CRITICAL POINT THEOREMS

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Received 12 December 2000

Let H be a Hilbert space such that $H = V \oplus W$, where V and W are two closed subspaces of H . We generalize an abstract theorem due to Lazer et al. (1975) and a theorem given by Moussaoui (1990-1991) to the case where V and W are not necessarily finite dimensional. We give two mini-max theorems where the functional $\Phi : H \rightarrow \mathbb{R}$ is of class $\mathcal{C}^2$ and $\mathcal{C}^1$, respectively.

2000 Mathematics Subject Classification: 58E05.

1. Introduction. Our purpose in this note is to generalize a mini-max theorem due to Lazer et al. [3]. Their theorem is as follows.

THEOREM 1.1. *Let X and Y be two closed subspaces of a real Hilbert space H such that X is finite dimensional and $H = X \oplus Y$ (X and Y not necessarily orthogonal). Let $\Phi : H \rightarrow \mathbb{R}$ be a C^2 functional and let $\nabla\Phi$ and $D^2\Phi$ denote the gradient and Hessian of Φ , respectively. Suppose that there exist two positive constants m_1 and m_2 such that*

$$(D^2\Phi(u)h, h) \leq -m_1\|h\|^2, \quad (D^2\Phi(u)k, k) \geq m_2\|k\|^2 \quad (1.1)$$

for all $u \in H$, $h \in X$, and $k \in Y$. Then Φ has a unique critical point, that is, there exists a unique $v_0 \in H$ such that $\nabla\Phi(v_0) = 0$. Moreover, this critical point is characterized by the

$$\Phi(v_0) = \max_{x \in X} \min_{y \in Y} \Phi(x + y). \quad (1.2)$$

Bates and Ekeland in [1] generalized Theorem 1.1 to the case where X and Y are not necessarily finite dimensional. Via a reduction method, Manasevich considered the same case in [4], but he supposed weaker conditions on Hessian of Φ . On the other hand, Tersian [7] studied the case where X and Y are not necessarily finite dimensional, $\nabla\Phi : H \rightarrow H$ is everywhere defined and hemicontinuous on H , which means that

$$\lim_{t \rightarrow 0} \nabla\Phi(u + tv) = \nabla\Phi(u) \quad \forall u, v \in H. \quad (1.3)$$

Instead of the conditions on the Hessian of Φ , they supposed

- (1) $(\nabla\Phi(h_1 + y) - \nabla\Phi(h_2 + y), h_1 - h_2) \leq -m_1\|h_1 - h_2\|^2$ $h_1, h_2 \in X$, $y \in Y$,
- (2) $(\nabla\Phi(x + k_1) - \nabla\Phi(x + k_2), k_1 - k_2) \geq m_2\|k_1 - k_2\|^2$ $k_1, k_2 \in Y$, $x \in X$,

where $H = X \oplus Y$, m_1 and m_2 are strictly positive.

Their result rests heavily upon two theorems on α -convex functionals and an existence theorem for a class of monotone operators due to Browder. By a completely

different method, the second author gave another version of Theorem 1.1 (see [5]) with convexity conditions that are weaker than those assumed above.

THEOREM 1.2. *Let H be a Hilbert space such that $H = V \oplus W$ where V is a finite-dimensional subspace of H and W its orthogonal. Let $\Phi : H \rightarrow \mathbb{R}$ be a functional such that*

- (i) Φ is of class $\mathcal{C}^1$.
  - (ii) Φ is coercive on W .
  - (iii) For fixed $w \in W$, $v \mapsto \Phi(v + w)$ is concave on V .
  - (iv) For fixed $w \in W$, $\Phi(v + w) \rightarrow -\infty$ when $\|v\| \rightarrow +\infty$, $v \in V$; and the convergence is uniform on bounded subsets of W .
  - (v) For all $v \in V$, Φ is weakly lower semicontinuous on $W + v$.
- Then Φ admits a critical point in H .

We consider the case where X and Y are not necessarily finite dimensional. Our proofs contain many steps used in [5] and our convexity conditions are weaker than those given by other authors. First, we prove a mini-max theorem where $\Phi : H \rightarrow \mathbb{R}$ is of class $\mathcal{C}^2$. Next, we prove the existence theorem for a particular class of $\mathcal{C}^1$ functional $\Phi : H \rightarrow \mathbb{R}$.

