ON INCLUSION RELATIONS FOR ABSOLUTE SUMMABILITY

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We obtain necessary and (different) sufficient conditions for a series summable $|\tilde{N}, p_n|_k$, $1 < k \leq s < \infty$, to imply that the series is summable $|T|_s$, where $(\tilde{N}, p_n)$ is a weighted mean matrix and T is a lower triangular matrix. As corollaries of this result, we obtain several inclusion theorems.

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Let $\sum a_n$ be a given series with partial sums s_n , (C, α) the Cesàro matrix of order α . If σ_n^α denotes the n th term of the (C, α) -transform of $\{s_n\}$ then, from Flett [4], $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty. \quad (1)$$

For any sequence $\{u_n\}$, the forward difference operator Δ is defined by $\Delta u_n = u_n - u_{n+1}$.

An appropriate extension of (1) to arbitrary lower triangular matrices T is

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta t_{n-1}|^k < \infty, \quad (2)$$

where

$$t_n := \sum_{k=0}^n t_{nk} s_k. \quad (3)$$

Such an extension is used, for example, in Bor [2]. But Sarigöl, Sulaiman, and Bor and Thorpe [3] make the following extension of (1).

They define a series $\sum a_n$ to be summable $|\tilde{N}, p_n|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} \left(\frac{p_n}{p_n} \right)^{k-1} |\Delta Z_{n-1}|^k < \infty, \quad (4)$$

where Z_n denotes the n th term of the weighted mean transform of $\{s_n\}$; that is,

$$Z_n = \frac{1}{P_n} \sum_{k=0}^n p_k s_k. \quad (5)$$

Apparently they have interpreted the n in (1) to represent the reciprocal of the n th diagonal term of the matrix $(\tilde{N}, p_n)$. (See, e.g., Sarigöl [6], where this is explicitly the case.)

Unfortunately, this interpretation cannot be correct. For if it were, then, since the n th diagonal entry of (C, α) is $O(n^{-\alpha})$, (1) would take the form

$$\sum_{n=1}^{\infty} (n^{\alpha})^{(k-1)} |\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}|^k < \infty. \quad (6)$$

However, Flett stays with (1). Thus (2) is an appropriate extension of (1) to lower triangular matrices.

Given any lower triangular matrix T , we can associate the matrices $\bar{T}$ and $\hat{T}$ with entries defined by

$$\bar{t}_{nk} = \sum_{i=k}^n t_{ni}, \quad \hat{t}_{nk} = \bar{t}_{nk} - \bar{t}_{n-1,k}, \quad (7)$$

respectively.

Thus, from (3),

$$\begin{aligned} t_n &= \sum_{k=0}^n t_{nk} s_k = \sum_{k=0}^n t_{nk} \sum_{i=0}^k a_i = \sum_{i=0}^n a_i \sum_{k=i}^n t_{nk} = \sum_{i=0}^n \bar{t}_{nk} a_i, \\ Y_n &:= t_n - t_{n-1} = \sum_{i=0}^n \bar{t}_{ni} a_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i = \sum_{i=0}^n \hat{t}_{ni} a_i, \quad \text{since } \bar{t}_{n-1,n} = 0. \end{aligned} \quad (8)$$

For a weighted mean matrix $A = (\bar{N}, p_n)$,

$$\bar{a}_{nk} = \sum_{i=k}^n \frac{p_k}{p_n} = \frac{1}{p_n} (p_n - p_{k-1}) = 1 - \frac{p_{k-1}}{p_n}. \quad (9)$$

Thus

$$\hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k} = 1 - \frac{p_{k-1}}{p_n} - 1 + \frac{p_{k-1}}{p_{n-1}} = \frac{p_n p_{k-1}}{p_n p_{n-1}}, \quad (10)$$

so that, from (5),

$$X_n := \Delta Z_{n-1} = \frac{p_n}{p_n p_{n-1}} \sum_{k=0}^{n-1} p_{k-1} a_k = \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} p_{v-1} a_v, \quad (11)$$

since $p_{-1} = 0$.

We will always assume that $\{p_n\}$ is a positive sequence with $p_n \rightarrow \infty$. Also, $\Delta_v \hat{t}_{nv} := \hat{t}_{nv} - \hat{t}_{n,v+1}$.

