mathbb{R}$ for all v in $\mathbb{R}$.

(H_2) There exists some unbounded increasing subsequence $(m_k)_k$, satisfying

$$\lim_{k\to+\infty} \frac{pF\left(m_k^{1/p}, m_k^{1/q}\right)}{m_k} < \mu_p, \qquad \lim_{k\to+\infty} \frac{qG\left(m_k^{1/p}, m_k^{1/q}\right)}{m_k} < \mu_q, \tag{2.1}$$

$$\lim_{k\to+\infty} \frac{pF\left(-m_k^{1/p}, -m_k^{1/q}\right)}{m_k} < \mu_p, \qquad \lim_{k\to+\infty} \frac{qG\left(-m_k^{1/p}, -m_k^{1/q}\right)}{m_k} < \mu_q, \tag{2.2}$$

where F and G are the following functions:

$$F(u,v)=\int_0^u f(s,v)ds, \qquad G(u,v)=\int_0^v g(u,t)dt, \tag{2.3}$$

and where we denote by μ_p and μ_q the following constants:

$$\mu_p=(p-1)\left[\frac{2}{b-a}\int_0^1 \frac{ds}{\sqrt[p]{1-s^p}}\right]^p, \qquad \mu_q=(q-1)\left[\frac{2}{b-a}\int_0^1 \frac{dt}{\sqrt[q]{1-t^q}}\right]^q, \tag{2.4}$$

with $b-a=\min(b_i-a_i)$ and $P=\Pi[a_i,b_i]$ is the smallest cube such that $P\supset\Omega$.

Observe that for $N=1$, μ_p and μ_q are, respectively, the first eigenvalue of $-\Delta_p$ and $-\Delta_q$ when $\Omega=]a,b[$. It is clear that μ_p is better than μ_p' defined in (1.5). In particular, it is interesting when Ω is a rectangle or a triangle, because $\mu_p\gg\mu_p'$ and $\mu_p\approx\lambda_{1,p}(\Omega)$.

The main result of this paper is the following statement.

Theorem 2.1. *Under hypotheses (H_1) and (H_2). Problem (1.1) has a solution (u,v) in $(W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega))\cap(L^{+\infty}(\Omega)\times L^{+\infty}(\Omega))$ for any (h_1,h_2) in $L^{+\infty}(\Omega)\times L^{+\infty}(\Omega)$.*

Example 2.2. Consider

$$f(u,v)=a(x)u|u|^{\alpha-1}|v|^{\beta+1}, \qquad g(u,v)=b(x)|u|^{\gamma+1}v|v|^{\delta-1}. \tag{2.5}$$

(1) Assume that $\|a\|_\infty<\mu_p$, $\|b\|_\infty<\mu_q$, and

$$\frac{\alpha+1}{p}+\frac{\beta+1}{q}\le 1, \qquad \frac{\gamma+1}{p}+\frac{\delta+1}{q}\le 1. \tag{2.6}$$

Then we conclude the existence of solutions.

(2) If

$$\frac{\alpha+1}{p}+\frac{\beta+1}{q}< 1, \qquad \frac{\gamma+1}{p}+\frac{\delta+1}{q}< 1, \tag{2.7}$$

we have the existence for all (a,b) in $L^{+\infty}(\Omega)$.

The method used in this paper is a shooting technique. In Section 3, we are concerned with the existence of a negative subsolution (u_0,v_0) and a nonnegative supersolution (u^0,v^0) in the sense of Hernandez's definition [13]. In Section 4, we consider some compact operator T and some invariant set K. And, we look for solutions of problem (1.1) as fixed points of the operator T. We will be in the conditions of the Schauder fixed point theorem.

