ON INCLUSION RELATIONS FOR ABSOLUTE SUMMABILITY

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We obtain necessary and (different) sufficient conditions for a series summable \( |\tilde{N}, p_n|_k \), \( 1 < k \leq s < \infty \), to imply that the series is summable \( |T|_s \), where \((\tilde{N}, p_n)\) is a weighted mean matrix and \( T \) is a lower triangular matrix. As corollaries of this result, we obtain several inclusion theorems.

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Let \( \sum a_n \) be a given series with partial sums \( s_n \), \( (C, \alpha) \) the Césaro matrix of order \( \alpha \). If \( \sigma_n^\alpha \) denotes the \( n \)th term of the \( (C, \alpha) \)-transform of \( \{s_n\} \) then, from Flett [4], \( \sum a_n \) is said to be summable \( |C, \alpha|_k \), \( k \geq 1 \) if

\[
\sum_{n=1}^\infty n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty, \tag{1}
\]

For any sequence \( \{u_n\} \), the forward difference operator \( \Delta \) is defined by \( \Delta u_n = u_n - u_{n-1} \).

An appropriate extension of (1) to arbitrary lower triangular matrices \( T \) is

\[
\sum_{n=1}^\infty n^{k-1} |\Delta t_{n-1}|^k < \infty, \tag{2}
\]

where

\[
t_n := \sum_{k=0}^n t_{nk}s_k. \tag{3}
\]

Such an extension is used, for example, in Bor [2]. But Sarigöl, Sulaiman, and Bor and Thorpe [3] make the following extension of (1).

They define a series \( \sum a_n \) to be summable \( |\tilde{N}, p_n|_k \), \( k \geq 1 \) if

\[
\sum_{n=1}^\infty \left( \frac{p_n}{p_{n+1}} \right)^{k-1} |\Delta Z_{n-1}|^k < \infty, \tag{4}
\]

where \( Z_n \) denotes the \( n \)th term of the weighted mean transform of \( \{s_n\} \); that is,

\[
Z_n = \frac{1}{p_n} \sum_{k=0}^n p_k s_k. \tag{5}
\]

Apparently they have interpreted the \( n \) in (1) to represent the reciprocal of the \( n \)th diagonal term of the matrix \((\tilde{N}, p_n)\). (See, e.g., Sarigöl [6], where this is explicitly the case.)
Unfortunately, this interpretation cannot be correct. For if it were, then, since the $n$th diagonal entry of $(C, \alpha)$ is $O(n^{-\alpha})$, (1) would take the form

$$\sum_{n=1}^{\infty} (n^\alpha)^{(k-1)} |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty.$$  \hfill (6)

However, Flett stays with (1). Thus (2) is an appropriate extension of (1) to lower triangular matrices.

Given any lower triangular matrix $T$, we can associate the matrices $\bar{T}$ and $\hat{T}$ with entries defined by

$$\bar{t}_{nk} = \sum_{i=k}^{n} t_{ni}, \quad \hat{t}_{nk} = \bar{t}_{nk} - \bar{t}_{n-1,k},$$  \hfill (7)

respectively.

Thus, from (3),

$$t_n = \sum_{k=0}^{n} t_{nk}s_k = \sum_{k=0}^{n} t_{nk} \sum_{i=0}^{k} a_i = \sum_{i=0}^{n} a_i \sum_{k=i}^{n} \bar{t}_{nk}a_i,$$

$$Y_n := t_n - t_{n-1} = \sum_{i=0}^{n} \bar{t}_{ni}a_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i}a_i = \sum_{i=0}^{n} \hat{t}_{ni}a_i, \quad \text{since} \: \bar{t}_{n-1,n} = 0.$$  \hfill (8)

For a weighted mean matrix $A = (\tilde{N}, p_n)$,

$$\tilde{a}_{nk} = \sum_{i=k}^{n} \frac{p_k}{p_n} = \frac{1}{p_n} (p_n - P_{k-1}) = 1 - \frac{P_{k-1}}{p_n}.$$  \hfill (9)

Thus

$$\tilde{a}_{nk} = \tilde{a}_{n,k} - \tilde{a}_{n-1,k} = 1 - \frac{P_{k-1}}{p_n} - 1 + \frac{P_{k-1}}{p_{n-1}} = \frac{p_nP_{k-1}}{p_nP_{n-1}},$$  \hfill (10)

so that, from (5),

$$X_n := \Delta Z_{n-1} = \frac{p_n}{P_{n-1}} \sum_{k=0}^{n-1} P_{k-1}a_k = \frac{p_n}{P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu-1}a_{\nu},$$  \hfill (11)

since $P_{-1} = 0$.

