Doubt Fuzzy BCI-Algebras

ZHAN JIANMING and TAN ZHISONG

Received 10 August 2001

The aim of this note is to introduce the notion of doubt fuzzy $p$-ideals in BCI-algebras and to study their properties. We also solve the problem of classifying doubt fuzzy $p$-ideals and study fuzzy relations on BCI-algebras.

2000 Mathematics Subject Classification: 03G25, 06F35, 03E72.

1. Introduction and preliminaries. The concept of a fuzzy set is applied to generalize some of the basic concepts of general topology [2]. Rosenfeld [6] constituted a similar application to the elementary theory of groupoids and groups. Xi [7] applied the concept of fuzzy set to BCK-algebras. Jun [4] defined a doubt fuzzy subalgebra, doubt fuzzy ideal, doubt fuzzy implicative ideal, and doubt fuzzy prime ideal in BCI-algebras, and got some results about it. In this note, we define a doubt fuzzy $p$-ideal of a BCI-algebra and investigate its properties.

A mapping $f : X \to Y$ of BCI-algebras is called homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$. A nonempty subset $I$ of a BCI-algebra $X$ is called an ideal of $X$ if (i) $0 \in I$, (ii) $x \ast y \in I$ and $y \in I$ imply that $x \in I$. We recall that a fuzzy subset $\mu$ of $X$ is a function $\mu$ from $X$ into $[0, 1]$. Let $\text{Im} \mu$ denote the image set of $\mu$. We will write $a \land b$ for $\min \{a, b\}$, and $a \lor b$ for $\max \{a, b\}$, where $a$ and $b$ are any real numbers.

Given a fuzzy set $\mu$ and $t \in [0, 1]$, let $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ and $\mu^t = \{x \in X \mid \mu(x) \leq t\}$. These could be empty sets. The set $\mu_t \neq \emptyset$ (resp., $\mu^t \neq \emptyset$) is called the $t$-confidence (resp., $t$-doubt) set of $\mu$ (see [3]).

**Definition 1.1** (see [7]). For any $x, y$ in a BCI-algebra $X$,

(i) if $\mu(x \ast y) \geq \mu(x) \land \mu(y)$, then $\mu$ is called a fuzzy subalgebra of $X$;

(ii) if $\mu(0) \geq \mu(x)$ and $\mu(x) \geq \mu(x \ast y) \land \mu(y)$, then $\mu$ is called a fuzzy ideal of $X$.

**Definition 1.2** (see [4]). Let $X$ be a BCI-algebra. A fuzzy set $\mu$ in $X$ is called (i) a doubt fuzzy subalgebra (briefly, DF-subalgebra) of $X$ if $\mu(x \ast y) \leq \mu(x) \lor \mu(y)$ for all $x, y \in X$; and (ii) a doubt fuzzy ideal (briefly, DF-ideal) of $X$ if $\mu(0) \leq \mu(x)$ and $\mu(x) \leq \mu((x \ast y) \ast (y \ast z)) \land \mu(y)$ for all $x, y, z \in X$.

**Definition 1.3** (see [5]). A nonempty subset $I$ of BCI-algebra $X$ is called $p$-ideal if

(i) $0 \in I$;

(ii) $(x \ast z) \ast (y \ast z) \in I$ and $y \in I$ imply that $x \in I$ for all $x, y, z \in X$.

**Definition 1.4** (see [5]). A fuzzy subset $\mu$ of a BCI-algebra $X$ is called a fuzzy $p$-ideal of $X$ if

(i) $\mu(0) \geq \mu(x)$ for any $x \in X$;

(ii) $\mu(x) \geq \mu((x \ast z) \ast (y \ast z)) \land \mu(y)$ for any $x, y, z \in X$. 
2. Doubt fuzzy $p$-ideals

**Definition 2.1.** A fuzzy subset $\mu$ of a BCI-algebra $X$ is called a doubt fuzzy $p$-ideal (briefly, DF $p$-ideal) of $X$ if

(i) $\mu(0) \leq \mu(x)$ for any $x \in X$;

(ii) $\mu(x) \leq \mu((x*z)*(y*z)) \lor \mu(y)$ for any $x, y, z \in X$.