2. First abstract result. The next two propositions are used in this work. For a proof of Proposition 2.1, see [2], and for a proof of Proposition 2.2, see [6].

PROPOSITION 2.1. *Let X be a reflexive Banach space and let $\Phi : X \rightarrow \mathbb{R}$ be a functional such that*

- (i) Φ is weakly lower semicontinuous on X ,
  - (ii) Φ is coercive, that is, $\Phi(u) \rightarrow +\infty$ when $\|u\| \rightarrow +\infty$,
- then Φ is lower bounded and there exists $u_0 \in X$ such that

$$\Phi(u_0) = \inf_X \Phi. \quad (2.1)$$

PROPOSITION 2.2. *Let H be a real Hilbert space and let L be a bounded linear operator on H . Suppose that*

$$(Lx, x) \geq a\|x\|^2, \quad (2.2)$$

for all $x \in H$ and a is a strictly positive real number. Then L is an isomorphism onto H and $\|L^{-1}\| \leq a^{-1}$.

THEOREM 2.3. *Let H be a Hilbert space such that $H = V \oplus W$ where V and W are two closed and orthogonal subspaces of H . Let $\Phi : H \rightarrow \mathbb{R}$ be a functional such that*

- (i) Φ is of class $\mathcal{C}^2$.
- (ii) There exists a continuous nonincreasing function $\gamma : [0, +\infty) \rightarrow]0, \infty)$ such that

$$\langle D^2\Phi(v + w)g, g \rangle \leq -\gamma(\|v\|)\|g\|^2 \quad (2.3)$$

for all $v \in V$, $w \in W$, and $g \in V$.

- (iii) Φ is coercive on W .
- (iv) For all $w \in W$, $\Phi(v + w) \rightarrow -\infty$ when $\|v\| \rightarrow +\infty$, $v \in V$.
- (v) Φ is weakly lower semicontinuous on $W + v$.

Then Φ admits at least a critical point $u \in H$. Moreover, this critical point of Φ is characterized by the equality

$$\Phi(u) = \min_{w \in W} \max_{v \in V} \Phi(v + w). \quad (2.4)$$

In the proof of [Theorem 2.3](#), we will use the following three lemmas.

LEMMA 2.4. *For all $w \in W$, there exists a unique $v \in V$ such that*

$$\Phi(v + w) = \max_{g \in V} \Phi(g + w). \quad (2.5)$$

PROOF. From [Theorem 2.3\(ii\)](#), for w fixed in W , $v \mapsto \Phi(v + w)$ is continuous and strictly concave on V . Then, it is weakly upper semicontinuous on V . Moreover, from [Theorem 2.3\(iv\)](#), it is anticoercive on V . So that it admits a maximum on V . We affirm that this maximum is unique, otherwise we suppose that there exists two maximums v_1 and v_2 . Let $v_\lambda = \lambda v_1 + (1 - \lambda)v_2$ for $0 < \lambda < 1$, then

$$\Phi(v_\lambda + w) > \lambda \Phi(v_1 + w) + (1 - \lambda)\Phi(v_2 + w) = \Phi(v_1 + w) = \Phi(v_2 + w). \quad (2.6)$$

For the rest of the note, we will adopt the notations

$$\begin{aligned} \bar{V}(w) &= \left\{ v \in V : \Phi(v + w) = \max_{g \in V} \Phi(g + w) \right\}, \\ S &= \{ u = v + w, w \in W, v \in \bar{V}(w) \}. \end{aligned} \quad (2.7) \quad \square$$

LEMMA 2.5. *There exists $u \in S$ such that*

$$\Phi(u) = \inf_S \Phi. \quad (2.8)$$

PROOF. There exists a sequence (u_n) of S such that $\Phi(u_n) \rightarrow \inf_S \Phi = a$. For all n , $u_n = v_n + w_n$ with $w_n \in W$, and $v_n \in \bar{V}(w_n)$.

