MULTIPLIERS ON L(S), L(S)**, AND LUC(S)* FOR A LOCALLY COMPACT TOPOLOGICAL SEMIGROUP

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We study compact and weakly compact multipliers on L(S), L(S)**, and LUC(S)*, where the latter is the dual of LUC(S). We show that a left cancellative semigroup S is left amenable if and only if there is a nonzero compact (or weakly compact) multiplier on L(S)**. We also prove that S is left amenable if and only if there is a nonzero compact (or weakly compact) multiplier on LUC(S)*.

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1. Introduction. Let S be a locally compact, Hausdorff topological semigroup. Let M(S) be the space of all complex Borel measures on S. It is known that M(S) = C₀(S)*, therefore, M(S) is a Banach space and with convolution μ * ν(ψ) = ∫∫ ψ(xy)dμ(x)dν(y) (μ, ν ∈ M(S), ψ ∈ C₀(ψ)), M(S) is a Banach algebra. The subalgebra L(S) of M(S) is defined by L(S) = {μ ∈ M(S) | x → |μ| * δₓ, x → δₓ * |μ| from S to M(S) are norm continuous} [1]. A semigroup S is called foundation if S = ∪µ∈L(S) supp µ. A trivial example is a topological group and in this case L(S) = L¹(G). Let C₀(S) be the set of all bounded continuous function on S. Let LUC(S) = {f | f ∈ C₀(S), x → rₓf is norm continuous} where rₓf(y) = f(yx) and LUC(S)* be the dual of LUC(S). Similarly, LUC(S)* is a Banach algebra. A linear map on a Banach algebra A is called a multiplier if

2. Preliminaries. For a Banach algebra A, we denote by A* and A** the first and second dual of A, respectively. On A** we define the first Arens product by

\[ \langle mn, f \rangle = \langle m, nf \rangle, \quad \langle nf, a \rangle = \langle n, fa \rangle, \quad \langle fa, b \rangle = f(ab) \quad (2.1) \]

(m, n ∈ A**; f ∈ A*; a, b ∈ A). With this product A** is a Banach algebra. We can also define a similar product on LUC(S)* such that \( \langle mn, f \rangle = \langle m, nf \rangle, \quad n f(\chi) = n( l_\chi f), \quad l_\chi f(y) = f(\chi y) \quad \text{for } (m, n ∈ LUC(S)*; f ∈ LUC(S); \chi, y ∈ S) \). Clearly, LUC(S)* is a Banach algebra. A linear map on a Banach algebra A is called a multiplier if
Therefore, \( T(xy) = T(x)y = xT(y) \) \((x, y \in A)\). The left (right) multiplier on \( L(S)^{**} \) is defined by \( l_m(n) = mn, (l_m(n) = nm)\). In general, \( LUC(S) \) and \( RUC(S) \) are different subalgebras of \( C_0(S) \) and \( LUC(S) = RUC(S) \) if and only if \( LUC(S) \) (resp., \( RUC(S) \)) is right \( \pi \)-ideal, (see [2, Theorem 4.4.5]). For example, if \( S \) is a compact semitopological semigroup or a totally bounded topological group, then \( LUC(S) = RUC(S) \) [2].

The semigroup \( S \) is called left amenable if there is a positive functional \( m \) on \( LUC(S) \) such that \( m(l_a f) = m(f), \| m \| = 1 \) for all \( f \in LUC(S), a \in S \). Such \( m \) is called a left invariant mean on \( LUC(S) \) [7]. Let \( A \) be a Banach algebra and \( B \) a closed subalgebra of \( A \) and \( i : B \rightarrow A \) the inclusion mapping, then \( \pi : A^* \rightarrow B^* \) is the restriction mapping which is norm decreasing and onto (by the Hahn-Banach theorem). Following Ghahramani and Lau [3], we have the following lemma (see [3, Lemmas 1.1, 1.2, 1.4, Proposition 1.3]).

**Lemma 2.1.** (a) Let \( f \in A^*, b \in B \). Then \( b\pi(f) = \pi(i(b)f) \).
(b) The mapping \( \pi^*: B^{**} \rightarrow A^{**} \) is a homeomorphism whenever \( B^{**} \) has the weak*-topology and \( \pi^*(B^{**}) \) has the relative weak*-topology.

**Lemma 2.2.** Let \( B \) be a closed ideal in \( A, n \in A^{**} \). If \( (a_\alpha) \) is a bounded net in \( A \) such that \( a_\alpha \rightarrow n, \) then \( i(b)a_\alpha \rightarrow \pi^*(b)n \) \((b \in B)\).

