DUALITY OF MEASURE AND CATEGORY IN INFINITE-DIMENSIONAL SEPARABLE HILBERT SPACE ℓ_2

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We prove that an analogy of the Oxtoby duality principle is not valid for the concrete nontrivial σ -finite Borel invariant measure and the Baire category in the classical Hilbert space ℓ_2 .

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As usual, we equip an infinite-dimensional separable Hilbert space ℓ_2 by such nonzero σ -finite Borel measures which are invariant with respect to everywhere dense vector subspaces and study duality between such measures and Baire category.

[Section 1](#) contains constructions of nontrivial σ -finite Borel measures, which are defined in the infinite-dimensional separable Hilbert space ℓ_2 and are invariant with respect to some everywhere dense vector subspaces. The duality between invariant Borel measures and Baire category in the classical Hilbert space ℓ_2 is studied in [Section 2](#). An idea applied in the process of proving of the main assertions allows us to obtain more general results for sufficiently large class of infinite-dimensional topological vector spaces.

1. Invariant Borel measures in classical Hilbert space ℓ_2 . Let $\mathbb{R}^N$ be the space of all sequences of real numbers equipped with the Tychonoff topology. Denote by $B(\mathbb{R}^N)$ the σ -algebra of all Borel subsets in $\mathbb{R}^N$.

Let $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ be sequences of real numbers such that

$$(\forall i) \quad (i \in \mathbb{N} \rightarrow a_i < b_i). \quad (1.1)$$

We put

$$A_n = R_0 \times \cdots \times R_n \times \left(\prod_{i>n} \Delta_i \right) \quad (n \in \mathbb{N}), \quad (1.2)$$

where

$$(\forall i) \quad (i \in \mathbb{N} \rightarrow R_i = R, \Delta_i = [a_i, b_i]). \quad (1.3)$$

For an arbitrary natural number $i \in \mathbb{N}$, consider the Lebesgue measure μ_i defined on the space R_i and satisfying the condition $\mu_i(\Delta_i) = 1$. Denote by λ_i the normed Lebesgue measure defined on the interval Δ_i .

For an arbitrary $n \in \mathbb{N}$, we denote by ν_n the measure defined by

$$\nu_n = \prod_{1 \leq i \leq n} \mu_i \times \prod_{i > n} \lambda_i, \quad (1.4)$$

and by $\bar{\nu}_n$ the Borel measure in the space $\mathbb{R}^N$ defined by

$$(\forall X) \quad (X \in B(\mathbb{R}^N) \rightarrow \bar{\nu}_n(X) = \nu_n(X \cap A_n)). \quad (1.5)$$

The following assertion is valid.

LEMMA 1.1. *For an arbitrary Borel set $X \subseteq \mathbb{R}^N$, there exists a limit*

$$\nu_\Delta(X) = \lim_{n \rightarrow \infty} \bar{\nu}_n(X). \quad (1.6)$$

Moreover, the functional ν_Δ is a nontrivial σ -finite measure defined on the Borel σ -algebra $B(\mathbb{R}^N)$.

PROOF. First, observe that, for an arbitrary natural number n , the condition $A_n \subset A_{n+1}$ is valid. By the property of σ -additivity of the measure ν_{n+1} , we obtain

$$\begin{aligned} \bar{\nu}_{n+1}(X) &= \nu_{n+1}(X \cap A_{n+1}) = \nu_{n+1}(X \cap [A_{n+1} \setminus A_n] \cup A_n) \\ &= \nu_{n+1}[X \cap (A_{n+1} \setminus A_n)] + \nu_{n+1}(X \cap A_n). \end{aligned} \quad (1.7)$$

Note that the restriction $\nu_{n+1}|_{A_n}$ of the measure ν_{n+1} to the set A_n coincides with the measure ν_n .