**Example 2.2.** Let $X = \{0, a, b, c\}$ in which $*$ is defined by

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X; *, 0)$ is a BCI-algebra. Let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 < t_1 < t_2$. Define $\mu : X \rightarrow [0, 1]$ by $\mu(0) = t_0$, $\mu(a) = t_1$, and $\mu(b) = \mu(c) = t_2$. Routine calculations give that $\mu$ is a DF $p$-ideal of $X$.

**Proposition 2.3.** If $\mu$ is a DF $p$-ideal of a BCI-algebra $X$, then $\mu(x) \leq \mu(0*(0*x))$ for all $x \in X$.

**Proof.** Since $\mu$ is a DF $p$-ideal of $X$, we have $\mu(x) \leq \mu((x*x)*(0*x)) \lor \mu(0) = \mu(0*(0*x)) \lor \mu(0) = \mu(0*(0*x))$. \hfill $\square$

**Proposition 2.4.** Every DF $p$-ideal is a DF-ideal.

**Proof.** Let $\mu$ be a DF $p$-ideal of $X$. We have $\mu(x) \geq \mu((x*0)*(y*0)) \lor \mu(y) = \mu(x*y) \lor \mu(y)$ for all $x, y \in X$. Hence $\mu$ is a DF-ideal. \hfill $\square$

**Remark 2.5.** The converse of Proposition 2.4 is not true in general as shown in the following example.

**Example 2.6.** Let $X = \{0, a, 1, 2, 3\}$ in which $*$ is defined by

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $X$ is a BCI-algebra. Let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 < t_1 < t_2$. Define $\mu : X \rightarrow [0, 1]$ by $\mu(0) = t_0$, $\mu(a) = t_1$, and $\mu(1) = \mu(2) = \mu(3) = t_3$. Routine calculations give that $\mu$ is a DF-ideal of $X$. But $\mu$ is not a DF $p$-ideal of $X$, because $\mu(a) = t_1$, and $\mu((a*1)*(0*1)) \lor \mu(0) = \mu(0) = t_0$, that is, $\mu(a) > \mu((a*1)*(0*1)) \lor \mu(0)$.

**Proposition 2.7.** If $\mu$ is a DF $p$-ideal of a BCI-algebra $X$, then $\mu(x*y) \geq \mu((x*z)*(y*z))$ for all $x, y, z \in X$.

**Proof.** Note that in a BCI-algebra $X$ the inequality $(x*z)*(y*z) \leq x*y$ holds. It follows that $((x*z)*(y*z))*(x*y) = 0$. Since $\mu$ is a DF-ideal by Proposition 2.4.
We have $\mu((x \ast z) \ast (y \ast z)) \geq \mu(((x \ast z) \ast (y \ast z)) \ast (x \ast y)) \vee \mu(x \ast y) = \mu(0) \vee \mu(x \ast y) = \mu(x \ast y)$. This completes the proof.

**Proposition 2.8.** Let $\mu$ be a DF-ideal of a BCI-algebra $X$. If $\mu$ satisfies $\mu(x \ast y) \leq \mu((x \ast z) \ast (y \ast z))$ for any $x, y, z \in X$, then $\mu$ is a DF $p$-ideal of $X$.

**Proof.** Let $\mu$ be a DF-ideal of $X$ satisfying $\mu(x \ast y) \leq \mu((x \ast z) \ast (y \ast z))$ for all $x, y, z \in X$. Then $\mu((x \ast z) \ast (y \ast z)) \vee \mu(y) \geq \mu(x)$. This completes the proof.

**Proposition 2.9.** Let $\mu$ be a DF-ideal of a BCI-algebra $X$. Then $\mu(0 \ast (0 \ast x)) \leq \mu(x)$ for all $x \in X$.

**Proof.** We have that $\mu(0 \ast (0 \ast x)) \leq \mu((0 \ast (0 \ast x)) \ast x) \vee \mu(x) = \mu(0) \vee \mu(x) = \mu(x)$ for all $x \in X$.