3. Construction of sub-supersolutions

Definition 3.1. A pair $[(u_0,v_0),(u^0,v^0)]$ is a weak sub-supersolution for the Dirichlet problem (1.1), if the following conditions are satisfied:

$$
\begin{gathered}
(u_0,v_0)\in(W^{1,p}(\Omega)\times W^{1,q}(\Omega))\cap(L^{+\infty}(\Omega)\times L^{+\infty}(\Omega)),\\
(u^0,v^0)\in(W^{1,p}(\Omega)\times W^{1,q}(\Omega))\cap(L^{+\infty}(\Omega)\times L^{+\infty}(\Omega)),\\
-\Delta_p u_0-f(x,u_0,v)\le 0\le -\Delta_p u^0-f(x,u^0,v)\quad \text{in } \Omega,\ \forall v\in[v_0,v^0],\\
-\Delta_q v_0-f(x,u,v_0)\le 0\le -\Delta_q v^0-f(x,u,v^0)\quad \text{in } \Omega,\ \forall u\in[u_0,u^0],\\
u_0\le u^0,\quad v_0\le v^0\quad \text{in } \Omega,\qquad u_0\le 0\le u^0,\quad v_0\le 0\le v^0\quad \text{on } \partial\Omega.
\end{gathered}
\tag{3.1}
$$

Similar definitions can be found in Diaz and Hernández [7], and Diaz and Herrero [8]. For all $M>0$, we note that

$$
\begin{aligned}
\hat{f}(u,v)&=f(u,v)+M, & \hat{g}(u,v)&=g(u,v)+M,\\
\hat{F}(u,v)&=F(u,v)+Mu, & \hat{G}(u,v)&=G(u,v)+Mv.
\end{aligned}
\tag{3.2}
$$

Notice that if F and G satisfy the assumption (2.1) of (H_2), then the same holds for $\hat{F}$ and $\hat{G}$.

Proposition 3.2 [6]. *Under hypothesis (2.1) of (H_2), there exist two sequences d_k and d'_k such that*

(a) $m_k^{1/p}\ge d_k\ge 0$, *for all $k\in\mathbb{N}$ and*

$$
\int_0^{d_k}\frac{ds}{\sqrt[p]{p\hat{F}\left(d_k,m_k^{1/q}\right)-p\hat{F}\left(s,m_k^{1/q}\right)}}>\int_0^1\frac{ds}{\sqrt[p]{1-s^p}}[\mu_p]^{-1/p}.
\tag{3.3}
$$

(b) $m_k^{1/p}\ge d'_k\ge 0$, *for all $k\in N$ and*

$$
\int_0^{d'_k}\frac{dt}{\sqrt[q]{q\hat{G}\left(m_k^{1/p},d'_k\right)-q\hat{G}\left(m_k^{1/p},t\right)}}>\int_0^1\frac{dt}{\sqrt[q]{1-t^q}}[\mu_q]^{-1/q}.
\tag{3.4}
$$

Remark 3.3. We have

$$
\sqrt[p]{p-1}\int_0^1\frac{ds}{\sqrt[p]{1-s^p}}[\mu_p]^{-1/p}=\sqrt[q]{q-1}\int_0^1\frac{dt}{\sqrt[q]{1-t^q}}[\mu_q]^{-1/q}=\frac{b-a}{2}.
\tag{3.5}
$$

Proof of Proposition 3.2. We only prove (a); the proof of (b) is similar.

(1) From (2.1) of hypothesis (H_2), there exists some $\mu>0$ such that

$$
\lim_{k\to+\infty}\frac{p\hat{F}\left(m_k^{1/p},m_k^{1/q}\right)}{m_k}<\mu<\mu_p,
\tag{3.6}
$$

then

$$
\lim_{k\to+\infty}\mu m_k-p\hat{F}\left(m_k^{1/p},m_k^{1/q}\right)=+\infty.
\tag{3.7}
$$

(2) We consider the functions $[H(\cdot, m_k)]_k$, where

$$H(s, m_k) = \mu s - p\hat{F}\left(s^{1/p}, m_k^{1/q}\right). \tag{3.8}$$

For all $k > 0$, we have

$$\begin{aligned} H(0, m_k) &= -p\hat{F}\left(0, m_k^{1/q}\right) = 0, \\ H(m_k, m_k) &= \mu m_k - p\hat{F}\left(m_k^{1/p}, m_k^{1/q}\right) > 0. \end{aligned} \tag{3.9}$$