Indeed, we have

$$\begin{aligned} \nu_{n+1}(A_n \cap X) &= \left(\prod_{1 \leq i \leq n+1} \mu_i \times \prod_{i > n+1} \lambda_i \right) (A_n \cap X) \\ &= \left\{ \prod_{1 \leq i \leq n} \mu_i \times [\mu_{n+1} | \Delta_{n+1} + \mu_{n+1} | \{R \setminus \Delta_{n+1}\}] \times \prod_{i > n+1} \lambda_i \right\} (A_n \cap X) \\ &= \left(\prod_{1 \leq i \leq n} \mu_i \times \prod_{i > n} \lambda_i \right) (A_n \cap X) \\ &\quad + \left(\prod_{1 \leq i \leq n} \mu_i \times (\mu_{n+1} | \{R \setminus \Delta_{n+1}\}) \times \prod_{i > n+1} \lambda_i \right) (A_n \cap X) \\ &= \nu_n(A_n \cap X). \end{aligned} \quad (1.8)$$

Since for an arbitrary $n \in \mathbb{N}$, the inclusion $A_n \subset A_{n+1}$ holds, we have

$$(\forall X) \quad (X \in B(\mathbb{R}^N) \rightarrow \nu_n(A_n \cap X) \leq \nu_{n+1}(A_n \cap X)). \quad (1.9)$$

Hence there exists a limit $\lim_{n \rightarrow \infty} \bar{\nu}_n(X)$ which we denote by $\nu_\Delta(X)$.

The proof of the fact that the measure ν_Δ is countably additive is trivial.

Establish the following properties of ν_Δ .

(I) The measure ν_Δ is nontrivial, since

$$\nu_\Delta \left(\prod_{i \in \mathbb{N}} \Delta_i \right) = 1. \quad (1.10)$$

(II) The measure ν_Δ is σ -finite. Indeed, we have

$$\mathbb{R}^N = \left(\mathbb{R}^N \setminus \bigcup_{n \in \mathbb{N}} A_n \right) \cup \left(\bigcup_{n \in \mathbb{N}} A_n \right). \quad (1.11)$$

Since $\mathbb{R}^N \setminus \bigcup_{n \in \mathbb{N}} A_n \in B(\mathbb{R}^N)$, by the definition of the measure ν_Δ we have

$$\nu_n \left(\mathbb{R}^N \setminus \bigcup_{k \in \mathbb{N}} A_k \right) = \nu_n \left(\left(\mathbb{R}^N \setminus \bigcup_{k \in \mathbb{N}} A_k \right) \cap A_n \right) = \nu_n(\emptyset) = 0. \quad (1.12)$$

Since, for an arbitrary natural number $n \in \mathbb{N}$, the measure $\bar{\nu}_n$ is σ -finite, there exists a countable family $(B_k^{(n)})_{k \in \mathbb{N}}$ of Borel measurable subsets of the space $\mathbb{R}^N$ such that

$$\begin{aligned} (\forall k) \quad & (k \in \mathbb{N} \rightarrow \bar{\nu}_n(B_k^{(n)}) < +\infty); \\ (\forall n) \quad & \left(n \in \mathbb{N} \rightarrow A_n = \bigcup_{k \in \mathbb{N}} B_k^{(n)} \right). \end{aligned} \quad (1.13)$$

Consider the family $(B_k^{(n)})_{k \in \mathbb{N}, n \in \mathbb{N}}$.

It is clear that

$$(\forall k) \quad (\forall n) \quad (k \in \mathbb{N}, n \in \mathbb{N} \rightarrow \nu_\Delta(B_k^{(n)}) = \bar{\nu}_n(B_k^{(n)}) < +\infty). \quad (1.14)$$

On the other hand, we have

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} B_k^{(n)}, \quad (1.15)$$

that is,

$$\mathbb{R}^N = \left(\mathbb{R}^N \setminus \bigcup_{n \in \mathbb{N}} A_n \right) \cup \left(\bigcup_{n \in \mathbb{N}, k \in \mathbb{N}} B_k^{(n)} \right). \quad (1.16)$$

The proof is completed. $\square$

REMARK 1.2. The measure ν_Δ described in [Lemma 1.1](#) can be regarded as an inductive limit of the family $(\bar{\nu})_{n \in \mathbb{N}}$ of invariant measures.

Recall that an element $h \in \mathbb{R}^N$ is called an admissible translation (in the sense of invariance) of the measure ν_Δ if

$$(\forall X) \quad (X \in B(\mathbb{R}^N) \rightarrow \nu_\Delta(X+h) = \nu_\Delta(X)). \quad (1.17)$$

We define

$$G_\Delta = \{h : h \in \mathbb{R}^N, h \text{ is an admissible translation for } \nu_\Delta\}. \quad (1.18)$$

It is easy to show that G_Δ is a vector subspace of the space $\mathbb{R}^N$.

