A GENERALIZATION RELATED TO SCHRÖDINGER OPERATORS WITH A SINGULAR POTENTIAL

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The purpose of this note is to generalize a result related to the Schrödinger operator $L = -\Delta + Q$, where Q is a singular potential. Indeed, we show that $D(L) = \{0\}$ in $L^2(\mathbb{R}^d)$ for $d \geq 4$. This fact answers to an open question that we formulated.

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1. Introduction. Let L be the operator of Schrödinger, defined in $L^2(\mathbb{R}^d)$ as $L = A + B$, where

$$\begin{aligned} A\phi &= -\Delta\phi, & D(A) &= H^2(\mathbb{R}^d), \\ B\phi &= Q\phi, & D(B) &= \{\phi \in L^2(\mathbb{R}^d) : Q\phi \in L^2(\mathbb{R}^d)\}. \end{aligned} \quad (1.1)$$

We suppose that the potential Q , verifies the following conditions, see, for example, [1],

$$Q > 0, \quad Q \in L^1(\mathbb{R}^d), \quad Q \notin L^2_{\text{loc}}(\mathbb{R}^d). \quad (1.2)$$

Under these conditions we show that $D(L) = \{0\}$, for $d \geq 4$, this fact extends the author's result (the case where $d \leq 3$, see the details in [1]). For that we use approximations of functions of $H^2(\mathbb{R}^d)$ (when $d \geq 4$) by continuous functions in connection with BMO space (where $\text{BMO}(\mathbb{R}^d)$ is the space of functions of Bounded Mean Oscillation), see [2]. Let $\phi \in L^2(\mathbb{R}^d)$, we denote by I_α the operator defined by

$$I_\alpha\phi = (-\Delta)^{-\alpha/2}\phi = \sqrt{(-\Delta)}^{(-\alpha)}\phi. \quad (1.3)$$

Thus, we know that

$$\|I_\alpha\phi\|_{L^q(\mathbb{R}^d)} \leq C_{p,q,d}\|\phi\|_{L^p(\mathbb{R}^d)}, \quad \text{if } \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d} \text{ with } \frac{1}{p} > \frac{\alpha}{d}. \quad (1.4)$$

In the case where $1/p = \alpha/d$, $\alpha = p = 2$, and $d = 4$, then

$$\|I_2\phi\|_{\text{BMO}} \leq C\|\phi\|_2. \quad (1.5)$$

We also have

$$I_2(C_0^\infty(\mathbb{R}^d)) \subseteq \text{VMO}, \quad (1.6)$$

where $\text{VMO}(\mathbb{R}^d)$ is the space of functions of Vanishing Mean Oscillation. We can find details in [2].

2. Generalization. Let H be a Hilbert space given by $H = L^2(\mathbb{R}^d)$, thus we have the following proposition.

PROPOSITION 2.1. *Under the previous hypotheses on the singular potential Q and if $d \geq 4$, then*

$$D(L) = \{0\}. \quad (2.1)$$

PROOF. Let $u \in D(A) \cap D(B)$, suppose that $u \neq 0$, then there exists an open subset Ω of $\mathbb{R}^d$ such that $|u(x)| > a$, for all $x \in \Omega \subseteq \text{supp } u$. Let $\Omega' \subseteq \Omega$, be a compact subset of Ω .

STEP 1. When $d \leq 3$, done in [1].

STEP 2. Suppose $d = 4$. Then there exists $(u_k) \in C_0^\infty(\mathbb{R}^4)$ such that u_k converges to u into $H^2(\mathbb{R}^4)$, thus, we can write $u_k = I_2 v_k$ and $u = I_2 v$ and $v \in L^2(\mathbb{R}^4)$. It follows that

$$\|u_k - u\|_{\text{BMO}} \leq C \|v_k - v\|_2 \rightarrow 0 \quad (2.2)$$

because v_k converges to v into $L^2(\mathbb{R}^4)$, thus u_k converges to u into BMO. Consider u_k and u as functions defined on Ω' , then $|Q|_{|\Omega'} = (|Qu_k|/|u_k|)_{|\Omega'}$, on passing to the limit in BMO and by the fact that B is a closed operator. It follows that $Q \in L^2(\Omega')$, that is impossible according to the hypothesis on the potential, $Q \notin L_{\text{loc}}^2(\mathbb{R}^4)$. And then, we conclude that $u = 0$.

STEP 3. Suppose $d > 4$, and write $u_k = I_2 v_k$ and $u = I_2 v$ where v_k converges to v into $L^2(\mathbb{R}^d)$. Thus, $\alpha = p = 2$ and $1/q = 1/2 - 2/d$ where $d > 4$, therefore,

$$\|u_k - u\|_q = \|I_2 v_k - I_2 v\|_q \leq C \|v_k - v\|_2, \quad (2.3)$$

then u_k converges to u into $L^q(\mathbb{R}^d)$. We also write, $Q = Qu_k/u_k$, and consider this function on Ω' and by passing to the limit into $L^q(\mathbb{R}^d)$, we get a contradiction. $\square$

CONCLUSION. The domain of the algebraic sum of A and B is always zero, that is, $D(A) \cap D(B) = \{0\}$, without restriction on d .

REMARKS. The dimensional d of $\mathbb{R}^d$ has no impact on the sum form of A and B , $(-\Delta \oplus Q)$. This operator is always defined and verifies Kato's condition and is given as

$$\begin{aligned} D((-\Delta \oplus Q)) &= \{u \in H^1(\mathbb{R}^d) : Q|u|^2 \in L^1(\mathbb{R}^d), -\Delta u + Qu \in L^2(\mathbb{R}^d)\}, \\ (-\Delta \oplus Q)u &= -\Delta u + Qu \end{aligned} \quad (2.4)$$

therefore, Kato's condition is satisfied, that is,

$$D\left(\sqrt{(-\Delta \oplus Q)}\right) = D(\sqrt{-\Delta}) \cap D(\sqrt{Q}) = D\left(\sqrt{(-\Delta \oplus Q)*}\right). \quad (2.5)$$

The example of singular potential given in [1] is always valid for all d ,

$$Q(x) = \sum_{k=0}^{+\infty} \frac{G(x - \alpha_k)}{k^2}, \quad (2.6)$$

where G is a function defined on the compact subset Ω of $\mathbb{R}^d$ and verifying

$$G > 0, \quad G \in L^1(\Omega), \quad G \notin L^2(\Omega), \quad G = 0 \quad \text{on } \mathbb{R}^d - \Omega, \quad (2.7)$$

where $\alpha_k = (\alpha_k^1, \alpha_k^2, \dots, \alpha_k^d) \in Q^d$ is a rational sequence.

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