Then for all $k \in \mathbb{N}$ there exists $d_k > 0$ such that $d_k^p \le m_k$ and for all $s \in [0, d_k^p]$, we have

$$H(s, m_k) \le H(d_k^p, m_k), \tag{3.10}$$

that is,

$$\mu s - p\hat{F}\left(s^{1/p}, m_k^{1/q}\right) \le \mu d_k^p - pF\left(d_k, m_k^{1/q}\right), \tag{3.11}$$

then

$$p\hat{F}\left(d_k, m_k^{1/q}\right) - p\hat{F}\left(s^{1/p}, m_k^{1/q}\right) \le \mu(d_k^p - s). \tag{3.12}$$

Let $s = \omega^p$, where $\omega \in [0, d_k] \subset [0, m_k^{1/p}]$. We obtain

$$p\hat{F}\left(d_k, m_k^{1/q}\right) - p\hat{F}\left(\omega, m_k^{1/q}\right) \le \mu(d_k^p - \omega^p), \tag{3.13}$$

that is,

$$\frac{1}{\sqrt[p]{d_k^p - \omega^p}}[\mu]^{-1/p} \le \frac{1}{\sqrt[p]{p\hat{F}\left(d_k, m_k^{1/q}\right) - p\hat{F}\left(\omega, m_k^{1/q}\right)}}. \tag{3.14}$$

Then integrating on $[0, d_k]$ we obtain

$$\int_0^1 \frac{d\omega}{\sqrt[p]{1 - \omega^p}}[\mu]^{-1/p} \le \int_0^{d_k} \frac{d\omega}{\sqrt[p]{p\hat{F}\left(d_k, m_k^{1/q}\right) - p\hat{F}\left(\omega, m_k^{1/q}\right)}}. \tag{3.15}$$

This proves (a). □

3.1. Construction of supersolution (u^0, v^0). In the following step we suppose that for all $k \in \mathbb{N}$ and for all $s \in [0, m_k^{1/p}]$

$$f\left(s, m_k^{1/q}\right) + M \ge 0. \tag{3.16}$$

Denote by $(\hat{f}_k)_k$ the sequence of functions defined by

$$\hat{f}_k(s) = \begin{cases} f\left(m_k^{1/p}, m_k^{1/q}\right) + M & \text{for } s \text{ in } \left[m_k^{1/p}, +\infty\right[, \\ f\left(s, m_k^{1/q}\right) + M & \text{for } s \text{ in } \left[0, m_k^{1/p}\right], \\ f\left(0, m_k^{1/q}\right) + M & \text{for } s \text{ in }]-\infty, 0]. \end{cases} \tag{3.17}$$

For all $k \in \mathbb{N}$, we associate to the function $\hat{f}_k$, the following problem:

$$-\left(|u'|^{p-2}u'\right)'(t) = \hat{f}_k(u(t)), \quad u(t) \geq 0 \text{ for } t \text{ in } [a,b]. \tag{3.18}$$

For all $k \in \mathbb{N}$, we define the nonlinear operator T_k such that

$$T_k : C([a,b]) \longrightarrow C([a,b]) \tag{3.19}$$

in the following way:

$$T_k(u)(t) = d_k - \int_a^t \left[\int_a^r \hat{f}_k(u(s))ds \right]^{1/(p-1)} dr. \tag{3.20}$$

Since $\hat{f}_k$ is a nonnegative function, the operator T_k is well defined.

Lemma 3.4. *For all $k \geq 0$,*
(i) *the operator T_k is completely continuous,*
(ii) *there exists a fixed point for T_k.*

Proof. Let $k \in \mathbb{N}$,
(1) the continuity is immediate,
(2) let $(u_n)_n$ be a bounded sequence in $C([a,b])$ such that the sequence $(T_k(u_n))_n$ is also bounded in $C([a,b])$.

