A NOTE ON LOGHARMONIC MAPPINGS

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We consider the problem of minimizing the moments of order p for a subclass of logharmonic mappings.

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1. Introduction. Let $H(U)$ be the linear space of all analytic functions defined on the unit disc $U = \{z = x + iy : |z| < 1\}$. A logharmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_z} = (a\overline{f}/f)f_z, \quad (1.1)$$

where the second dilatation function a is in $H(U)$ and $|a(z)| < 1$ for all $z \in U$. If f does not vanish on U , then f is of the form

$$f = H \cdot \overline{G}, \quad (1.2)$$

where H and G are in $H(U)$. On the other hand, if f vanishes at 0 but has no other zeros in U , then f admits the representation

$$f(z) = z^m |z|^{2\beta m} h(z) \overline{g(z)}, \quad (1.3)$$

where

- (a) m is nonnegative integer,
- (b) $\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$ and therefore, $\Re \beta > -1/2$,
- (c) h and g are analytic in U , $g(0) = 1$, and $h(0) \neq 0$.

Univalent logharmonic mappings on the unit disc have been studied extensively. For details see [1, 2, 3, 4, 5, 6, 7, 8]. Suppose that f is a univalent logharmonic mapping defined on the unit disc U . Then, if $f(0) = 0$, the function $F(\zeta) = \log(f(e^\zeta))$ is univalent and harmonic on the half plane $\{\zeta : \Re \zeta < 0\}$. For more details on univalent harmonic mappings defined in the unit disc U , see [9, 10, 11, 12].

In this note, we consider the problem of minimizing the moments of order p over a subclass of the class logharmonic mappings defined over the unit disc U . It is interesting to note that the extremal functions are univalent starlike logharmonic mappings.

2. Moments of order p

THEOREM 2.1. Let $f = zh(z)\overline{g(z)}$ be logharmonic mapping defined on the unit disc U such that $h(0) = g(0) = 1$. Let $M_p(r, f)$ denote the moment of order p , $p \geq 0$. Then,

$$M_p(r, f) \geq 2\pi \left(\frac{r^{p+2}}{p+2} - \frac{r^{p+4}}{p+4} \right). \quad (2.1)$$

Equality holds if and only if

$$f_1(z) = z \frac{(1 + ((p+2)/(p+4))\bar{z})}{(1 + ((p+2)/(p+4))z)} \quad (2.2)$$

or one of its rotations $\bar{\eta}f_1(\eta z)$.

REMARK 2.2. If $p = 0$ in [Theorem 2.1](#), then we have the problem of minimizing the area. Moreover, if $p = 2$, then we obtain the minimum of the moment of inertia.

PROOF. Let $f = zh(z)\overline{g(z)}$ be logharmonic mapping defined on the unit disc U . Then, f satisfies [\(1.1\)](#) for some $a \in H(U)$ such that $|a(z)| < 1$ and $a(0) = 0$. Hence, using Schwarz's lemma, we have

$$\begin{aligned} M_p(r, f) &\geq \int_0^r \int_0^{2\pi} |f|^p (|f_z|^2 - |f_{\bar{z}}|^2) \rho d\theta d\rho \\ &= \int_0^r \int_0^{2\pi} |f|^p |f_z|^2 (1 - |a|^2) \rho d\theta d\rho \\ &\geq \int_0^r \rho (1 - \rho^2) \int_0^{2\pi} |f|^p |f_z|^2 d\theta d\rho. \end{aligned} \quad (2.3)$$

Writing $(h \cdot g)^{p/2} \cdot (zh)' \cdot g = 1 + \sum_{k=1}^{\infty} c_k z^k$, we have

$$\int_0^{2\pi} |f|^p |f_z|^2 d\theta = 2\pi \rho^p \left(1 + \sum_{k=1}^{\infty} |c_k|^2 \rho^{2k} \right) \quad (2.4)$$

and therefore,

$$M_p(r, f) \geq 2\pi \int_0^{2\pi} \rho^p (1 - \rho^2) d\rho = 2\pi \left(\frac{r^{p+2}}{p+2} - \frac{r^{p+4}}{p+4} \right). \quad (2.5)$$

Equality holds if and only if

$$(h)^{p/2} \cdot (g)^{(p+2)/2} \equiv 1 \quad (2.6)$$

and $a(z) = \eta z$, $|\eta| = 1$. This implies that

$$(h)^{(p+2)/2} \cdot (g)^{p/2} g' = \eta \quad (2.7)$$

and then,

$$z \cdot \frac{\partial (h \cdot g)^{(p+2)/2}}{\partial z} = \frac{(p+2)(1 - (h \cdot g)^{(p+2)/2} + \eta z)}{2}. \quad (2.8)$$

The solution of the differential equation

$$z \cdot u(z)' + \frac{(p+2) \cdot u(z)}{2} = \frac{(p+2)(1 + \eta z)}{2}; \quad u(0) = 1 \quad (2.9)$$

is $u(z) = (h(z)g(z))^{(p+2)/2} = 1 + ((p+2)/(p+4))\eta z$. Together with (2.6), we get

$$\frac{g(z)'}{g(z)} = \frac{\eta}{(1 + ((p+2)/(p+4))\eta z)} \quad (2.10)$$

and therefore,

$$g(z) = \left(1 + \frac{p+2}{p+4}\eta z\right)^{(p+4)/(p+2)},$$

$$zh(z) = \frac{z}{(1 + ((p+2)/(p+4))\eta z)}, \quad (2.11)$$

which leads to the solution $\bar{\eta}f_1(\eta z)$. Since

$$\phi(z) = \frac{zh(z)}{g(z)} = \frac{z}{(1 + ((p+2)/(p+4))\eta z)^{(2p+6)/(p+2)}} \quad (2.12)$$

is a starlike univalent analytic, it follows from [4, Theorem 2.1] that f_1 is a starlike univalent logharmonic mapping. $\square$

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