We will always assume that $\{p_n\}$ is a positive sequence with $P_n \rightarrow \infty$. Also, $\Delta_{\nu} \hat{t}_{n\nu} := \hat{t}_{n\nu} - \hat{t}_{n_{\nu+1}}$.

**Theorem 1.** Let $1 < k \leq s < \infty$, $\{p_n\}$ satisfying

$$\sum_{n=s+1}^{\infty} n^{k-1} \left( \frac{p_n}{p_nP_{n-1}} \right)^k = O \left( \frac{1}{p_v^k} \right).$$  \hfill (12)

Let $T$ be a lower triangular matrix. Then, the necessary conditions for $\sum a_n$ summable $|\tilde{N}, p_n|$ to imply $\sum a_n$ is summable $|T|_s$ are

(i) $P_{\nu} |t_{\nu\nu}| / p_{\nu} = O(\nu^{1/s-1/k})$;
(ii) $\sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta_{\nu} \hat{t}_{n\nu}|^s = O(\nu^{s-s/k}(p_{\nu}/p_{\nu})^s)$;
(iii) $\sum_{n=\nu+1}^{\infty} n^{s-1} |\hat{t}_{n_{\nu+1}}|^s = O(1)$. 


PROOF. We are given that
\[ \sum_{n=1}^{\infty} n^{s-1} |Y_n|^s < \infty, \quad (13) \]
whenever
\[ \sum_{n=1}^{\infty} n^{k-1} |X_n|^k < \infty. \quad (14) \]

Now, the space of sequences \( \{a_n\} \) satisfying (14) is a Banach space if normed by
\[ \|X\| = \left( |X_0|^k + \sum_{n=1}^{\infty} n^{k-1} |X_n|^k \right)^{1/k}. \quad (15) \]

We also consider the space of those sequences \( \{Y_n\} \) that satisfy (13). This is also a BK-space with respect to the norm
\[ \|Y\| = \left( |Y_0|^s + \sum_{n=1}^{\infty} n^{s-1} |Y_n|^s \right)^{1/s}. \quad (16) \]

Observe that (8) transforms the space of sequences satisfying (14) into the space of sequences satisfying (13). Applying the Banach-Steinhaus theorem, there exists a constant \( K > 0 \) such that
\[ \|Y\| \leq K \|X\|. \quad (17) \]

Applying (11) and (8) to \( a_\nu = e_\nu - e_{\nu+1} \), where \( e_\nu \) is the \( \nu \)th coordinate vector, we have, respectively,
\[ X_n = \begin{cases} 0, & \text{if } n < \nu, \\ \frac{p_\nu}{P_\nu}, & \text{if } n = \nu, \\ \frac{-p_\nu p_n}{P_n P_{n-1}}, & \text{if } n > \nu, \end{cases} \quad (18) \]
\[ Y_n = \begin{cases} 0, & \text{if } n < \nu, \\ \hat{t}_{n\nu}, & \text{if } n = \nu, \\ \Delta_\nu \hat{t}_{n\nu}, & \text{if } n > \nu. \end{cases} \]

By (15) and (16), it follows that
\[ \|X\| = \left\{ v^{k-1} \left( \frac{p_\nu}{P_\nu} \right)^k + \sum_{n=1}^{\infty} n^{k-1} \left( \frac{p_\nu p_n}{P_n P_{n-1}} \right)^k \right\}^{1/k}, \quad (19) \]
\[ \|Y\| = \left\{ v^{s-1} |t_{\nu\nu}|^s + \sum_{n=1}^{\infty} n^{s-1} |\Delta_\nu \hat{t}_{n\nu}|^s \right\}^{1/s}, \quad (20) \]
recalling that \( \hat{t}_{\nu\nu} = \bar{t}_{\nu\nu} = t_{\nu\nu} \).
Using (19) and (20) in (17), along with (12), it follows that