**Proposition 2.10.** Let $\mu$ be a DF-ideal of a BCI-algebra $X$ satisfying $\mu((0 \ast (0 \ast x))) \geq \mu(x)$ for all $x \in X$.

**Proof.** Let $x, y, z \in X$. Then

$$
\mu((x \ast z) \ast (y \ast z)) \geq \mu((0 \ast (0 \ast x)) \ast (x \ast y))
= \mu((0 \ast y) \ast (0 \ast x))
= \mu(0 \ast (0 \ast (x \ast y)))
\geq \mu(x \ast y).
$$

It follows from Proposition 2.8 that $\mu$ is a DF $p$-ideal of $X$.

**Theorem 2.11.** Let $\mu$ be a fuzzy subset of a BCI-algebra $X$. If $\mu$ is a DF $p$-ideal of $X$, then the set $I = \{x \in X \mid \mu(x) = \mu(0)\}$ is a $p$-ideal of $X$.

**Proof.** Assume that $\mu$ is a DF $p$-ideal of $X$. Clearly $0 \in I$. Let $(x \ast z) \ast (y \ast z) \in I$ and $y \in I$. Then $\mu(x) \leq \mu((x \ast z) \ast (y \ast z)) \vee \mu(y) = \mu(0)$. But $\mu(0) \leq \mu(x)$ for all $x \in X$. Thus $\mu(0) = \mu(x)$. Hence $x \in I$. This completes the proof.

**Definition 2.12** (see [6]). Let $f$ be a mapping defined on a set $X$. If $\mu$ is a fuzzy subset of $X$, then the fuzzy subset $v$ of $f(x)$, defined by

$$
v(y) = \inf_{x \in f^{-1}(y)} \mu(x)
$$

for all $y \in f(x)$, is called the image of $\mu$ under $f$. Similarly, if $v$ is a fuzzy subset of $f(x)$, then the fuzzy subset $\mu = v \circ f$ in $X$ (i.e., the fuzzy subset defined by $\mu(x) = v(f(x))$ for all $x \in X$) is called the preimage of $v$ under $f$.

**Theorem 2.13.** An onto homomorphic preimage of a DF $p$-ideal is also a DF $p$-ideal.

**Proof.** Let $f : X \to X'$ be an onto homomorphism of BCI-algebras, $v$ a DF $p$-ideal of $X'$, and $\mu$ the preimage of $v$ under $f$. Then $v(f(x)) = \mu(x)$ for all $x \in X$. Since $f(x) \in X'$ and $v$ is a DF $p$-ideal of $X'$, it follows that $v(0') \leq v(f(x)) = \mu(x)$ for all $x \in X$, where $0'$ is the zero element of $X'$. But $v(0') = v(f(0) = \mu(0))$, and so $\mu(0) \leq \mu(x)$ for all $x \in X$. 
Since \( v \) is a DF \( p \)-ideal, we have \( \mu(x) = v(f(x)) \leq v((f(x) * z') * (y' * z')) \cup v(y') \) for any \( y', z' \in X' \). Since \( f \) is onto, there exist \( y, z \in X \) such that \( f(y') = y' \) and \( f(z) = z' \). Then

\[
\mu(x) \leq v((f(x) * z') * (y' * z')) \cup v(y') \\
= v((f(x) * f(z)) * (f(y) * f(z))) \\
= v(f(x * z) * f(y * z)) \cup v(f(y')) \\
= \mu((x * z) * (y * z)) \cup \mu(y).
\]

(2.3)

Since \( y' \) and \( z' \) are arbitrary elements of \( X' \), the above result is true for all \( y, z \in X \), that is, \( \mu(x) \leq \mu((x * z) * (y * z)) \cup \mu(y) \) for all \( x, y, z \in X \). This completes the proof. \( \square \)

**Definition 2.14** (see [6]). A fuzzy subset \( \mu \) of \( X \) has inf property if for any subset \( T \) of \( X \), there exists \( t_0 \in T \) such that

\[
\mu(t_0) = \inf_{t \in I} \mu(t).
\]