By the continuity of the function $\hat{f}_k$, there exists some constant C_k such that

$$t' \in [a,b], \quad \forall t, \tag{3.21}$$

for all $n \in \mathbb{N}$ we have

$$\left| T_k(u_n)(t) - T_k(u_n)(t') \right| \leq C_k \left| t - t' \right|. \tag{3.22}$$

So $(T_k(u_n))_n$ is uniformly equicontinuous and by Ascoli theorem the sequence $(T_k(u_n))_n$ is relatively compact in $C([a,b])$.

(3) Using the Leray-Schauder theorem we deduce that T_k has a fixed point $u_k \in C([a,b])$, that is, $T_k(u_k) = u_k$. □

Remark 3.5. By definition of the operator T_k, we have
(i) $-|u_k'|^{p-2}u_k'(t) = \int_a^t \hat{f}_k(u_k(s))ds$,
(ii) $u_k'(a) = 0$,
(iii) $u_k(a) = d_k$.

Since $\hat{f}_k$ is a nonnegative function we have
(iv) $u_k'(t) \leq 0$ for t in $[a,b]$.

That is, u_k is a nonincreasing function on $[a,b]$.

Lemma 3.6. *From (2.1), choose $(d_k)_k$ such that*

$$u_k(t) \geq 0 \quad \textit{in} \left[a, \frac{a+b}{2}\right] \forall k \in \mathbb{N}, \tag{3.23}$$

where u_k is the fixed point of the operator T_k.

Proof. Let $(d_k)_k$ be some sequence such that $d_k \in [0, m_k^{1/p}]$ for all $k \in \mathbb{N}$.

We denote by t_k a real number such that $u_k(t_k) = 0$ and $u_k(t) \geq 0$ on $[a, t_k]$.

Then, from Remark 3.5, since u_k is a nonincreasing function and $d_k \in [0, m_k^{1/p}]$, we have

$$m_k^{1/p} \geq u_k \geq 0 \quad \forall t \in [a, t_k]. \tag{3.24}$$

Consequently, for all $t \in [a, t_k]$ we have

$$-\left(|u_k'|^{p-2}u_k'\right)'(t) = \hat{f}_k(u_k(t)) = f\left(u_k(t), m_k^{1/q}\right) + M. \tag{3.25}$$

Multiplying (3.25) by u_k' we obtain

$$\frac{p-1}{p}\left(-|u_k'(t)|^p\right)' = \frac{d}{dt}\left(\hat{F}\left(u_k(t), m_k^{1/q}\right)\right), \tag{3.26}$$

where

$$\hat{F}\left(u, m_k^{1/q}\right) = F(u, v) + Mu. \tag{3.27}$$

Integrating (3.26) on $[a, t] \subset [a, t_k]$, we obtain

$$-\sqrt[p]{p-1}\,u_k'(t) = \sqrt[p]{p\hat{F}\left(d_k, m_k^{1/q}\right) - p\hat{F}\left(u_k(t), m_k^{1/q}\right)}. \tag{3.28}$$

Integrating (3.28) again on $[a, t_k]$ we deduce that

$$\sqrt[p]{p-1}\int_a^{t_k} \frac{-u_k'(t)}{\sqrt[p]{p\hat{F}\left(d_k, m_k^{1/q}\right) - p\hat{F}\left(u_k(t), m_k^{1/q}\right)}}\,dt \leq t_k - a. \tag{3.29}$$

Then, we obtain

$$\sqrt[p]{p-1}\int_0^{d_k} \frac{1}{\sqrt[p]{p\hat{F}\left(d_k, m_k^{1/q}\right) - p\hat{F}\left(s, m_k^{1/q}\right)}}\,ds \leq t_k - a. \tag{3.30}$$