\[
\nu^{s-1} |t_{\nu\nu}|^s + \sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta \hat{t}_{n\nu}|^s \leq K^s \left( \nu^{k-1} \left( \frac{p_\nu}{P_\nu} \right)^k + \sum_{n=\nu+1}^{\infty} n^{k-1} \left( \frac{p_\nu p_n}{P_\nu P_{n-1}} \right)^{k/s} \right)
\]

\[
\leq K^s \left( \nu^{k-1} \left( \frac{p_\nu}{P_\nu} \right)^k + O(1) \left( \frac{p_\nu}{P_\nu} \right)^{k/s} \right)
\]

\[
= O \left( \left( \frac{p_\nu}{P_\nu} \right)^k \nu^{k-1} \right)^{s/k}.
\]

(21)

The above inequality will be true if and only if each term on the left-hand side is \( O((p_\nu/P_\nu)^k \nu^{k-1})^{s/k} \).

Taking the first term,

\[
\nu^{s-1} |t_{\nu\nu}|^s = O \left( \frac{p_\nu}{P_\nu} \right)^k \nu^{k-1},
\]

\[
|t_{\nu\nu}|^s = O \left( \frac{p_\nu}{P_\nu} \right)^s \nu^{1-s/k},
\]

\[
|t_{\nu\nu}| = O \left( \frac{p_\nu}{P_\nu} \right)^s \nu^{1-s/k}^{1/s},
\]

\[
= O \left( \frac{p_\nu}{P_\nu} \right)^s \nu^{1/2-s/k},
\]

(22)

which verifies that (i) is necessary.

Using the second term, we have

\[
\sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta \hat{t}_{n\nu}|^s = O \left( \frac{p_\nu}{P_\nu} \right)^k \nu^{k-1} \left( \nu^{s-s/k} \right) = O \left( \frac{p_\nu}{P_\nu} \right)^s \nu^{s-s/k}.
\]

(23)

which is condition (ii).

If we now apply (11) and (8) to \( a_\nu = e^{\nu+1} \), we have, respectively,

\[
X_n = \begin{cases} 
0, & \text{if } n \leq \nu, \\
\frac{p_\nu p_n}{P_\nu P_{n-1}}, & \text{if } n > \nu,
\end{cases}
\]

\[
Y_n = \begin{cases} 
0, & \text{if } n \leq \nu, \\
\hat{t}_{n,\nu+1}, & \text{if } n > \nu.
\end{cases}
\]

(24)
The corresponding norms are
\[
\|X\| = \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} \left( \frac{P_{\nu}P_n}{P_nP_{\nu-1}} \right)^k \right\}^{1/k},
\]
\[
\|Y\| = \left\{ \sum_{n=\nu+1}^{\infty} n^{s-1} |\hat{t}_{n,\nu+1}|^s \right\}^{1/s},
\]
(25)

Applying (17) and (12),
\[
\sum_{n=\nu+1}^{\infty} n^{s-1} |\hat{t}_{n,\nu+1}|^s \leq K^s \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} \left( \frac{P_{\nu}P_n}{P_nP_{\nu-1}} \right)^k \right\}^{s/k},
\]
(26)
which is condition (iii).

**COROLLARY 2.** Let \(T\) be a lower triangular matrix, \(\{p_n\}\) satisfying (12). Then the necessary conditions for \(\sum a_n\) summable \(|\tilde{N},p_n|_k\) to imply \(\sum a_n\) summable \(|T|_k\) are
(i) \(P_{\nu}/p_{\nu} = O(1)\);
(ii) \(\sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{t}_{n,\nu}|^k = O(\nu^{k-1}(p_{\nu}/P_{\nu})^k)\);
(iii) \(\sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{t}_{n,\nu+1}|^k = O(1)\).

To prove Corollary 2, simply set \(s = k\) in Theorem 1.