(2.4)

**Theorem 2.15.** An onto homomorphic image of a DF \( p \)-ideal with inf property is a DF \( p \)-ideal.

**Proof.** Let \( f : X \to X' \) be an onto homomorphism of BCI-algebras, \( \mu \) a DF \( p \)-ideal of \( X \) with inf property, and \( v \) the image of \( \mu \) under \( f \). Since \( \mu \) is a DF \( p \)-ideal of \( X \), we have \( \mu(0) \leq \mu(x) \) for all \( x \in X \). Note that \( 0 \in f^{-1}(0') \), where \( 0 \) and \( 0' \) are the zero elements of \( X \) and \( X' \), respectively. Thus \( v(0') = \inf_{t \in f^{-1}(0')} \mu(t) = \mu(0) \leq \mu(x) \) for all \( x \in X \), which implies that \( v(0') \leq \inf_{t \in f^{-1}(x')} \mu(t) = v(x') \) for any \( x' \in X' \). For any \( x', y', z' \in X' \), let \( x_0 \in f^{-1}(x') \), \( y_0 \in f^{-1}(y') \), and \( z_0 \in f^{-1}(z') \) be such that

\[
\mu(x_0) = \inf_{t \in f^{-1}(x')} \mu(t), \quad \mu(y_0) = \inf_{t \in f^{-1}(y')} \mu(t), \\
\mu((x_0 * z_0) * (y_0 * z_0)) = \inf_{t \in f^{-1}(x' * z') * (y' * z')} \mu(t).
\]

(2.5)

Then

\[
v(x') = \inf_{t \in f^{-1}(x')} \mu(t) \\
= \mu(x_0) \leq \mu((x_0 * z_0) * (y_0 * z_0)) \cup \mu(y_0) \\
= \inf_{t \in f^{-1}(x' * z') * (y' * z')} \mu(t) \cup \inf_{t \in f^{-1}(y')} \mu(t) \\
= v((x' * z') * (y' * z')) \cup v(y').
\]

(2.6)

Hence \( v \) is a DF \( p \)-ideal of \( X' \). \( \square \)

**Theorem 2.16.** A fuzzy subset \( \mu \) of a BCI-algebra \( X \) is a DF \( p \)-ideal if and only if, for every \( t \in [0,1] \), \( \mu^t \) is a \( p \)-ideal of \( X \), when \( \mu^t \neq \emptyset \).
**Proof.** Assume that μ is a DF p-ideal of X. By Definition 2.1, we have μ(0) ≤ μ(x) for any x ∈ X. Therefore, μ(0) ≤ μ(x) ≤ t for x ∈ µ, and so 0 ∈ µt. Let (x * z) * (y * z) ∈ µt and y ∈ µt. Since μ is a DF p-ideal, it follows that μ(x) ≤ μ((x * z) * (y * z)) ∨ μ(y) ≤ t, and that x ∈ µt. Hence μt is a p-ideal of X. Conversely, we only need to show that μ is a DF p-ideal of X. If Definition 2.1(i) is not true, then there exists x′ ∈ X such that μ(x′) > 0. If we take t′ = (μ(x′) + μ(0))/2, then μ(x′) > t′ and 0 ≤ μ(x′) < t′ ≤ 1. Thus x′ ∈ µt′ and µt′ ≠ ∅. As µt′ is a p-ideal of X, we have 0 ∈ µt′, and so μ(0) ≤ t′. This is a contradiction. Now assume that Definition 2.1(ii) is not true. Suppose that there exist x′, y′, z′ ∈ X such that μ(x′) > μ((x′ * z′) * (y′ * z′)) ∨ μ(y′). Putting t′ = (μ(x′) + μ((x′ * z′) * (y′ * z′)) ∨ μ(y′))/2, then μ(x′) > t′ and 0 ≤ μ((x′ * z′) * (y′ * z′)) ∨ μ(y′) ≤ 1. Hence, μ((x′ * z′) * (y′ * z′)) < t′ and μ(y′) < t′, which imply that (x′ * z′) * (y′ * z′) ∈ µt′ and y′ ∈ µt′, since µt′ is a p-ideal, it follows that x′ ∈ µt′, and μ(x′) ≤ t′. This is also a contradiction. Hence, μ is a DF p-ideal of X.