It follows from Proposition 3.2 and Remark 3.3 that one can choose the sequence (d_k) such that for all $k \geq k_0$ we have

$$\frac{b-a}{2} < \sqrt[p]{p-1}\int_0^{d_k} \frac{1}{\sqrt[p]{p\hat{F}\left(d_k, m_k^{1/q}\right) - p\hat{F}\left(s, m_k^{1/q}\right)}}\,ds. \tag{3.31}$$

Consequently, from (3.30) and (3.31), we obtain that for all $k \geq k_0$, there exists t_k satisfying $t_k > (b+a)/2$. □

Proposition 3.7. *Suppose that the sequence $(m_k)_k$ satisfies (2.1), and that for all $k > 0$ we have*

$$\inf_{s \in [0, m_k^{1/p}]} f\left(s, m_k^{1/q}\right) + M \geq 0. \tag{3.32}$$

Then, there exists some number $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the problem

$$\begin{gathered} -(|u'|^{p-2}u')' = f\left(u, m_k^{1/q}\right) + M \quad \text{in } (a,b), \\ u \geq 0 \quad \text{on } [a,b], \end{gathered} \tag{3.33}$$

has a solution $\hat{u}_k$ satisfying $\hat{u}_k \in C^1([a,b])$, $(|\hat{u}_k'|^{p-2}\hat{u}_k')' \in C([a,b])$ and $m_k^{1/p} \geq \hat{u}_k \geq 0$ for all $k \geq k_0$.

Proof. Let $(u_k)_k$ be the sequence defined in Lemma 3.6. This sequence satisfies that for all $k \geq k_0$,

$$\begin{gathered} u_k \in C^1\left(\left[a, \frac{a+b}{2}\right]\right), \qquad \left(|u_k'|^{p-2}u_k'\right)' \in C\left(\left[a, \frac{a+b}{2}\right]\right), \\ -\left(|u_k'|^{p-2}u_k'\right)'(t) = f\left(u_k(t), m_k^{1/q}\right) + M \quad \text{in } \left[a, \frac{a+b}{2}\right], \end{gathered} \tag{3.34}$$

$$\begin{gathered} m_k^{1/p} \geq \hat{u}_k \geq 0 \quad \text{in } \left[a, \frac{a+b}{2}\right], \\ u_k'(a) = 0. \end{gathered} \tag{3.35}$$

We denote by $\hat{u}_k$ the following function:

$$\hat{u}_k(t) = \begin{cases} u_k\left(\dfrac{3a+b}{2} - t\right) & \text{if } t \in \left[a, \dfrac{a+b}{2}\right], \\ u_k\left(t - \dfrac{a+b}{2}\right) & \text{if } t \in \left[\dfrac{a+b}{2}, b\right]. \end{cases} \tag{3.36}$$

Then, from (3.34), it is easy to see that

$$\begin{gathered} \forall k \geq k_0, \quad \hat{u}_k \in C^1([a,b]), \quad \left(|\hat{u}_k'|^{p-2}\hat{u}_k'\right)' \in C([a,b]), \\ -\left(|\hat{u}_k'|^{p-2}\hat{u}_k'\right)'(t) = f\left(\hat{u}_k(t), m_k^{1/q}\right) + M \quad \text{in } [a,b], \\ m_k^{1/p} \geq \hat{u}_k \geq 0 \quad \text{in } [a,b]. \end{gathered} \tag{3.37}$$

Then the conclusion holds. □

Proposition 3.8. *Let $M > 0$. From (2.1), there exist some $m > 0$ and $(\hat{u}_m, \hat{v}_m) \in (C^1([a,b]))^2$ such that*

$$\begin{gathered} \left(\left(|\hat{u}_m'|^{p-2}\hat{u}_m'\right)', \left(|\hat{v}_m'|^{p-2}\hat{v}_m'\right)'\right) \in (C[a,b])^2, \\ -\left(|\hat{u}_m'|^{p-2}\hat{u}_m'\right)' \geq f(\hat{u}_m, m^{1/q}) + M \quad \text{in } (a,b), \\ -\left(|\hat{v}_m'|^{q-2}\hat{v}_m'\right)' \geq g(m^{1/p}, \hat{v}_m) + M \quad \text{in } (a,b), \\ m^{1/p} \geq \hat{u}_m \geq 0, \quad m^{1/q} \geq \hat{v}_m \geq 0 \quad \text{on } [a,b]. \end{gathered} \tag{3.38}$$

Proof. We study three cases.