Claim

$$\|w_n\| \leq c. \quad (2.9)$$

Otherwise,

$$\Phi(u_n) = \Phi(v_n + w_n) \geq \Phi(w_n). \quad (2.10)$$

From [Theorem 2.3\(iii\)](#), $\Phi(w_n) \rightarrow +\infty$, hence $\Phi(u_n) \rightarrow +\infty$. This gives a contradiction. Moreover, from (2.9), there exists a subsequence also denoted w_n such that $w_n \rightharpoonup w$. Take v in V , by [Theorem 2.3\(v\)](#), we have

$$\Phi(v + w) \leq \liminf_n \Phi(v + w_n) \leq \liminf_n \Phi(v_n + w_n) = a. \quad (2.11)$$

This is true for all $v \in V$, in particular, for $v \in \bar{V}(w)$. Then $u = v + w$ satisfies (2.8). $\square$

LEMMA 2.6. *The application $\bar{V} : W \rightarrow V$ such that*

$$\Phi(w + \bar{V}(w)) = \max_{g \in V} \Phi(g + w) \quad (2.12)$$

is of class C^1 .

PROOF OF THEOREM 2.3. For each $w \in W$, let $\Phi_w : V \rightarrow \mathbb{R}$ be defined by $\Phi_w(v) = \Phi(v + w)$. Then $\Phi_w \in C^2(V, \mathbb{R})$ and for $v' \in V$, we have

$$\begin{aligned} (\nabla \Phi_w(v), v') &= (\nabla \Phi(v + w), v'), \\ (D^2 \Phi_w(v) v', v') &= (D^2 \Phi(v + w) v', v'). \end{aligned} \quad (2.13)$$

By Lemma 2.4, we conclude that for all $w \in W$, there exists a unique v_w in V such that $\nabla \Phi_w(v_w) = 0$. To prove that $\tilde{V} \in C^1(W, V)$, we will use the implicit function theorem. To see this, let P denote the orthogonal projection of H onto V . Then

$$v = \tilde{V}(w) \quad \text{iff} \quad P \nabla \Phi(w + v) = 0. \quad (2.14)$$

Next, we define $E : W \times V \rightarrow V$ by

$$E(w, v) = P \nabla \Phi(w + v). \quad (2.15)$$

Then E is of class C^1 and given any pair $w_0 \in W$, $v_0 \in V$ such that $E(w_0, v_0) = 0$, it follows that $v_0 = \tilde{V}(w_0)$.

If E_v denotes the partial derivative of E with respect to v , and if $v' \in V$, we have

$$E_v(w_0, v_0) v' = P D^2 \Phi(w_0 + v_0) v'. \quad (2.16)$$

The mapping $E_v(w_0, v_0) : V \rightarrow V$ is linear and bounded we have from Theorem 2.3(ii)

$$(E_v(w_0, v_0) v', v') = (D^2 \Phi(w_0 + v_0) v', v') \leq -\gamma(\|v_0\|) \|v'\|^2, \quad (2.17)$$

for all $v' \in V$. By Proposition 2.2, $E_v(w_0, v_0)$ is an isomorphism onto V . Then from the implicit function theorem [2], there exists a C^1 mapping f from a neighborhood U of w_0 in W into V such that $E(w, f(w)) = 0$ for all $w \in U$. Moreover, from (2.14) and (2.15), $f(w) = \tilde{V}(w)$ for all $w \in W$. Hence, since w_0 was arbitrarily chosen, it follows that f can be defined over all of W . Then we conclude that $\tilde{V} \in C^1(W, V)$. $\square$

REMARK 2.7. The proof of Lemma 2.6 relies on the implicit function theorem. This theorem was used by Thews in [8] to prove the existence of a critical point for a particular class of functionals. It was also used by Manasevich in [4].