**Corollary 3.17.** If a fuzzy subset μ of a BCI-algebra X is a DF p-ideal, then for every t ∈ Imμ, μt is a p-ideal of X, when μt ≠ ∅.

3. Doubt Cartesian product of doubt fuzzy p-ideals

**Definition 3.1** (see [1]). A fuzzy relation on any set S is a fuzzy subset μ : S × S → [0, 1].

**Definition 3.2.** If μ is a fuzzy relation on a set S and v is a fuzzy subset of S, then μ is a doubt fuzzy relation on v if μ(x, y) ≥ μ(x) ∨ μ(y) for all x, y ∈ S.

**Definition 3.3.** Let μ and v be fuzzy subsets of a set S. The doubt Cartesian product of μ and v is defined by (μ × v)(x, y) = μ(x) ∨ v(y) for all x, y ∈ S.

**Lemma 3.4.** Let μ and v be fuzzy subsets of a set S. Then (i) μ × v is a fuzzy relation on S; (ii) (μ × v)t = μt × vt for all t ∈ [0, 1].

**Definition 3.5.** If v is a fuzzy subset of a set S, the smallest doubt fuzzy relation on S that is a doubt fuzzy relation on v is μv, given by μv(x, y) = v(x) ∨ v(y) for all x, y ∈ S.

**Lemma 3.6.** For a given fuzzy subset v of a set S, let μv be the smallest doubt fuzzy relation on a set S. Then for t ∈ [0, 1], (μv)t = vt v t.

**Proposition 3.7.** For a given fuzzy subset v of a BCI-algebra X, let μv be the smallest doubt fuzzy relation on X. If μv is a DF p-ideal of X × X, then v(x) ≥ v(0) for all x ∈ X.

**Proof.** Since μv is a DF p-ideal of X × X, it follows that μv(x, x) ≥ μv(0, 0), where (0, 0) is the zero element of X × X. But this means that v(x) ∨ v(x) ≥ v(0) ∨ v(0), which implies that v(x) ≥ v(0).

**Theorem 3.8.** Let μ and v be DF p-ideal of a BCI-algebra X. Then μ × v is a DF p-ideal of X × X.
PROOF. Note first that for every \((x, y) \in X \times X\), \((\mu \times v)(0, 0) = \mu(0) \vee v(0) \leq v(x) \vee v(y) = (\mu \times v)(x, y)\). Now let \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X\). Then

\[
(\mu \times v)((x_1, x_2) \ast (z_1, z_2)) \ast (y_1, y_2) \leq (\mu \times v)(x_1, y_1) \vee (\mu \times v)(y_1, y_2)
\]

\[
= (\mu \times v)((x_1 \ast z_1, x_2 \ast z_2) \ast (y_1 \ast z_1, y_2 \ast z_2)) \vee (\mu \times v)(y_1, y_2)
\]

\[
= (\mu \times v)((x_1 \ast z_1) \ast (y_1 \ast z_1), (x_2 \ast z_2) \ast (y_2 \ast z_2)) \vee (\mu \times v)(y_1, y_2)
\]

\[
= (\mu ((x_1 \ast z_1) \ast (y_1 \ast z_1)) \vee (\mu \times v)((x_2 \ast z_2) \ast (y_2 \ast x_2)) \vee v(y_2))
\]

\[
\leq \mu(x_1) \vee v(x_2)
\]

\[
= (\mu \times v)(x_1, x_2).
\]

This completes the proof. \(\square\)

**Theorem 3.9.** Let \(\mu\) and \(v\) be fuzzy subsets of a BCI-algebra \(X\) such that \(\mu \times v\) is a DF \(p\)-ideal of \(X \times X\). Then

(i) either \(\mu(x) \geq \mu(0)\) or \(v(x) \geq v(0)\) for all \(x \in X\);

(ii) if \(\mu(x) \geq \mu(0)\) for all \(x \in X\), then either \(\mu(x) \geq v(0)\) or \(v(x) \geq v(0)\);

(iii) if \(v(x) \geq v(0)\) for all \(x \in X\), then either \(\mu(x) \geq \mu(0)\) or \(v(x) \geq \mu(0)\);

(iv) either \(\mu \) or \(v\) is a DF \(p\)-ideal of \(X\).