**THEOREM 3.** Let \(1 < k \leq s < \infty\). Let \(T\) be a triangle with bounded entries such that \(T\) and \(\{p_n\}\) satisfy the following:
(i) \(t_{\nu\nu} = O((p_{\nu}/P_{\nu})^{1/s-1/k})\);
(ii) \(n|X_{\nu}|)^{s-k} = O(1)\);
(iii) \(\sum_{n=\nu+1}^{\infty} |\hat{t}_{n,\nu}| = O(|t_{\nu\nu}|)\);
(iv) \(\sum_{n=\nu+1}^{\infty} (n|t_{\nu\nu}|)^{s-1} |\hat{t}_{n,\nu}| = O(\nu^{s-1}|t_{\nu\nu}|^s)\);
(v) \(\sum_{n=\nu+1}^{\infty} |t_{\nu\nu}| = O(|t_{\nu\nu}|)\);
(vi) \(\sum_{n=\nu+1}^{\infty} (n|t_{\nu\nu}|)^{s-1} |\hat{t}_{n,\nu+1}| = O(\nu|t_{\nu\nu}|)^{s-1}\).

Then \(\sum a_n\) is \(\tilde{N},p_n|_k\).

**PROOF.** Solving (11) for \(\{a_n\}\) and substituting into (8) give
\[
Y_n = \sum_{\nu=1}^{n} \hat{t}_{\nu\nu} \left( X_{\nu}p_{\nu} - \frac{X_{\nu-1}p_{\nu-2}}{p_{\nu-1}} \right)
\]
\[
= \sum_{\nu=1}^{n} \hat{t}_{\nu\nu} X_{\nu}p_{\nu} - \sum_{\nu=1}^{n} \hat{t}_{\nu\nu} \frac{X_{\nu-1}p_{\nu-2}}{p_{\nu-1}}
\]
\[
= \sum_{\nu=1}^{n} \hat{t}_{\nu\nu} X_{\nu}p_{\nu} - \sum_{\nu=1}^{n} \hat{t}_{\nu,\nu+1} X_{\nu}p_{\nu-1}
\]
\[
= \hat{t}_{\nu\nu} X_{\nu}p_{\nu} + \sum_{\nu=1}^{n} (-\hat{t}_{\nu\nu}p_{\nu} + \hat{t}_{\nu,\nu+1}p_{\nu-1}) X_{\nu}p_{\nu}
\]
\[
\begin{align*}
&= t_{nn} p_n X_n + \sum_{v=1}^{n-1} \left[ P_v (\hat{t}_{nv} - \hat{t}_{n,v+1}) + \hat{t}_{n,v+1} (P_v - P_{v-1}) \right] \frac{X_v}{p_v} \\
&= \frac{p_n t_{nn} X_n}{p_n} + \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \Delta_v \hat{t}_{nv} + \hat{t}_{n,v+1} \right) X_v \\
&= T_{n1} + T_{n2} + T_{n3}.
\end{align*}
\] (27)

From Minkowski's inequality, it is sufficient to show that
\[
\sum_{n=1}^{\infty} n^{s-1} |T_{ni}|^s < \infty, \quad i = 1, 2, 3.
\] (28)

Using condition (i) of Theorem 3,
\[
J_1 := \sum_{n=1}^{\infty} n^{s-1} |T_{n1}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \frac{t_{nn} p_n X_n}{p_n} \right|^s
\]
\[
= O(1) \sum_{n=1}^{\infty} n^{s-1} (n^{1/s-1/k})^s |X_n|^s
\]
\[
= O(1) \sum_{n=1}^{\infty} n^{k-1} |X_n|^k \left( n^{s-1/k-k+1} |X_n|^{s-k} \right).
\] (29)

But
\[
n^{s-1/k-k+1} |X_n|^{s-k} = (n^{1/k} |X_n|)^{s-k} = O \left( (n |X_n|)^{s-k} \right) = O(1),
\] (30)

from (ii) of Theorem 3.

Since \( \sum a_n \) is summable, \(|\tilde{N}, p_n| k \), \( J_1 = O(1) \).