Case 1. We suppose that for all $k \in \mathbb{N}$ we have

$$\inf_{s\in g[0,m_k^{1/p}]} f\left(s, m_k^{1/q}\right) + M < 0, \qquad \inf_{t\in[0,m_k^{1/q}]} g\left(m_k^{1/p}, t\right) + M < 0. \tag{3.39}$$

Then for all $k \in \mathbb{N}$, there exist $s_{m_k} \in [0, m_k^{1/p}]$ and $t_{m_k} \in [0, m_k^{1/q}]$ satisfying

$$f\left(s_{m_k}, m_k^{1/q}\right) + M < 0, \qquad g\left(m_k^{1/p}, t_{m_k}\right) + M < 0. \tag{3.40}$$

Consequently, for $m = m_k$, the couple $(\hat{u}_m, \hat{v}_m) = (s_{m_k}, t_{m_k})$ satisfies the result.

Case 2. Assume that for all $k \in \mathbb{N}$ we have,

$$\inf_{s\in[0,m_k^{1/p}]} f\left(s, m_k^{1/q}\right) + M \geq 0, \tag{3.41}$$

$$\inf_{t\in[0,m_k^{1/q}]} g\left(m_k^{1/p}, t\right) + M < 0. \tag{3.42}$$

(a) From (3.41) and Proposition 3.7 there exist some $k_0 \in \mathbb{N}$ and some sequence $(\hat{u}_k)_k$ such that, for all $k \geq k_0$, we have

$$\begin{gathered} \hat{u}_k \in C^1([a,b]), \quad \left(|\hat{u}_k'|^{p-2}\hat{u}_k'\right)' \in C([a,b]), \\ -\left(|\hat{u}_k'|^{p-2}\hat{u}_k'\right)' \geq f\left(\hat{u}_k, m_k^{1/q}\right) + M \quad \text{in } (a,b), \\ m_k^{1/p} \geq \hat{u}_k \geq 0 \quad \text{in } [a,b]. \end{gathered} \tag{3.43}$$

(b) From (3.42), there exists a sequence $(t_{m_k})_k$ such that

$$m_k^{1/p} \geq t_{m_k} \geq 0, \quad g\left(m_k^{1/p}, t_{m_k}\right) + M < 0 \quad \forall k \geq k_0. \tag{3.44}$$

Consequently, for $m = m_k$ with $k > k_0$, the pair $(\hat{u}_{m_k}, t_{m_k})$ satisfies the result.

Case 3. Assume that for all $k \in \mathbb{N}$, we have

$$\inf_{s\in[0,m_k^{1/p}]} f\left(s, m_k^{1/q}\right) + M \geq 0, \qquad \inf_{t\in[0,m_k^{1/q}]} g\left(m_k^{1/p}, t\right) + M \geq 0. \tag{3.45}$$

Then, from Proposition 3.7, for all $k \geq k_0$, there exists $(\hat{u}_k, \hat{v}_k) \in (C^1([a,b]))^2$ such that

$$\begin{gathered} \left(\left(|\hat{u}_k'|^{p-2}\hat{u}_k'\right)', \left(|\hat{v}_k'|^{p-2}\hat{v}_k'\right)'\right) \in (C[a,b])^2, \\ -\left(|\hat{u}_k'|^{p-2}\hat{u}_k'\right)' \geq f\left(\hat{u}_k, m_k^{1/q}\right) + M \quad \text{in } (a,b), \\ -\left(|\hat{v}_k'|^{p-2}\hat{v}_k'\right)' \geq g\left(m_k^{1/p}, \hat{v}_k\right) + M \quad \text{in } (a,b), \\ m_k^{1/p} \geq \hat{u}_k \geq 0, \quad m_k^{1/q} \geq \hat{v}_k \geq 0 \quad \text{on } [a,b]. \end{gathered} \tag{3.46}$$

This proves the results. □

Now, for problem (1.1), we consider a smooth bounded domain Ω in $\mathbb{R}^N$, and we have the following result.