REMARK 1.3. The construction of the measure ν_Δ belongs to Kharazishvili [1].

Our next theorem gives a representation of the algebraic structure of the vector subspace G_Δ of all admissible translations for ν_Δ .

THEOREM 1.4. *The following conditions are equivalent:*

$$g = (g_1, g_2, \dots) \in G_\Delta, \quad (1.19)$$

$$(\exists n_g) \left(n_g \in \mathbb{N} \rightarrow \text{the series } \sum_{i=n_g}^{\infty} \ln \left(1 - \frac{|g_i|}{b_i - a_i} \right) \text{ is convergent} \right). \quad (1.20)$$

PROOF. Assume that for an element $g = (g_1, g_2, \dots) \in \mathbb{R}^{\mathbb{N}}$, the condition (1.19) is satisfied. Then we have

$$\nu_\Delta(\Delta + g) = \nu_\Delta(\Delta) = 1. \quad (1.21)$$

On the other hand, we have

$$\begin{aligned} \nu_\Delta(\Delta + g) &= \nu_\Delta(\Delta + g) \\ &= \nu_\Delta \left(\prod_{i \in \mathbb{N}} [a_i + g_i, b_i + g_i[\right) \\ &= \lim_{n \rightarrow \infty} \tilde{\nu}_n(A_n \cap (\Delta + g)) \\ &= \lim_{n \rightarrow \infty} \left(\prod_{1 \leq i \leq n} \mu_i \times \prod_{i > n} \lambda_i \right) \left(\left(\prod_{1 \leq i \leq n} R_i \times \prod_{i > n} [a_i, b_i[\right) \cap \prod_{i \in \mathbb{N}} [a_i + g_i, b_i + g_i[\right) \\ &= \lim_{n \rightarrow \infty} \left(\prod_{1 \leq i \leq n} \mu_i \left(\prod_{1 \leq i \leq n} [a_i + g_i, b_i + g_i[\right) \right) \times \left(\prod_{i > n} \lambda_i ([a_i + g_i, b_i + g_i[) \right) \\ &= \lim_{n \rightarrow \infty} \prod_{i > n} \lambda_i ([a_i, b_i[\cap [a_i + g_i, b_i + g_i[) = 1. \end{aligned} \quad (1.22)$$

We show that

$$(\forall g) \left(g = (g_1, g_2, \dots) \in G_\Delta \rightarrow \lim_{i \rightarrow \infty} \frac{|g_i|}{|b_i - a_i|} = 0 \right). \quad (1.23)$$

Indeed, if we assume the contrary, then there exist a countable subset $(n_k)_{k \in \mathbb{N}}$ of $\mathbb{N}$ and a positive real number $\epsilon > 0$, such that

$$(\forall k) \left(k \in \mathbb{N} \rightarrow \frac{|g_{n_k}|}{b_{n_k} - a_{n_k}} > \epsilon \right). \quad (1.24)$$

Choose a number $m > 0$ such that $\epsilon \cdot m > 1$. Since $g \in G_\Delta$, we have

$$m \cdot g = (m \cdot g_1, m \cdot g_2, \dots) \in G_\Delta. \quad (1.25)$$

In view of the property of σ -additivity of the measure ν_Δ , we obtain

$$\nu_\Delta(\Delta) = \nu_\Delta(\Delta + m \cdot g) = 1. \quad (1.26)$$

But note that

$$(\Delta + m \cdot g) \cap \left(\bigcup_{n \in \mathbb{N}} A_n \right) = \emptyset. \quad (1.27)$$

Indeed, assume the contrary and take

$$(x_i)_{i \in \mathbb{N}} \in (\Delta + m \cdot g) \cap \left(\bigcup_{n \in \mathbb{N}} A_n \right). \quad (1.28)$$

Then it is clear that, for the n_k th coordinate, we have

$$(\exists k_0) \quad (k_0 \in \mathbb{N} \text{ and } (\forall k) \quad (k \geq k_0 \rightarrow (a_{n_k} + m \cdot g_{n_k} \leq x_{n_k} < b_{n_k} + m \cdot g_{n_k}), \\ (a_{n_k} \leq x_{n_k} < b_{n_k}))). \quad (1.29)$$