PROOF. Let $w \in W$ and $u \in S_w$. We will prove that if u satisfies (2.8), then u is a critical point of Φ . By Lemma 2.4, it is easy to see that $(\nabla \Phi(u), g) = 0$ for all $g \in V$, so it suffices to prove that

$$(\nabla \Phi(u), h) = 0 \quad \forall h \in W. \quad (2.18)$$

Recall that $u \in S$ can be written $u = w + v$ where $w \in W$ and $v \in \tilde{V}(w)$. Take $h \in W$ and let $w_t = w + th$ for $|t| \leq 1$. For each t such that $0 < |t| \leq 1$, there exists a unique $v_t \in V(w_t)$. By Lemma 2.6, we conclude that v_{t_n} converge to a certain v_0 and that $v_0 \in \tilde{V}(w)$. Then, by Lemma 2.4, $v_0 = v$. For $t > 0$, we have

$$\frac{\Phi(w_t + v_t) - \Phi(v_t + w)}{t} \geq \frac{\Phi(w_t + v_t) - \Phi(v_0 + w)}{t} \geq 0. \quad (2.19)$$

Then,

$$(\nabla\Phi(v_t + w + \lambda_t th), h) \geq 0 \quad \text{for } 0 < \lambda_t < 1. \quad (2.20)$$

At the limit, we obtain

$$(\nabla\Phi(u), h) = 0 \quad \forall h \in W. \quad (2.21)$$

Hence, u is a critical point of Φ . $\square$

3. Second abstract result. Let H be a Hilbert space such that $H = V \oplus W$ where V and W are two closed and orthogonal subspaces of H . Let $\Phi : H \rightarrow \mathbb{R}$ be such that

$$\begin{aligned} \Phi &= q + \psi, \\ q(v + w) &= q(v) + q(w) \quad \forall (v, w) \in V \times W \\ \psi &\text{ is weakly continuous on } H. \end{aligned} \quad (3.1)$$

THEOREM 3.1. *Let H be a Hilbert space such that $H = V \oplus W$ where V and W are two closed and orthogonal subspaces of H . Let $\Phi : H \rightarrow \mathbb{R}$ be a functional verifying (3.1) such that*

- (i) q and ψ are of class $\mathcal{C}^1$.
- (ii) $\nabla\Phi$ is weakly continuous on H .
- (iii) Φ is coercive on W .
- (iv) For a fixed $w \in W$, $v \mapsto \Phi(v + w)$ is concave on V .
- (v) For a fixed $w \in W$, $\Phi(v + w) \rightarrow -\infty$ when $\|v\| \rightarrow +\infty$, $v \in V$; and the convergence is uniform on the bounded sets of W .
- (vi) For a fixed $v \in V$, Φ is weakly lower semicontinuous on $W + v$.

Then Φ admits a critical point $u \in H$. Moreover, this critical point is characterized by the equality

$$\Phi(u) = \min_{w \in W} \max_{v \in V} \Phi(v + w). \quad (3.2)$$

For the proof of [Theorem 3.1](#), we use some results of [Lemmas 2.4](#) and [2.5](#) and we need also the following lemmas.

LEMMA 3.2. *For each $w \in W$, $\tilde{V}(w)$ is convex.*

PROOF. Take $v_1, v_2 \in \tilde{V}(w)$ and $v_\lambda = \lambda v_1 + (1 - \lambda)v_2$ with $\lambda \in [0, 1]$. So that from [Theorem 3.1\(iv\)](#), we have $\Phi(v_\lambda + w) \geq \lambda\Phi(v_1 + w) + (1 - \lambda)\Phi(v_2 + w) = \Phi(v_1 + w) = \Phi(v_2 + w)$. Then

$$\Phi(v_\lambda + w) = \Phi(v_1 + w) = \Phi(v_2 + w). \quad (3.3)$$

So $v_\lambda \in \tilde{V}(w)$. $\square$

LEMMA 3.3. *Let $L(w) = \{\nabla\Phi(v + w) : v \in \tilde{V}(w)\}$. For each $w \in W$,*

- (i) $L(w)$ is convex.
- (ii) $L(w)$ is closed.

PROOF. (i) Let $h \in W$ and $v_1, v_2 \in \tilde{V}(w)$. From [Theorem 3.1](#)(iv) and [Lemma 3.2](#), we have for all $t > 0$,

$$\begin{aligned} \Phi(v_\lambda + w + th) - \Phi(v_\lambda + w) &\geq \lambda(\Phi(v_1 + th + w) - \Phi(v_1 + w)) \\ &\quad + (1 - \lambda)(\Phi(v_2 + th + w) - \Phi(v_2 + w)). \end{aligned} \quad (3.4)$$

Divide by t and let t tend to 0, then

$$(\nabla\Phi(v_\lambda + w), h) \geq \lambda(\nabla\Phi(v_1 + w), h) + (1 - \lambda)(\nabla\Phi(v_2 + w), h). \quad (3.5)$$

Since this is true for all $h \in W$, we conclude that

$$\nabla\Phi(v_\lambda + w) = \lambda\nabla\Phi(v_1 + w) + (1 - \lambda)\nabla\Phi(v_2 + w). \quad (3.6)$$

(ii) For $w \in W$, let $S_w = \{v + w : v \in \tilde{V}(w)\}$.