Proposition 3.9. *Under hypotheses (H_1) and (2.1) of (H_2), problem (1.1) has a nonnegative supersolution (u^0, v^0) in $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$.*

Proof. Let $M \geq \| h_1 \|_\infty + \| h_2 \|_\infty \cdot P = \prod[a_i, b_i]$ is a cube containing Ω and

$$b - a = \inf_{1\leq i\leq N} b_i - a_i = b_1 - a_1. \tag{3.47}$$

From (2.1) of hypothesis (H_2) and Proposition 3.8, there exist $m > 0$ and $(\hat{u}_m, \hat{v}_m) \in (C^1([a,b]))^2$ such that

$$\left(\left(|\hat{u}_m'|^{p-2}\hat{u}_m'\right)', \left(|\hat{v}_m'|^{p-2}\hat{v}_m'\right)'\right) \in (C[a,b])^2, \tag{3.48}$$

and $(\hat{u}_m, \hat{v}_m)$ satisfies

$$\begin{gathered} -\left(|\hat{u}_m'|^{p-2}\hat{u}_m'\right)' \geq f\left(\hat{u}_m, m^{1/q}\right) + M \quad \text{in } (a,b), \\ -\left(|\hat{v}_m'|^{q-2}\hat{v}_m'\right)' \geq g(m^{1/p}, \hat{v}_m) + M \quad \text{in } (a,b), \\ m^{1/p} \geq \hat{u}_m \geq 0, \quad m^{1/q} \geq \hat{v}_m \geq 0 \quad \text{on } [a,b]. \end{gathered} \tag{3.49}$$

We denote by u^0 and v^0 the functions such that for all $x \in \Omega$ with $x = (x_1, x_2, \dots, x_N)$

$$u^0(x) = \hat{u}_m(x_1), \qquad v^0(x) = \hat{v}_m(x_1), \tag{3.50}$$

where (u^0, v^0) is clearly in $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$, moreover by hypothesis (H_1), we easily obtain

$$\begin{gathered} -\Delta_p u^0 \geq f(u^0, v) + h_1 \quad \text{for } v \leq v^0 \text{ a.e. on } \Omega, \\ -\Delta_q v^0 \geq g(u, v^0) + h_2 \quad \text{for } u \leq u^0 \text{ a.e. on } \Omega, \\ u^0 \geq 0, \quad v^0 \geq 0 \quad \text{on } \Omega. \end{gathered} \tag{3.51}$$

Then the result follows. □

3.2. Construction of a subsolution (u_0, v_0). Similar to the construction of a supersolution we can prove the following result.

Proposition 3.10. *Under hypotheses (H_1) and (2.2) of (H_2), problem (1.1) has a subsolution (u_0, v_0) in $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$.*

4. Proof of Theorem 2.1. We proceed in the following steps.

(i) From Propositions 3.9 and 3.10, there exists a pair $[(u_0, v_0); (u^0, v^0)]$ of sub-supersolution of problem (1.1).

(ii) Construction of an invariant set. In order to apply Schauder's fixed point theorem, we introduce the set $K = [u_0, u^0]\times[v_0, v^0]$. Next we define the following nonlinear operator T: for all $(u_1, v_1) \in W^{1,p}(\Omega)\times W^{1,q}(\Omega)$, we associate $(u_2, v_2) = T((u_1, v_1))$, where (u_2, v_2) is the solution of the system

$$\begin{gathered} -\Delta_p u = \hat{f}(x, u, v_1), \quad -\Delta_q v = \hat{g}(x, u_1, v) \quad \text{in } \Omega, \\ u = 0, \quad v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

where

$$\hat{f}(x, u, v) = f(U, V) + h_1(x), \qquad \hat{g}(x, u, v) = g(U, V) + h_2(x), \tag{4.2}$$

with

$$\begin{aligned} U(x) &= u(x) + (u_0 - u)_+ - (u - u^0)_+, \\ V(x) &= v(x) + (v_0 - v)_+ - (v - v^0)_+. \end{aligned} \tag{4.3}$$