THEOREM 1. *Let $1 < k \leq s < \infty$, $\{p_n\}$ satisfying*

$$\sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{p_n}{p_n p_{n-1}} \right)^k = O\left(\frac{1}{p_v^k} \right). \quad (12)$$

Let T be a lower triangular matrix. Then, the necessary conditions for $\sum a_n$ summable $|\bar{N}, p_n|_k$ to imply $\sum a_n$ is summable $|T|_s$ are

- (i) $p_v |t_{vv}| / p_v = O(v^{1/s-1/k})$;
- (ii) $\sum_{n=v+1}^{\infty} n^{s-1} |\Delta_v \hat{t}_{nv}|^s = O(v^{s-s/k} (p_v / P_v)^s)$;
- (iii) $\sum_{n=v+1}^{\infty} n^{s-1} |\hat{t}_{n,v+1}|^s = O(1)$.

PROOF. We are given that

$$\sum_{n=1}^{\infty} n^{s-1} |Y_n|^s < \infty, \quad (13)$$

whenever

$$\sum_{n=1}^{\infty} n^{k-1} |X_n|^k < \infty. \quad (14)$$

Now, the space of sequences $\{a_n\}$ satisfying (14) is a Banach space if normed by

$$\|X\| = \left(|X_0|^k + \sum_{n=1}^{\infty} n^{k-1} |X_n|^k \right)^{1/k}. \quad (15)$$

We also consider the space of those sequences $\{Y_n\}$ that satisfy (13). This is also a BK-space with respect to the norm

$$\|Y\| = \left(|Y_0|^k + \sum_{n=1}^{\infty} n^{s-1} |Y_n|^s \right)^{1/s}. \quad (16)$$

Observe that (8) transforms the space of sequences satisfying (14) into the space of sequences satisfying (13). Applying the Banach-Steinhaus theorem, there exists a constant $K > 0$ such that

$$\|Y\| \leq K \|X\|. \quad (17)$$

Applying (11) and (8) to $a_v = e_v - e_{v+1}$, where e_v is the v th coordinate vector, we have, respectively,

$$X_n = \begin{cases} 0, & \text{if } n < v, \\ \frac{p_v}{P_v}, & \text{if } n = v, \\ -\frac{p_v p_n}{P_n P_{n-1}}, & \text{if } n > v, \end{cases} \quad (18)$$

$$Y_n = \begin{cases} 0, & \text{if } n < v, \\ \hat{t}_{nv}, & \text{if } n = v, \\ \Delta_v \hat{t}_{nv}, & \text{if } n > v. \end{cases}$$

By (15) and (16), it follows that

$$\|X\| = \left\{ v^{k-1} \left(\frac{p_v}{P_v} \right)^k + \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{p_v p_n}{P_n P_{n-1}} \right)^k \right\}^{1/k}, \quad (19)$$

$$\|Y\| = \left\{ v^{s-1} |\hat{t}_{vv}|^s + \sum_{n=v+1}^{\infty} n^{s-1} |\Delta_v \hat{t}_{nv}|^s \right\}^{1/s}, \quad (20)$$

recalling that $\hat{t}_{vv} = \bar{t}_{vv} = t_{vv}$.

Using (19) and (20) in (17), along with (12), it follows that

$$\begin{aligned}
 \nu^{s-1} |t_{\nu\nu}|^s + \sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta \hat{t}_{n\nu}|^s &\leq K^s \left(\nu^{k-1} \left(\frac{p_\nu}{P_\nu} \right)^k + \sum_{n=\nu+1}^{\infty} n^{k-1} \left(\frac{p_\nu p_n}{P_n P_{n-1}} \right)^k \right)^{s/k} \\
 &\leq K^s \left(\nu^{k-1} \left(\frac{p_\nu}{P_\nu} \right)^k + O(1) \left(\frac{p_\nu}{P_\nu} \right)^k \right)^{s/k} \\
 &= O \left(\left(\frac{p_\nu}{P_\nu} \right)^k \nu^{k-1} \right)^{s/k}.
 \end{aligned} \tag{21}$$

The above inequality will be true if and only if each term on the left-hand side is $O((p_\nu/P_\nu)^k \nu^{k-1})^{s/k}$.

Taking the first term,

$$\begin{aligned}
 \nu^{s-1} |t_{\nu\nu}|^s &= O \left(\left(\frac{p_\nu}{P_\nu} \right)^k \nu^{k-1} \right)^{s/k}, \\
 |t_{\nu\nu}|^s &= O \left(\left(\frac{p_\nu}{P_\nu} \right)^s \nu^{1-s/k} \right), \\
 |t_{\nu\nu}| &= O \left(\left(\frac{p_\nu}{P_\nu} \right)^s \nu^{1-s/k} \right)^{1/s} \\
 &= O \left(\left(\frac{p_\nu}{P_\nu} \right) \nu^{1/s-1/k} \right),
 \end{aligned} \tag{22}$$

which verifies that (i) is necessary.