On the other hand, the validity of the condition

$$(\forall k) \quad \left(k \in \mathbb{N} \rightarrow \frac{|g_{n_k}|}{b_{n_k} - a_{n_k}} > \epsilon \right) \quad (1.30)$$

implies the validity of the relation

$$(\forall k) \quad (k \in \mathbb{N} \rightarrow m \cdot |g_{n_k}| > b_{n_k} - a_{n_k}), \quad (1.31)$$

which shows that the intervals $[a_{n_k}, b_{n_k}[$ and $[a_{n_k} + m \cdot g_{n_k}, b_{n_k} + m \cdot g_{n_k}[$ have an empty intersection. Hence the condition $\lim_{i \rightarrow \infty} (|g_i| / (b_i - a_i)) = 0$ holds.

From the validity of the condition $\lim_{i \rightarrow \infty} (|g_i| / (b_i - a_i)) = 0$, we conclude that there exists a natural number n_g such that

$$(\forall i) \quad \left(i > n_g \rightarrow \frac{|g_i|}{b_i - a_i} < 1 \right), \quad (1.32)$$

since

$$(\forall i) \quad \left(i > n_g \rightarrow \lambda_i([a_i, b_i[\cap [a_i + g_i, b_i + g_i]) = \frac{b_i - a_i - |g_i|}{b_i - a_i} = 1 - \frac{|g_i|}{b_i - a_i} \right). \quad (1.33)$$

Keeping in mind that

$$\lim_{p \rightarrow \infty} \prod_{i \geq n_g + p} \left(1 - \frac{|g_i|}{b_i - a_i} \right) = 1 \quad (1.34)$$

and considering the logarithms of both sides, we have

$$\lim_{p \rightarrow \infty} \sum_{i \geq n_g + p} \ln \left(1 - \frac{|g_i|}{b_i - a_i} \right) = 0. \quad (1.35)$$

This means that the series $\sum_{i \geq n_g} \ln(1 - |g_i| / (b_i - a_i))$ is convergent.

The validity of the implication (1.19)→(1.20) is proved.

Now we prove (1.20)→(1.19). Let n_g be a natural number such that the series $\sum_{i \geq n_g} \ln(1 - |g_i|/(b_i - a_i))$ is convergent.

Consider an arbitrary element X having the form

$$X = B \times \prod_{i > n} \Delta_i, \quad (1.36)$$

where $B \in B(\mathbb{R}^N)$ ($n \in \mathbb{N}$).

The sets of these forms generate the σ -algebra $B(A_n)$ of the space A_n , and the condition $B(A_n) = B(\mathbb{R}^N) \cap A_n$ holds. To prove the implication (1.20)→(1.19), it is sufficient to show the validity of the condition

$$\begin{aligned} \nu_\Delta(X + g) &= \nu_\Delta \left\{ \left[\left(B \times \prod_{n+1 \leq i \leq n_g+n} \Delta_i \right) + (g_1, \dots, g_{n_g}) \right] \times \prod_{i > n_g+n} [a_i + g_i, b_i + g_i] \right\} \\ &= \lim_{n \rightarrow \infty} \prod_{i=1}^{n_g+n} \mu_i \left(B \times \prod_{n+1 \leq i \leq n_g+n} \Delta_i \right) \\ &\quad \times \prod_{i > n_g+n} \lambda_i([a_i + g_i, b_i + g_i] \cap [a_i, b_i]) = \nu_\Delta \left(B \times \prod_{i > n} \Delta_i \right) \\ &\quad \times \lim_{n \rightarrow \infty} \prod_{i > n_g+n} \left(1 - \frac{|g_i|}{b_i - a_i} \right) = \nu_\Delta \left(B \times \prod_{i > n} \Delta_i \right) = \nu_\Delta(X). \end{aligned} \quad (1.37)$$

We have used the well-known result from mathematical analysis

$$\begin{aligned} &\left(\text{the series } \sum_{i \geq n_g} \ln \left(1 - \frac{|g_i|}{b_i - a_i} \right) \text{ is convergent} \right) \\ &\Leftrightarrow \lim_{n \rightarrow \infty} \prod_{i \geq n_g+n} \left(1 - \frac{|g_i|}{b_i - a_i} \right) = \ln 1 \\ &\Leftrightarrow \lim_{n \rightarrow \infty} \prod_{i > n_g+n} \left(1 - \frac{|g_i|}{b_i - a_i} \right) = 1. \end{aligned} \quad (1.38)$$