**Proposition 2.3.** Let \( B \) be a right (or left) ideal of \( A \). Then \( \pi^*(B^{**}) \) is a right (resp., left) ideal of \( A^{**} \).

**Lemma 2.4.** Let \( A \) be a commutative Banach algebra. Then any weak*-closed right ideal in \( A^{**} \) is an ideal. If \( X = \text{spec } A \), then \( h(n) = (n, \delta_X) \) is a multiplicative on \( A^{**} \), where \( \delta_X(\psi) = (x, \psi) \).

3. Multipliers on \( LUC(S)^* \) and \( L(S)^{**} \). First we prove a theorem which is new even for topological groups.

**Theorem 3.1.** Let \( S \) be a right cancellative topological semigroup with identity \( e \). Then the following are equivalent:
(a) \( S \) is left amenable.
(b) There is a nonzero compact (or weakly compact) right multiplier on \( LUC(S)^* \).

**Proof.** (a)\( \Rightarrow \) (b). Let \( S \) be left amenable and \( m \) be a left invariant mean on \( LUC(S) \).
Then \( \langle nm, f \rangle = \langle n, mf \rangle, \) \( mf(x) = m(l_xf) = m(f) \) \((f \in LUC(S)^*, f \in LUC(S))\).
Therefore, \( \langle nm, f \rangle = \langle n, m(f) \rangle = m(f)\langle n, 1 \rangle \), that is, \( nm = \langle n, 1 \rangle m \). Thus \( l_m(n) = \langle n, 1 \rangle m \) is a rank one operator and hence compact.

(b)\( \Rightarrow \) (a). Let \( T \) be a nonzero weakly compact right multiplier on \( LUC(S)^* \). Then \( T(m) = T(m\delta_e) = mT(\delta_e) = l_T(\delta_e)m \). So, \( T = l_n \) where \( n = T(\delta_e) \). Note that \( \delta_e \in LUC(S)^* \) and \( \delta_e(f) = f(e) \) \((f \in LUC(S))\). Now, let \( A = \{ \delta_X n \mid x \in S \} = \{ \delta_X T(\delta_e) \mid x \in S \} = \{ T(\delta_X) \mid x \in S \} \) which is weakly compact. By Krein-Smulian’s theorem \( K = \overline{\omega^o A} \) is weakly compact [2]. Now, we show that if \( k \neq k' \in K \), then \( \| \delta_X k_1 \| \leq \| k_1 \| \). On the other hand, if we define

\[
g(y) = \begin{cases} 
  f(t), & y = tx, \\
  0, & \text{otherwise},
\end{cases}
\] (3.1)
then $g$ is well defined and belongs to $\beta(S)$ (the space of bounded functions on $S$), then $\delta_x g(t) = \delta_x (l_x g) = g(tx) = r_x g(t) = f(t)$. Let $\hat{k}_1$ be the extension of $k_1$ to $\beta(S)$ (by the Hahn-Banach theorem). Then

$$
||k_1|| = ||\hat{k}|| \leq \sup \{ ||\langle \hat{k}_1, f \rangle \mid f \in \beta(S) \}
= \sup \{ ||\langle \hat{k}_1, \delta_x g \rangle \mid g \in \beta(S) \}
= \sup \{ ||\delta_x \hat{k}_1, g \rangle \mid g \in \beta(S) \}
= ||\delta_x k_1||
= ||\delta_x k||.
$$

It follows that $||\delta_x k_1|| = ||k_1|| \neq 0$. Now, we show that if $k, k' \in \co(A)$, and $k \neq k'$, then a similar argument shows that $||\delta_x (k - k')|| \neq 0$. Finally, we show that $0 \notin \{ \delta_x (k - k') \mid x \in S \}$ since, by a completely similar argument, we have $||\delta_x (k - k')|| = ||k - k'|| \neq 0$. Therefore, $0 \notin \{ \delta_x (k - k') \mid x \in S \}^-$. Hence, by Ryll-Nardzewski fixed point theorem [2], there exists a point $q \in K$ such that $\delta_x q = q$. It follows that $\delta_x |q| = |\delta_x q| = |q|$, and $||q|| = ||n|| \neq 0$. Now, if we take $m = |q|/||q||$, then clearly $\delta_x m = m$, so, $m(f) = \delta_x m(f) = \delta_x (m f) = m(f(x)) = m(x f)$. Therefore, $m$ is a left invariant mean on $LUC(S)$, that is, $S$ is left amenable.