First, we show that S_w is closed. Let $v_n + w \in S_w$ such that $v_n + w \rightarrow v_0 + w$. $\Phi(v_n + w) \rightarrow \Phi(v_0 + w)$ and $\Phi(v_n + w) = \max_{g \in \tilde{V}} \Phi(g + w)$. Then $v_0 + w \in S_w$.

Next, we affirm that S_w is bounded. If not, there exists v_n of $\tilde{V}(w)$ such that $\|v_n\| \rightarrow +\infty$, and we conclude from [Theorem 3.1](#)(v) that $\Phi(v_n + w) \rightarrow -\infty$. This gives a contradiction.

Consequently, S_w is closed and bounded. Since S_w is convex, we conclude that S_w is weakly compact. From [Theorem 3.1](#)(ii), it follows that $L(w)$ is weakly compact. Then $L(w)$ is weakly closed. Thus $L(w)$ is closed. $\square$

PROOF OF THEOREM 3.1. Let $w \in W$ and $u \in S_w$. If u satisfies (2.8), we will show that $L(w)$ contains 0 and there exists $v \in \tilde{V}(w)$ such that

$$\nabla\Phi(v + w) = 0. \quad (3.7)$$

By contradiction, suppose that $L(w)$ does not contain 0. Since it is convex and closed in the Hilbert space, there exists $h_1 \in L(w)$ such that

$$0 \neq \|h_1\| = \inf \{\|h\| : h \in L(w)\}. \quad (3.8)$$

Let $h \in L(w)$, $h_1 + \lambda(h - h_1) \in L(w)$ for $\lambda \in [0, 1]$, thus

$$(h_1 + \lambda(h - h_1), h_1 + \lambda(h - h_1)) \geq \|h_1\|^2. \quad (3.9)$$

Hence

$$\|h_1\|^2 + 2\lambda(h - h_1, h_1) + \lambda^2\|h - h_1\|^2 \geq \|h_1\|^2, \quad (3.10)$$

so

$$2(h - h_1, h_1) + \lambda\|h - h_1\|^2 \geq 0. \quad (3.11)$$

When λ tends to 0. We obtain $(h - h_1, h_1) \geq 0$. So that $(h, h_1) \geq \|h_1\|^2 > 0$. Equivalently,

$$(\nabla\Phi(v + w), h_1) > 0 \quad \forall v \in \tilde{V}(w). \quad (3.12)$$

Denote $w_t = w + th_1$ for $|t| \leq 1$. We note that $w_t \in W$. By [Lemma 2.4](#), for each $0 < |t| \leq 1$, there exists $v_t \in V(w_t)$. Since $\|w_t\| \leq \|w\| + \|h_1\|$, [Theorem 3.1\(v\)](#) implies that there exists a constant $A > 0$ such that

$$\Phi(v + w_t) < \inf_W \Phi \leq \Phi(w_t), \quad (3.13)$$

for $v \in V$, $\|v\| \geq A$, and $|t| \leq 1$. (Since Φ is coercive and weakly lower semicontinuous in the reflexive space W , it reaches its minimum.) It follows that

$$\|v_t\| \leq A. \quad (3.14)$$

Otherwise, we would have

$$\Phi(v_t + w_t) < \Phi(w_t), \quad (3.15)$$

which contradicts the fact that $v_t \in \bar{V}(w_t)$. We conclude then as V is reflexive that there exists a subsequence $t_n \rightarrow 0$ and $t_n < 0$ such that $v_{t_n} \rightharpoonup v_0 \in V$.