The proof is completed. $\square$

REMARK 1.5. Let $\mathbb{R}^{(N)}$ be the space of all finite sequences, that is,

$$\mathbb{R}^{(N)} = \{(g_i)_{i \in \mathbb{N}} \mid (g_i)_{i \in \mathbb{N}} \in \mathbb{R}^N, \text{ card } \{i \mid g_i \neq 0\} < \aleph_0\}. \quad (1.39)$$

It is clear that, on the one hand, for an arbitrary compact infinite-dimensional parallelepiped $\Delta = \prod_{k \in \mathbb{N}} [a_k, b_k]$, we have

$$\mathbb{R}^{(N)} \subset G_\Delta. \quad (1.40)$$

On the other hand, $G_\Delta \setminus \mathbb{R}^{(N)} \neq \emptyset$, since the element $(g_i)_{i \in \mathbb{N}}$ defined by

$$(\forall i) \quad \left(i \in \mathbb{N} \rightarrow g_i = \left(1 - \exp \left\{ -\frac{b_i - a_i}{2^i} \right\} \times (b_i - a_i) \right) \right) \quad (1.41)$$

belongs to the difference $G_\Delta \setminus \mathbb{R}^{(N)}$.

It is easy to show that the vector space G_Δ is everywhere dense in $\mathbb{R}^N$ with respect to the Tychonoff topology, since $\mathbb{R}^{(N)} \subset G_\Delta$.

In the sequel, we will need the following result.

THEOREM 1.6. *In the separable Hilbert space ℓ_2 , there exists a σ -finite Borel measure λ such that*

- (1) $\lambda(\Delta_0) = 1$;
- (2) *a group G_{Δ_0} of all admissible translations of the measure λ has the form*

$$G_{\Delta_0} = \left\{ (c_k)_{k \in \mathbb{N}} \mid (c_k)_{k \in \mathbb{N}} \in \ell_2, \right. \\ \left. (\exists n_p) \left(n_p \in \mathbb{N} \rightarrow \text{the series } \sum_{n=n_p}^{\infty} \ln(1 - |c_k|(i+1)) \text{ is convergent} \right) \right\}, \quad (1.42)$$

where $\Delta_0 = \prod_{i \in \mathbb{N}} [0; 1/(i+1)[$.

PROOF. According to Suslin's theorem we have $B(\ell_2) \subseteq B(\mathbb{R}^N)$. Now the proof of [Theorem 1.6](#) can be obtained easily if we put

$$(\forall X) \quad (X \in B(\ell_2) \Rightarrow \lambda(X) = \nu_{\Delta_0}(\ell_2 \cap X)). \quad (1.43)$$

□

2. Duality of measure and category in the infinite-dimensional separable Hilbert space ℓ_2 . In this section, we continue our discussion of some properties of invariant measures in the infinite-dimensional separable Hilbert space ℓ_2 and study the question of the duality between the Baire category and the above-constructed measure λ .

The following definitions are important for our investigation.

Let (E, T) be a nonempty topological vector space. Denote by $B(E)$ the Borel σ -algebra of subsets of the space E , generated by the topology T . Consider a nontrivial Borel measure μ defined on the σ -algebra $B(E)$. A subset $X \subseteq E$ is called small in the sense of measure if $\mu^*(X) = 0$. Analogously, a subset $Y \subseteq E$ is called small in the sense of category if it is the first category set in the topological space (E, T) . Further, let P be a such sentence in formulation of which the notions of measure zero and of the first category are used. We say that the duality between the measure μ and the Baire category is valid with respect to the sentence P if the sentence P is equivalent to the sentence P^* obtained from the sentence P by interchanging the notions of the above small sets. We also say that strict duality between the measure μ and Baire category is valid if the duality between the measure μ and the Baire category is valid for all the above P sentences formulated only by using the notions of measure zero, of first category and of purely set-theoretical notions.

The following result is known as the Erdős-Sierpiński duality principle.

THEOREM 2.1 (duality principle). *If the continuum hypothesis is true, then the strict duality between a linear Lebesgue measure and the Baire category of the real axis $\mathbb{R}$ is valid.*

The proof of [Theorem 2.1](#) can be found, for example, in [4].