For a foundation semigroup $S$, let $i : LUC(S) \to L(S)^*$ be such that $\langle i(f), \mu \rangle = \langle \mu, f \rangle$ ($f \in LUC(S), \mu \in L(S)$) is an embedding and $\pi = i^* : L(S)^* \to LUC(S)^*$ is onto. It is clear from the proof of [3, Lemma 2.2] for topological groups that $\pi(E) = \delta_e$ where $E$ is a right identity, $\pi$ is a homomorphism and $FG = F\pi(G)$. Also we have the following proposition which is similar to [6, Theorem 2.3].

We prove the following proposition for foundation semigroups with identity $e$.

**Proposition 3.2.** Let $E$ be a right identity in $L(S)^*$. Then $\pi$ is an isometric isomorphism of $EL(S)^*$ onto $LUC(S)^*$.

**Proof.** Let $I$ be the identity operator on $L(S)^*$. Then

$$
L(S)^* = EL(S)^* + (I - E)L(S)^*.
$$

Now, if $m \in L(S)^*$, then $\pi((I - E)m) = \pi(m) - \pi(E)\pi(m) = \pi(m) - \delta_e \pi(m) = \pi(m) - \pi(m) = 0$. Thus $(I - E)m \in \ker \pi$. On the other hand, if $m \in \ker \pi$, then $Em = E\pi(m) = 0$. So $m = m - Em = (I - E)m \in (I - E)L(S)^*$. Thus,

$$
\ker \pi = (I - E)L(S)^*.
$$

So, we have

$$
L(S)^* = EL(S)^* + \ker \pi.
$$

It follows that $\pi$ is injective from $EL(S)^*$ onto $L(S)^*/\ker \pi$, therefore $\pi$ is injective from $EL(S)^*$ onto $LUC(S)^*$, and so $\pi$ is an algebra isomorphism. We also have $||Em|| = ||E\pi(m)|| \leq ||E|| ||\pi(m)|| = ||\pi(m)|| \leq ||m||$, since $\pi$ is a quotient map. Thus $||\pi(Em)|| \leq ||\pi|| ||Em|| \leq ||Em|| \leq ||\pi(m)||$. So $||\pi(Em)|| = ||\pi(m)|| = ||Em||$, that is, $\pi$ is an isometry.
Now, we have another main theorem.

**Theorem 3.3.** Let $S$ be a right cancellative locally compact foundation semigroup with identity $e$. Then the following are equivalent:

(a) $S$ is left amenable.

(b) There is a nonzero compact (or weakly compact) right multiplier on $L(S)^{**}$.

**Proof.** (a)⇒(b). The proof of this part exactly reads the same line of the proof of (a)⇒(b) of Theorem 3.1, so it is omitted.

(b)⇒(a). Let $T$ be a nonzero weakly compact right multiplier on $L(S)^{**}$, so $T = l_n$ for some $n \in L(S)^{**}$. Now $l_{En}$ is also a nonzero right multiplier on $EL(S)^{**}$ where $E$ is a right identity of $L(S)^{**}$ with norm 1, since $l_{En}(Em) = EmEn = Emn$. Now by Proposition 3.2, $\pi(EL(S)^{**}) = (LUC(S))^*$ is isometrically isomorphic. If we define $l'_n = l_{En} \circ \pi$, then $l'_n$ is a nonzero right multiplier on $LUC(S)^*$. Therefore, $S$ is left amenable.

In [3, Theorem 2.1] it was also shown that a locally compact group $G$ is amenable if and only if there is a nonzero compact (weakly compact) right multiplier on $M(G)^{**}$. But we are not able to extend this result to $M(S)^{**}$.

**Proposition 3.4.** A right multiplier $l_n(m) = mn$ ($m \in LUC(S)^*$) is compact if and only if the restriction of $l_n$ to $M(S)$ is compact.

**Proof.** Let $l_n$ be compact, then clearly the restriction of $l_n$ to $M(S)$ is compact. Conversely, let $l_n : M(S) \to LUC(S)^*$ be compact, where $l_n(\mu) = \mu n$ ($\mu \in M(S)$). Let $m \in LUC(S)^*$ with $\|m\| \leq 1$. Since, the linear span of $\delta_x$'s is weak$^*$-dense in $LUC(S)^*$, there is a net $\mu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \delta_{x_{\alpha,i}}$ such that $\mu_\alpha \to m$ in weak$^*$-topology. By compactness of $l_n$, there is a subnet $(\mu_{\alpha(\beta)})$ such that $(\mu_{\alpha(\beta)}n)$ converges in norm.

Now, we have $mn = \omega^* - \lim \mu_{\alpha(\beta)}n$. Thus $mn = \lim \mu_{\alpha(\beta)}n$ with norm topology. It follows that

$$\{mn \mid \|m\| \leq 1\} \subseteq \{\mu n \mid \mu \in L(S), \|\mu\| \leq 1\}. \quad (3.6)$$

Thus, $l_n$ is compact.