**Proof.** (i) Suppose that \(\mu(x) < \mu(0)\) and \(v(y) < v(0)\) for some \(x, y \in X\). Then

\[
(\mu \times v)(x, y) = \mu(x) \vee v(y) < \mu(0) \vee v(0) = (\mu \times v)(0, 0).
\]

This is a contradiction and we obtain (i).

(ii) Assume that there exist \(x, y \in X\) such that \(\mu(x) < v(0)\) and \(v(y) < v(0)\). Then \((\mu \times v)(0, 0) = \mu(0) \vee v(0) = v(0)\). It follows that \((\mu \times v)(x, y) = \mu(x) \vee v(y) < v(0) = (\mu \times v)(0, 0)\), which is a contradiction. Hence (ii) holds.

(iii) Its proof follows by a similar method to (ii).

(iv) Since by (i) either \(\mu(x) \geq \mu(0)\) or \(v(x) \geq v(0)\) for all \(x \in X\); without loss of generality, we may assume that \(v(x) \geq v(0)\) for all \(x \in X\). From (iii) it follows that either \(\mu(x) \geq \mu(0)\) or \(v(x) \geq v(0)\). If \(v(x) \geq v(0)\) for any \(x \in X\), then \((\mu \times v)(0, x) = \mu(0) \vee v(x) = v(x)\). Let \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X\), since \(\mu \times v\) is a DF \(p\)-ideal of \(X \times X\). We have \((\mu \times v)(x_1, x_2) \leq (\mu \times v)((x_1, x_2) \ast (z_1, z_2)) \vee (\mu \times v)(y_1, y_2) = (\mu \times v)((x_1 \ast z_1) \ast (y_1 \ast z_1), (x_2 \ast z_2) \ast (y_2 \ast z_2)) \vee (\mu \times v)(y_1, y_2).\)

If we take \(x_1 = y_1 = z_1 = 0\), then \(v(x_2) = (\mu \times v)(0, x_2) \leq (\mu \times v)(0, (x_2 \ast z_2)) \vee (\mu \times v)(0, y_2) = (\mu(0) \vee v((x_2 \ast z_2) \ast (y_2 \ast z_2))) \vee (\mu(0) \vee v(y_2)) = v((x_2 \ast z_2) \ast (y_2 \ast z_2)) \vee v(y_2).\) This proves that \(v\) is a DF \(p\)-ideal of \(X\). Now we consider the case \(\mu(x) \geq \mu(0)\) for all \(x \in X\), suppose that \(v(y) < \mu(0)\) for some \(y \in X\). Then \(v(0) \leq v(y) < \mu(0)\), since \(\mu(x) \geq \mu(0)\). It follows that \(\mu(x) > v(0)\) for any \(x \in X\). Hence \((\mu \times v)(x, 0) = \mu(x) \vee v(0) = \mu(x)\). Taking \(x_2 = y_2 = x_2 = 0\), then \(\mu(x_1) = \mu(x) \vee v(0) = \mu(x)\).
\[(\mu \times v)(x_1,0) \leq (\mu \times v)((x_1 \ast z_1) \ast (y_1 \ast z_2),0) \lor (\mu \times v)(y_1,0) = \mu((x_1 \ast z_1) \ast (y_1 \ast z_2)) \lor \mu(y_1),\] which proves that \(\mu\) is a DF \(p\)-ideal of \(X\). Hence either \(\mu\) or \(v\) is a DF \(p\)-ideal of \(X\).

**Theorem 3.10.** Let \(v\) be a fuzzy subset of a BCI-algebra \(X\) and let \(\mu_v\) be the smallest doubt fuzzy relation on \(X\). Then \(v\) is a DF \(p\)-ideal of \(X\) if and only if \(\mu_v\) is a DF \(p\)-ideal of \(X \times X\).