Using Hölder's inequality and conditions (i), (ii), (iii), and (iv) of Theorem 3.

\[
J_2 := \sum_{n=1}^{\infty} n^{s-1} |T_{n2}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{v=1}^{n-1} \left( \frac{P_v}{P_v} \right) (\Delta_v \hat{t}_{nv}) X_v \right|^s
\]
\[
= O(1) \sum_{n=1}^{\infty} n^{s-1} \left( \sum_{v=1}^{n-1} \nu^{1/s-1/k} |t_{v,v}|^{-1} |\Delta_v \hat{t}_{nv}| |X_v| \right)^s
\]
\[
= O(1) \sum_{n=1}^{\infty} n^{s-1} \left( \sum_{v=1}^{n-1} \nu^{1-s/k} |t_{v,v}|^{-s} |\Delta_v \hat{t}_{nv}| |X_v| \right)^s \times \left( \sum_{v=1}^{n-1} |\Delta_v \hat{t}_{nv}| \right)^{s-1}
\]
\[
= O(1) \sum_{n=1}^{\infty} (n |t_{nn}|)^{s-1} \sum_{v=1}^{n-1} \nu^{1-s/k} |t_{v,v}|^{-s} |\Delta_v \hat{t}_{nv}| |X_v|^s
\]
\[
= O(1) \sum_{v=1}^{\infty} \nu^{1-s/k} |t_{vv}|^{-s} |X_v|^s \sum_{n=v+1}^{\infty} (n |t_{nn}|)^{s-1} |\Delta_v \hat{t}_{nv}|
\]
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\[ \sum_{\nu=1}^{\infty} v^{1-s/k} \left| t_{\nu \nu} \right|^{-s} \left| X_{\nu} \right|^s v^{s-1} \left| t_{\nu \nu} \right|^s = O(1) \]

\[ \sum_{\nu=1}^{\infty} v^{s-s/k} \left| X_{\nu} \right|^s = O(1) \]

\[ \sum_{\nu=1}^{\infty} v^{k-1} \left| X_{\nu} \right|^k \left( v^{s-s/k-1} \left| X_{\nu} \right|^{s-k} \right) = O(1) \]

\[ \sum_{\nu=1}^{\infty} v^{k-1} \left| X_{\nu} \right|^k = O(1). \]

(31)

By Hölder’s inequality and conditions (v), (vi), and (iii) of Theorem 3, we have

\[ J_3 := \sum_{n=1}^{\infty} n^{s-1} \left| T_{n3} \right|^s = \sum_{n=1}^{\infty} \sum_{\nu=1}^{n-1} \left| t_{n,\nu+1} X_{\nu} \right|^s \]

\[ \leq \sum_{n=1}^{\infty} n^{s-1} \left( \sum_{\nu=1}^{n-1} \left| t_{n,\nu+1} X_{\nu} \right|^s \right) \]

\[ \leq \sum_{n=1}^{\infty} n^{s-1} \left( \sum_{\nu=1}^{n-1} \left| t_{\nu \nu} \right|^{1-s} \left| \hat{t}_{n,\nu+1} X_{\nu} \right|^s \right) \]

\[ \times \left( \sum_{\nu=1}^{n-1} \left| t_{\nu \nu} \right| \left| \hat{t}_{n,\nu+1} X_{\nu} \right|^s \right)^{-1} \]

\[ = O(1) \sum_{n=1}^{\infty} \left( n \left| t_{nn} \right| \right)^{s-1} \sum_{\nu=1}^{n-1} \left| t_{\nu \nu} \right|^{1-s} \left| \hat{t}_{n,\nu+1} X_{\nu} \right|^s \]

(32)

\[ = O(1) \sum_{\nu=1}^{\infty} \left| t_{\nu \nu} \right|^{1-s} \left| X_{\nu} \right|^s \sum_{n=\nu+1}^{\infty} \left( n \left| t_{nn} \right| \right)^{s-1} \left| \hat{t}_{n,\nu+1} \right| \]

\[ = O(1) \sum_{\nu=1}^{\infty} \left| t_{\nu \nu} \right|^{1-s} \left| X_{\nu} \right|^s \left( v \left| t_{\nu \nu} \right| \right)^{s-1} \]

\[ = O(1) \sum_{\nu=1}^{\infty} v^{s-1} \left| X_{\nu} \right|^s \]

\[ = O(1) \sum_{\nu=1}^{\infty} v^{k-1} \left| X_{\nu} \right|^k \left( v \left| X_{\nu} \right| \right)^{s-k} \]

\[ = O(1) \sum_{\nu=1}^{\infty} v^{k-1} \left| X_{\nu} \right|^k = O(1). \]

\[ \boxed{ \sum_{\nu=1}^{\infty} \left( \left| t_{\nu \nu} \right| \left| X_{\nu} \right| \right)^s = O(1) } \]