Using the same argument applied in the process of the proving of [Theorem 2.1](#) (see [4, pages 129–131]), it is easy to conclude that if the continuum hypothesis is true, then the strict duality between the measure λ and Baire category of ℓ_2 is valid also.

Here we apply the well-known method to establish one important property of Baire second category subsets in the infinite-dimensional separable Hilbert space ℓ_2 .

THEOREM 2.2. *For an arbitrary second category Baire subset $X \subseteq \ell_2$, there exists a positive number $\delta > 0$ such that*

$$(\forall x) \quad (x \in \ell_2, \|x\| < \delta \rightarrow (X+x) \cap X \neq \emptyset). \quad (2.1)$$

PROOF. Since the set X has the Baire property, there exist an open subset $G \subseteq \ell_2$ and a first category subset $P \subseteq \ell_2$ such that the equality

$$X = G \Delta P \quad (2.2)$$

is fulfilled.

Evidently, there exists an open nonempty ball $B \subseteq G$.

Note that the inclusion

$$[(x+B) \cap B] \setminus [P \cup (x+P)] \subseteq (x+X) \cap X \quad (2.3)$$

holds for arbitrary $x \in \ell_2$. If $\|x\| < \text{diam}(B)$, then the set, the left-hand side of (2.3), is a nonempty open set minus a first category set.

Using the well-known Baire theorem, we complete the proof of [Theorem 2.2](#). $\square$

REMARK 2.3. The method considered in the proof of [Theorem 2.2](#) was worked out and applied by many authors, for example, Oxtoby who establishes an analogous result for linear Baire second category subsets in $\mathbb{R}$ (cf. [4]).

The following simple result (which is however important from the viewpoint of applications) is also essentially due to Steinhaus.

THEOREM 2.4. *Let X be an arbitrary linear Borel subset in $\mathbb{R}$ with a positive Lebesgue measure. Then there exists a positive number δ such that the condition*

$$(\forall x) \quad (x \in \mathbb{R}, |x| < \delta \rightarrow (x+X) \cap X \neq \emptyset) \quad (2.4)$$

holds.

The proof of [Theorem 2.4](#) can be found in [4].

The next theorem plays the main role in our further consideration.

THEOREM 2.5. *In the infinite-dimensional separable Hilbert space ℓ_2 , there exists a Borel subset $Y \subset \ell_2$ with $\lambda(Y) > 0$ such that*

$$(\forall \delta) \quad (\delta > 0 \rightarrow (\exists y) (\|y\| < \delta \rightarrow Y \cap (Y+y) = \emptyset)). \quad (2.5)$$

PROOF. Let

$$Y \equiv \Delta_0 = \prod_{i \in \mathbb{N}} \left[0, \frac{1}{i+1} \right]. \quad (2.6)$$

For an arbitrary positive real number $\delta > 0$, denote by n_δ a natural number such that

$$\sum_{i=n_\delta}^{\infty} \frac{1}{(i+1)^2} < \delta^2. \quad (2.7)$$

Assume that

$$\begin{aligned} (\forall k) \quad (1 \leq k < n_\delta \rightarrow h_k = 0), \\ (\forall k) \quad \left(k \geq n_\delta \rightarrow h_k = \frac{1}{k+1} \right). \end{aligned} \quad (2.8)$$

It is clear that $h = (h_k)_{k \in \mathbb{N}} \notin G_{\Delta_0}$, $\|h\| < \delta$, and $\Delta_0 \cap (\Delta_0 + h) = \emptyset$. [Theorem 2.5](#) is proved. $\square$

Summarizing all the above results, we obtain the following statement.

THEOREM 2.6. *The duality between the measure λ and the Baire category with respect to the sentence P_0 , where*

$$\begin{aligned} P_0 = (\forall X) \quad (X \subseteq \ell_2, X \text{ is a Baire subset of second category} \\ \rightarrow (\exists \delta) (\delta > 0 \rightarrow (\forall x) (\|x\| < \delta \rightarrow X \cap (X+x) \neq \emptyset))), \end{aligned} \quad (2.9)$$

is not valid.