The functions $\hat{f}$ and $\hat{g}$ are bounded, so the operator T is well defined. Furthermore, K is an invariant set for T. Let $(u_1, v_1) \in K$ and $(u_2, v_2) = T((u_1, v_1))$.

We show, for example, that $u_2 \le u^0$. From (3.51), (4.1), and (4.2) we have

$$\begin{aligned} 0 &\ge -\Delta_p u_2 - \hat{f}(x, u_2, v_1) \ge -\Delta_p u_2 - f(U_2, V_1) - h_1(x) \\ &\ge [-\Delta_p u_2 + \Delta_p u^0] + [f(u^0, v_1) - f(U_2, V_1)], \end{aligned} \tag{4.4}$$

multiplying (4.4) by $(u_2 - u^0)_+$ and integrating over Ω, we obtain

$$\begin{aligned} 0 \ge &\int_\Omega [\,|\nabla u_2|^{p-2} \nabla u_2 - |\nabla u^0|^{p-2} \nabla u^0] \nabla (u_2 - u^0)_+ dx \\ &+ \int_\Omega [f(u^0, v_1) - f(U_2, V_1)](u_2 - u^0)_+ dx. \end{aligned} \tag{4.5}$$

Since $v_1 \in [v_0, v^0]$, we have $V_1 = v_1$, where V_1 is associated with v_1 as in (4.3).

Denote by Ω_+ the set

$$\Omega_+ = \{x \in \Omega;\ u_2 - u^0 > 0\}. \tag{4.6}$$

We have $U_2 = u^0$ in Ω_+. Then

$$\begin{aligned} &\int_\Omega [f(u^0, v_1) - f(U_2, V_1)](u_2 - u^0)_+ dx \\ &\quad = \int_\Omega [f(u^0, v_1) - f(u_0, v_1)](u_2 - u^0)_+ dx = 0. \end{aligned} \tag{4.7}$$

By the monotonicity of $-\Delta_p$ in $L^p(\Omega)$, we get that $0 \ge \| (u_2 - u^0)_+ \|_{L^p(\Omega)}$.

Thus $u_2 \le u^0$ on Ω and similarly $v_2 \le v^0$ on Ω. So that the property, $T(K) \subset K$, holds.

(iii) The operator T is completely continuous.

(a) We prove that T is compact; let $(u_1^j, v_1^j)_j$ be a bounded sequence in $L^p(\Omega) \times L^q(\Omega)$. Let $(u_2^j, v_2^j) = T((u_1^j, v_1^j))$, so multiplying (4.1) by u_2^j, we obtain

$$\int_\Omega \left|\nabla u_2^j\right|^p dx = \int_\Omega \hat{f}\left(x, u_2^j, v_1^j\right) u_2^j\, dx \le C\left[\int_\Omega \left|u_2^j\right|^p dx\right]^{1/p}. \tag{4.8}$$

Therefore, $(u_2^j)_j$ is bounded in $W^{1,p}(\Omega)$ and it possesses a convergent subsequence in $L^p(\Omega)$. Analogously for $(v_2^j)_j$ in $L^q(\Omega)$.

(b) Now we prove the continuity of the operator T; from the continuity of the functions f and g associated at the bounded functions $\hat{f}, \hat{g}$, and by the dominated convergence theorem, we deduce easily the continuity of the operator T.

Since K is a convex, bounded, and closed subset, we apply Schauder's fixed point theorem and we obtain the existence of a fixed point for T which gives the existence of one solution of (1.1). □

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