**Corollary 4** (see [5]). Let $T$ be a nonnegative lower triangular matrix, $\{p_n\}$ a positive sequence satisfying

(i) $t_{ni} \geq t_{n+1,i}$, $n \geq i$, $i = 0, 1, 2, \ldots$;

(ii) $P_n t_{nn} = O(p_n)$;

(iii) $\hat{t}_{n0} = \hat{t}_{n-1,0}$, $n = 1, 2, \ldots$;
setting each $\lambda_n$ known. The following result comes from Theorem 2.1 of Rhoades and Savaş [5] by and condition (iii) of Theorem 3 is satisfied. Therefore, using conditions (i) and (iii) of Corollary 4, we should not expect to obtain a set of necessary and sufficient conditions when an arbitrary triangle is involved.

Finally, we state necessary and sufficient conditions when $k \geq 1$, (12) is automatically satisfied. Therefore, the necessity of the conditions follows from Theorem 1.

To prove the conditions sufficient, use [5, Corollary 4.1] by setting each $\lambda_n = 1$.

**Corollary 9.** $\sum a_n$ summable $|N, q_n|_k$ implies $\sum a_n$ summable $|T|, k \geq 1$.

**Proof.** Since $s = k$ and $T$ is nonnegative, condition (ii) of Theorem 3 is automatically satisfied, and conditions (ii), (iv), (v), and (vi) of Corollary 4 are equivalent to conditions (i), (v), (iv), and (vi) of Theorem 3, respectively

$$\Delta_v \hat{t}_{nv} = \hat{t}_{nv} - \hat{t}_{n,v+1} = \hat{t}_{nv} - \hat{t}_{n-1,v} + \hat{t}_{n-1,v+1} = t_{nv} - t_{n-1,v}. \tag{33}$$

Therefore, using conditions (i) and (iii) of Corollary 4,

$$\sum_{v=1}^{n-1} |\Delta_v \hat{t}_{nv}| = \sum_{v=1}^{n-1} (t_{n-1,v} - t_{nv}) = 1 - t_{n-1,0} - 1 + t_{nn} + t_{nn} \leq t_{nn}, \tag{34}$$

and condition (iii) of Theorem 3 is satisfied.

**Remark 5.** For $1 < k \leq s < \infty$, necessary and sufficient conditions for a triangle $A : \ell^k \to \ell^s$ are known only for factorable matrices (see Bennett [1]), which include weighted mean matrices. Therefore, we should not expect to obtain a set of necessary and sufficient conditions when an arbitrary triangle is involved.

However, necessary and sufficient conditions for a matrix $A : \ell \to \ell^s$, $1 \leq s < \infty$ are known. The following result comes from Theorem 2.1 of Rhoades and Savaş [5] by setting each $\lambda_n = 1$.

**Theorem 6.** Let $T$ be a lower triangular matrix. Then $\sum a_n$ summable $|N, p_n|_k$ implies $\sum a_n$ summable $|T|, s \geq 1$ if and only if

1. $P_v |t_{vv}| / p_v = O(v^{1/s-1})$,
2. $\sum_{n=v+1}^{\infty} n^{s-1} |\Delta_v \hat{t}_{nv}|^s = O((p_v / P_v)^s)$,
3. $\sum_{n=v+1}^{\infty} n^{s-1} |\hat{t}_{n,v+1}|^s = O(1)$.

**Remark 7.** In [5], it is assumed that $T$ has nonnegative entries and row sums one, but these restrictions are not used in the proofs.

Finally, we state necessary and sufficient conditions when $k = s = 1$.

**Theorem 8.** The series $\sum a_n$ summable $|N, p_n|$ implies $\sum a_n$ summable $T$ if and only if

1. $P_v |t_{vv}| / p_v = O(1)$;
2. $\sum_{n=v+1}^{\infty} n^{s-1} |\Delta_v \hat{t}_{nv}| = O(p_v / P_v)$;
3. $\sum_{n=v+1}^{\infty} |\hat{t}_{n,v+1}| = O(1)$.