**Proof.** Assume that \(v\) is a DF \(p\)-ideal of \(X\), we note that \(\mu_v(0,0) = v(0) \lor v(0) \leq v(x) \lor v(y)\) for all \((x,y) \in X \times X\)

\[
\mu_v(x_1,x_2) = v(x_1) \lor v(x_2)
\]

\[
\leq (v((x_1 \ast z_1) \ast (y_1 \ast z_2)) \lor v((x_2 \ast x_2) \ast (y_1 \ast z_2))) \lor v(y_2))
\]

\[
= (v((x_1 \ast z_1) \ast (y_1 \ast z_2))) \lor v((x_2 \ast x_2) \ast (y_2 \ast z_2))) \lor v(y_2))
\]

\[
= \mu_v((x_1 \ast z_1) \ast (y_1 \ast z_2), (x_2 \ast x_2) \ast (y_2 \ast z_2)) \lor \mu_v(y_1,y_2)
\]

\[
= \mu_v((x_1 \ast z_1, x_2 \ast x_2) \ast (y_1 \ast z_2, y_2 \ast z_2)) \lor \mu_v(y_1,y_2)
\]

\[
= \mu_v(((x_1 \ast x_2) \ast (z_1 \ast z_2)) \ast ((y_1 \ast y_2) \ast (z_1 \ast z_2))) \lor \mu_v(y_1,y_2),
\]

(3.3)

for all \((x_1,x_2), (y_1,y_2), (z_1,z_2) \in X \times X\). Hence \(\mu_v\) is a DF \(p\)-ideal of \(X \times X\).

Conversely, suppose that \(\mu_v\) is a DF \(p\)-ideal of \(X \times X\). Then for all \((x_1,x_2) \in X \times X\), \(v(0) \lor v(0) = \mu_v(0,0) \leq \mu_v(x,x) = v(x) \lor v(x)\). It follows that \(v(0) \leq v(x)\) for all \(x \in X\). Now let \((x_1,x_2), (y_1,y_2), (z_1,z_2) \in X \times X\). Then

\[
v(x_1) \lor v(x_2) = \mu_v(x_1,x_2)
\]

\[
\leq \mu_v(((x_1 \ast x_2) \ast (z_1 \ast z_2)) \ast ((y_1 \ast y_2) \ast (z_1 \ast z_2))) \lor \mu_v(y_1,y_2)
\]

\[
= \mu_v((x_1 \ast x_2, x_2 \ast z_2) \ast (y_1 \ast z_1, y_2 \ast z_2)) \lor \mu_v(y_1,y_2)
\]

\[
= \mu_v((x_1 \ast z_1) \ast (y_1 \ast z_1), (x_2 \ast x_2) \ast (y_2 \ast z_2)) \lor \mu_v(y_1,y_2)
\]

\[
= (v((x_1 \ast z_1) \ast (y_1 \ast z_1))) \lor v(y_1) \lor (v((x_2 \ast x_2) \ast (y_2 \ast x_2))) \lor v(y_2))
\]

(3.4)

In particular, if we take \(x_2 = y_2 = z_2 = 0\) (resp., \(x_1 = y_1 = z_1 = 0\)) then \(v(x_1) \leq v((x_1 \ast z_1) \ast (y_1 \ast z_1)) \lor v(y_1)\) (resp., \(v(x_2) \leq ((x_2 \ast x_2) \ast (y_2 \ast x_2)) \lor v(y_2))\). This completes the proof. \(\square\)

**References**


ZHAN JIANMING: DEPARTMENT OF MATHEMATICS, HUBEI INSTITUTE FOR NATIONALITIES, ENSHI CITY, HUBEI PROVINCE, 445000, CHINA

E-mail address: zhanjianming@hotmail.com

TAN ZHISONG: DEPARTMENT OF MATHEMATICS, HUBEI INSTITUTE ENSHI, HUBEI PROVINCE, 44500, CHINA
Submit your manuscripts at http://www.hindawi.com