**Theorem 3.6.** Let $S$ be a right cancellative semigroup with identity $e$ and $l_n$ a right multiplier on $LUC(S)^*$. Then $l_n$ can be written as a linear combination of four compact right multipliers $l_{n_i}$ ($i = 1, 2, 3, 4$), $n_i \geq 0$, $n_i \in LUC(S)^*$.

**Proof.** Let $e$ be the identity of $S$. Then $T(m) = T(m\delta_e) = mT(\delta_e)l_T(\delta_e)(m)$. So, $T = l_n$ ($n = T(\delta_e) \in LUC(S)^*$). Let $n = n^+_1 - n^-_1 + i(n^+_2 - n^-_2)$ where $n^+_k, n^-_k$ ($k = 1, 2$) are Hermitian. So, it suffices to show that $l_{n^+_1}$ and $l_{n^-_1}$ are compact. By Proposition 3.4 it suffices to prove that the restrictions of these operators to $M(S)$ are compact. Now since $l_n$ is compact on $LUC(S)^*$, $\{\delta_x n \mid x \in S\}$ is compact. So $\{|\delta_x n||x \in S\}$ is compact. Since, $\|n^+\| \leq \|n\|$, $\{|\delta_x n^+| \mid x \in S\}$ is compact. It follows that $\{|\delta_x n^+| \mid x \in S\}$ is compact. Since the linear span of $\delta_x$, $s$ is weak$^*$ dense in $LUC(S)^*$, $\{|\mu n^+| \mid \mu \in M(S), \|\mu\| \leq 1\}$ is compact. Therefore, $l_{n^+}$ is compact. This completes the proof.
We denote by $\beta S$ the space of all multiplicative linear functional on $LUC(S)$. We have another main theorem.

**Theorem 3.7.** Let $S$ be a finite topological semigroup. Then there exists $n \in \beta S$ such that $l_n$ is compact. Conversely, if $S$ is a subsemigroup of a topological group with identity, and there exists $n \in \beta S$ such that $l_n$ is compact, then $S$ is finite.

**Proof.** Let $S$ be finite, then by [2, Corollary 4.1.8], $AP(S) = C(S)$. Also, by [2, Proposition 4.4.8], $AP(S) = LUC(S) = RUC(S)$. Therefore, $LUC(S) = C(S)$. So $\beta S$ is topologically isomorphic to $S$. On the other hand, since $l_S S \subseteq S$ is compact, $l_S C(S)$ is compact. Hence, $l_n$ is compact.

Conversely, let $l_n$ be compact for some $n \in \beta S$, by Theorem 3.6, we may assume that $n$ is positive, then $T_n(f) = nf \ (f \in LUC(S))$ is compact. Now, let $F = \text{range } T_n$. Clearly $T_n$ is an algebra homomorphism, since, $T_n(fg) = nf(g)(x) = \langle n, l_x fg \rangle = n((l_x f)(l_x g)) = n(l_x f)n(l_x g) = T_n(f)T_n(g)$. Also $T_n$ preserves conjugation. So, by [8, Theorem 5.3], $\| T_n f \| \geq \| f \| \ (f \in LUC(S))$. So by open mapping theorem, $T_n$ is a homeomorphism. Since $T_n$ is compact, $F$ is closed. Also, $\{ T_n f \mid f \in LUC(S), \| T_n f \| \leq 1 \} \subseteq \{ T_n f \mid f \in LUC(S), \| f \| \leq 1 \}$, so $\{ T_n f \mid f \in LUC(S), \| T_n f \| \leq 1 \}$ is compact. Therefore $F$ is reflexive. It follows that $F$ is finite dimensional (see [8, Exercise 2]). Let $\{m_1, m_2, \ldots, m_k\}$ be the spectrum of $F$ and we can assume that $m_i$ is positive. If we define $m(f) = (1/k) \sum_{i=1}^{k} m_i(T_n f)$, then clearly, $m \geq 0, m(1) = 1$. Also, since $S$ is left cancellative, $l_x^{*} \{m_1, \ldots, m_k\} = \{m_1, \ldots, m_k\}$. Therefore, $\langle m_i, T_n l_x(f) \rangle = \langle m_i, l_x T_n(f) \rangle = \langle l_x m_i, T_n(f) \rangle = \langle m_j, T_n(f) \rangle$, for some $1 \leq j \leq k$. It follows that $m(l_x f) = m(f)$, that is, $m$ is a left-invariant mean on $LUC(S)$, so by [5, Theorem 3] $S$ is finite.

**References**


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