**Proof.** Note that, with $k = 1$, (12) is automatically satisfied. Therefore, the necessity of the conditions follows from Theorem 1.
Proof. With each $p_n = 1$, $T = (N, q_n)$, condition (i) of Theorem 8 reduces to condition (i) of Corollary 9.

Using (33),

\[
\sum_{n=\nu+1}^{\infty} |\Delta \hat{t}_{\nu,v} | = \sum_{n=\nu+1}^{\infty} |t_{\nu,v} - t_{\nu-1,v} | = \sum_{n=\nu+1}^{\infty} \left| \frac{p_v}{p_n} - \frac{p_v}{p_{n-1}} \right| = p_v \sum_{n=\nu+1}^{\infty} \frac{p_n}{p_n p_{n-1}} = \frac{p_v}{p_v},
\]

and condition (ii) of Theorem 8 is satisfied. Since $(N, p_n)$ has row sums one,

\[
\hat{t}_{n,v+1} = \hat{t}_{n,v+1} - \hat{t}_{n-1,v+1} = \sum_{i=\nu+1}^{n} t_{ni} - \sum_{i=\nu+1}^{n} t_{n-1,i}
\]

\[
= 1 - \sum_{i=0}^{\nu} t_{ni} - 1 + \sum_{i=0}^{\nu} t_{n-1,i}
\]

\[
= \sum_{i=0}^{\nu} (t_{n-1,i} - t_{ni}) = \sum_{i=0}^{\nu} \left( \frac{p_i}{p_{n-1}} - \frac{p_i}{p_n} \right)
\]

\[
= \frac{p_n}{p_n p_{n-1}} \sum_{i=0}^{\nu} p_i = \frac{p_n p_{\nu}}{p_n p_{n-1}}.
\]

Therefore

\[
\sum_{n=\nu+1}^{\infty} |\hat{t}_{n,v+1} | = p_v \sum_{n=\nu+1}^{\infty} \frac{p_n}{p_n p_{n-1}} = 1,
\]

and condition (iii) of Theorem 8 is satisfied.

Corollary 10. The series $\sum a_n$ summable $|N, p_n|_k$ implies $\sum a_n$ summable $|C, 1|_k$ if and only if

(i) $p_n/(np_n) = O(1)$.

Proof. Using $T = (C, 1)$ in Theorem 8, condition (i) of Theorem 8 reduces to condition (i) of Corollary 10.

From (33) and (i) of Corollary 10,

\[
\sum_{n=\nu+1}^{\infty} \left| \Delta \hat{t}_{\nu,v} \right| = \sum_{n=\nu+1}^{\infty} \left| t_{\nu-1,v} - t_{\nu,v} \right| = \sum_{n=\nu+1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]

\[
= \frac{1}{\nu+1} = \frac{p_v}{\nu p_v} \left( \frac{\nu}{\nu+1} \right) \left( \frac{p_v}{p_v} \right) = O\left( \frac{p_v}{p_v} \right),
\]

and condition (ii) of Theorem 8 is satisfied.
Using (36),
\[
\sum_{n=\nu+1}^{\infty} |\hat{t}_{n,\nu+1}| = \sum_{n=\nu+1}^{\infty} \left| \sum_{i=0}^{\nu} (t_{n-1,i} - t_{ni}) \right| \\
= \sum_{n=\nu+1}^{\infty} \left| \sum_{i=0}^{\nu} \left( \frac{1}{n} - \frac{1}{n+1} \right) \right| \\
= \sum_{n=\nu+1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)(\nu + 1) = (\nu + 1)\left( \frac{1}{\nu + 1} \right) = 1,
\]
and condition (iii) of Theorem 8 is satisfied.

Combining Corollaries 9 and 10, we have the following corollary.

**Corollary 11.** \(|\hat{N}, p_n|\) and \(|C, 1|\) are equivalent if and only if

(i) \(np_n/P_n = O(1)\);

(ii) \(P_n/(np_n) = O(1)\).

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**References**


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