Claim

$$v_0 \in \bar{V}(w). \quad (3.16)$$

We have $v_{t_n} \rightharpoonup v_0$ and $w_{t_n} \rightarrow w$, so

$$v_{t_n} + w_{t_n} \rightharpoonup v_0 + w. \quad (3.17)$$

Since ψ is weakly upper semicontinuous on H , we have

$$\psi(v_0 + w) \geq \limsup_{n \rightarrow \infty} \psi(v_{t_n} + w_{t_n}). \quad (3.18)$$

By [Lemma 2.4](#), Φ is weakly upper semicontinuous on V and we know that ψ is weakly lower semicontinuous on V , so $q = \Phi - \psi$ is weakly upper semicontinuous on V . Then

$$q(v_0) \geq \limsup_{n \rightarrow \infty} q(v_{t_n}). \quad (3.19)$$

Moreover, the continuity of q implies that

$$q(w) = \lim_{n \rightarrow \infty} q(w_{t_n}) = \limsup_{n \rightarrow \infty} q(w_{t_n}). \quad (3.20)$$

Then

$$\begin{aligned} q(v_0 + w) &= q(v_0) + q(w) \\ &\geq \limsup_{n \rightarrow \infty} q(v_{t_n}) + \limsup_{n \rightarrow \infty} q(w_{t_n}) \\ &\geq \limsup_{n \rightarrow \infty} (q(v_{t_n}) + q(w_{t_n})) \\ &\geq \limsup_{n \rightarrow \infty} q(v_{t_n} + w_{t_n}). \end{aligned} \quad (3.21)$$

On the other hand, $v_{t_n} \in V(w_{t_n})$ implies that

$$\Phi(v_{t_n} + w_{t_n}) \geq \Phi(v + w_{t_n}) \quad \forall v \in V. \quad (3.22)$$

We then obtain

$$\begin{aligned}
 q(v_0 + w) + \psi(v_0 + w) &\geq \limsup_{n \rightarrow \infty} q(v_{t_n} + w_{t_n}) + \limsup_{n \rightarrow \infty} \psi(v_{t_n} + w_{t_n}) \\
 &\geq \limsup_{n \rightarrow \infty} (q(v_{t_n} + w_{t_n}) + \psi(v_{t_n} + w_{t_n})) \\
 &\geq \limsup_{n \rightarrow \infty} (q(v + w_{t_n}) + \psi(v + w_{t_n})) \quad \forall v \in V \\
 &\geq q(v + w) + \psi(v + w) \quad \forall v \in V.
 \end{aligned} \tag{3.23}$$

Thus

$$\Phi(v_0 + w) \geq \Phi(v + w) \quad \forall v \in V. \tag{3.24}$$

Equivalently, $v_0 \in \tilde{V}(w)$.

Therefore, we have

$$-\frac{\Phi(w_{t_n} + v_{t_n}) - \Phi(v_{t_n} + w)}{t_n} \geq -\frac{\Phi(w_{t_n} + v_{t_n}) - \Phi(v_0 + w)}{t_n} \geq 0, \tag{3.25}$$

and so

$$(\nabla \Phi(v_{t_n} + w + \varepsilon_n t_n h_1), h_1) \leq 0 \quad \text{for } 0 < \varepsilon_n < 1. \tag{3.26}$$

When t_n tend to 0, by (ii), we deduce finally that

$$(\nabla \Phi(v_0 + w), h_1) \leq 0. \tag{3.27}$$

Which contradicts (3.8). Then there exists $v_1 \in \tilde{V}(w)$ such that $\nabla \Phi(v_1 + w) = 0$ and

$$\Phi(v_1 + w) = \min_{w \in W} \max_{v \in V} \Phi(v + w). \tag{3.28}$$

□

REMARK 3.4. In the proof of [Theorem 3.1](#), (3.1) allows us to show that $v_0 \in \tilde{V}(w)$. Or, we remark that we do not need to introduce ψ and q if $\Phi(v + w) = \Phi(v) + \Phi(w)$. Indeed, $w_{t_n} \rightarrow w$ and $v_{t_n} \rightarrow v_0$ imply that

$$\begin{aligned}
 \limsup \Phi(v_{t_n} + w_{t_n}) &= \limsup (\Phi(v_{t_n}) + \Phi(w_{t_n})) \\
 &\leq \limsup \Phi(v_{t_n}) + \limsup \Phi(w_{t_n}).
 \end{aligned} \tag{3.29}$$