REMARK 2.7. By [Remark 2.3](#) and [Theorem 2.4](#), it is easy to obtain the validity of the duality between the linear Lebesgue measure and the Baire category with respect to the sentence P_0 in $\mathbb{R}$. This result is essentially due to Oxtoby and may be called Oxtoby duality principle in $\mathbb{R}$ (cf. [4]).

REMARK 2.8. [Theorem 2.6](#) states that an analogy of the Oxtoby duality principle is not valid for the measure λ and the Baire category in the infinite-dimensional separable Hilbert space ℓ_2 .

There are also several important works devoted to the solution of analogous problems in various topological vector spaces (cf. [2, 3] and others).

The following notion is frequently useful in studying various questions of measure theory.

We say that the measure μ defined in a topological vector space (E, T) satisfies the axiom of Steinhaus if the following condition:

$$\begin{aligned} (\forall X) \quad (X \in \text{dom}(\mu), \mu(X) < \infty \\ \rightarrow (\forall \epsilon) (\epsilon > 0 \rightarrow ((\text{there exists a neighborhood } V_\epsilon \text{ of the zero vector } 0), \\ ((\forall h) (h \in V_\epsilon \rightarrow \mu((X+h) \triangle X) < \epsilon)))))) \end{aligned} \quad (2.10)$$

holds.

THEOREM 2.9. *The measure λ does not satisfy the axiom of Steinhaus.*

PROOF. Assume the contrary. Then for the set Δ_0 and for the number $\epsilon = 1/2$, there exists a number $\delta > 0$ such that

$$(\forall x) \left(\|x\| < \delta \rightarrow \lambda((\Delta_0 + x) \triangle \Delta_0) < \frac{1}{2} \right). \quad (2.11)$$

Consider the element $h = (h_k)_{k \in \mathbb{N}}$ constructed in [Theorem 2.5](#). Since $\|h\| < \delta$, $(\Delta_0 + h) \cap (\bigcup_{n \in \mathbb{N}} A_n) = \emptyset$, and the measure λ is concentrated on the set $\bigcup_{n \in \mathbb{N}} A_n$, where A_n is defined in [Section 1](#) for $\Delta = \Delta_0$, we have $\lambda((\Delta_0 + h) \triangle \Delta_0) = \lambda(\Delta_0) = 1$. This contradicts the condition

$$\lambda((\Delta_0 + h) \triangle \Delta_0) < \frac{1}{2}. \quad (2.12)$$

Thus, [Theorem 2.9](#) is proved. $\square$

REMARK 2.10. We must say that the analogies of Theorems [2.6](#) and [2.9](#) are valid for an arbitrary nontrivial σ -finite Borel measure and Baire category defined in infinite-dimensional Polish topological vector space, but this question will not concern us here.

EXAMPLE 2.11. Define the measure μ_0 by

$$(\forall B) \quad (B \in \mathcal{B}(\ell_2) \rightarrow \mu_0(B) = \begin{cases} \infty, & \text{if } B \text{ is of second category,} \\ 0, & \text{if } B \text{ is of first category.} \end{cases}) \quad (2.13)$$

It is proved that, on the one hand, the measure μ_0 satisfies Suslin's property and is invariant with respect to the vector space ℓ_2 (see [\[3\]](#)). On the other hand, using [Theorem 2.2](#), we conclude that the measure μ_0 (unlike the measure λ) satisfies

$$\begin{aligned} (\forall X) \quad (X \in \mathcal{B}(\ell_2), \mu(X) > 0 \\ \rightarrow (\exists \delta) (\delta > 0 \rightarrow (\forall h) (\|h\| < \delta \rightarrow (X + h) \cap X \neq \emptyset))). \end{aligned} \quad (2.14)$$

This means that the duality between the measure μ_0 (which is not σ -finite) and the Baire category, with respect to the property P_0 , is valid in the separable Hilbert space ℓ_2 . Also note that the measure μ_0 satisfies the axiom of Steinhaus.

REMARK 2.12. Clearly, it is not possible to define, in the space ℓ_2 , a translation-invariant nontrivial σ -finite Borel measure. But if we ignore the condition of σ -finiteness, then in some consistent system of axioms, the construction of such Borel measures is possible (cf. [\[5\]](#)). In connection with the above results, one can pose the problem of the validity of the duality between the translate-invariant Borel measure and the Baire category with respect to the property P_0 in the infinite-dimensional separable Hilbert space ℓ_2 .

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