By [Lemma 2.4](#), Φ is weakly upper semicontinuous on V , thus

$$\limsup \Phi(v_{t_n} + w_{t_n}) \leq \Phi(v_0) + \Phi(w) = \Phi(v_0 + w). \tag{3.30}$$

On the other hand, $v_{t_n} \in \tilde{V}(w_{t_n})$ implies that

$$\Phi(v_{t_n} + w_{t_n}) \geq \Phi(v + w_{t_n}) \quad \forall v \in V. \tag{3.31}$$

So

$$\limsup \Phi(v_{t_n} + w_{t_n}) \geq \Phi(v + w) \quad \forall v \in V. \tag{3.32}$$

Then

$$\Phi(v_0 + w) \geq \Phi(v + w) \quad \forall v \in V, \tag{3.33}$$

that is, $v_0 \in \tilde{V}(w)$.

REMARK 3.5. We can also prove [Theorem 3.1](#) for any functional $\Phi : H \rightarrow \mathbb{R}$ without introducing ψ and q if Φ is weakly upper semicontinuous on H .

ANOTHER VERSION OF THEOREM 3.1. Let A be a convex set. The function $f : A \rightarrow \mathbb{R}$ is quasiconcave if for all x_1, x_2 in A , and for all λ in $]0, 1[$, then

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \min(f(x_1), f(x_2)). \quad (3.34)$$

The function f is quasiconvex if $(-f)$ is quasiconcave, and it is strictly quasiconcave if the inequality above is strict.

It is clear that any strictly concave function is strictly quasiconcave.

PROPOSITION 3.6. Let E be a reflexive Banach space. If $\Phi : E \rightarrow \mathbb{R}$ is quasiconcave and upper semicontinuous, then Φ is weakly upper semicontinuous.

THEOREM 3.7. Let E be a reflexive Banach space such that $E = V \oplus W$ where V and W are two closed subspaces of E not necessarily orthogonal. Let $\Phi : H \rightarrow \mathbb{R}$ be a functional satisfying (3.1) such that

- (i) q and ψ are of class $\mathcal{C}^1$.
- (ii) $\nabla \Phi$ is weakly continuous.
- (iii) Φ is coercive on W .
- (iv) For all $w \in W$, $v \mapsto \Phi(v + w)$ is strictly quasiconcave on V .
- (v) For all $w \in W$, $\Phi(v + w) \rightarrow -\infty$ when $\|v\| \rightarrow +\infty$, $v \in V$; and the convergence is uniform on bounded subsets of W .
- (vi) For all $v \in V$, Φ is lower weakly semicontinuous on $W + v$.

Then Φ admits a critical point $u \in H$. Moreover, this critical point is characterized by the equality

$$\Phi(u) = \min_{w \in W} \max_{v \in V} \Phi(v + w). \quad (3.35)$$

In the proof of this theorem, we need [Lemmas 2.4](#) and [2.5](#). We note that by [Proposition 3.6](#), the result of [Lemma 2.4](#) is still true in this case.

PROOF. We will prove that $u \in S$ obtained in [Lemma 2.5](#) is a critical point of Φ . We have $\langle \Phi'(u), v \rangle$ for all $v \in V$, so it is sufficient to show that $\langle \Phi'(u), h \rangle = 0$ for all $h \in W$. Recall that $u \in S$ can be written as $u = v + w$ where $w \in W$ and $v \in \tilde{V}(w)$. Let $h \in W$ and $w_t = w + th$ for $|t| \leq 1$. For all t such that $0 < |t| \leq 1$, there exists a unique $v_t \in \tilde{V}(w_t)$. In the same way as in the proof of [Theorem 3.1](#), we can extract a subsequence v_{t_n} such that $v_{t_n} \rightharpoonup v_0$ and $v_0 \in \tilde{V}(w)$. By [Lemma 2.4](#), we deduce that $v_0 = v$. Hence for $t > 0$, we have

$$\langle \Phi'(u), h \rangle = 0 \quad \forall h \in W. \quad (3.36)$$

Then, u is a critical point of Φ . □

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