Using the second term, we have

$$\sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta \hat{t}_{n\nu}|^s = O \left(\left(\frac{p_\nu}{P_\nu} \right)^k \nu^{k-1} \right)^{s/k} = O \left(\left(\frac{p_\nu}{P_\nu} \right)^s \nu^{s-s/k} \right), \tag{23}$$

which is condition (ii).

If we now apply (11) and (8) to $a_\nu = e^{\nu+1}$, we have, respectively,

$$\begin{aligned}
 X_n &= \begin{cases} 0, & \text{if } n \leq \nu, \\ \frac{P_\nu p_n}{P_n P_{n-1}}, & \text{if } n > \nu, \end{cases} \\
 Y_n &= \begin{cases} 0, & \text{if } n \leq \nu, \\ \hat{t}_{n,\nu+1}, & \text{if } n > \nu. \end{cases}
 \end{aligned} \tag{24}$$

The corresponding norms are

$$\begin{aligned}\|X\| &= \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} \left(\frac{P_{\nu} p_n}{P_n P_{n-1}} \right)^k \right\}^{1/k}, \\ \|Y\| &= \left\{ \sum_{n=\nu+1}^{\infty} n^{s-1} |\hat{t}_{n,\nu+1}|^s \right\}^{1/s}.\end{aligned}\tag{25}$$

Applying (17) and (12),

$$\sum_{n=\nu+1}^{\infty} n^{s-1} |\hat{t}_{n,\nu+1}|^s \leq K^s \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} \left(\frac{P_{\nu} p_n}{P_n P_{n-1}} \right)^k \right\}^{s/k},\tag{26}$$

which is condition (iii). $\square$

COROLLARY 2. Let T be a lower triangular matrix, $\{p_n\}$ satisfying (12). Then the necessary conditions for $\sum a_n$ summable $|\bar{N}, p_n|_k$ to imply $\sum a_n$ summable $|T|_k$ are

- (i) $P_{\nu} |t_{\nu\nu}| / p_{\nu} = O(1)$;
- (ii) $\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_{\nu} \hat{t}_{n\nu}|^k = O(\nu^{k-1} (p_{\nu} / P_{\nu})^k)$;
- (iii) $\sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{t}_{n,\nu+1}|^k = O(1)$.

To prove Corollary 2, simply set $s = k$ in Theorem 1.

A lower triangular matrix T is called a triangle if each $t_{nn} \neq 0$.

THEOREM 3. Let $1 < k \leq s < \infty$. Let T be a triangle with bounded entries such that T and $\{p_n\}$ satisfy the following:

- (i) $t_{\nu\nu} = O((p_{\nu} / P_{\nu}) \nu^{1/s-1/k})$;
- (ii) $(n|X_n|)^{s-k} = O(1)$;
- (iii) $\sum_{\nu=1}^{n-1} |\Delta_{\nu} \hat{t}_{n\nu}| = O(|t_{nn}|)$;
- (iv) $\sum_{n=\nu+1}^{\infty} (n|t_{nn}|)^{s-1} |\Delta_{\nu} \hat{t}_{n\nu}| = O(\nu^{s-1} |t_{\nu\nu}|^s)$;
- (v) $\sum_{\nu=1}^{n-1} |t_{\nu\nu}| |\hat{t}_{n,\nu+1}| = O(|t_{nn}|)$;
- (vi) $\sum_{n=\nu+1}^{\infty} (n|t_{nn}|)^{s-1} |\hat{t}_{n,\nu+1}| = O(\nu |t_{\nu\nu}|)^{s-1}$.

Then $\sum a_n$ is $\bar{N}, p_n|_k$.

PROOF. Solving (11) for $\{a_n\}$ and substituting into (8) give

$$\begin{aligned}Y_n &= \sum_{\nu=1}^n \hat{t}_{n\nu} \left(\frac{X_{\nu} P_{\nu}}{p_{\nu}} - \frac{X_{\nu-1} P_{\nu-2}}{p_{\nu-1}} \right) \\ &= \sum_{\nu=1}^n \hat{t}_{n\nu} \frac{X_{\nu} P_{\nu}}{p_{\nu}} - \sum_{\nu=1}^n \hat{t}_{n\nu} \frac{X_{\nu-1} P_{\nu-2}}{p_{\nu-1}} \\ &= \sum_{\nu=1}^n \hat{t}_{n\nu} \frac{X_{\nu} P_{\nu}}{p_{\nu}} - \sum_{\nu=0}^{n-1} \hat{t}_{n,\nu+1} \frac{X_{\nu} P_{\nu-1}}{p_{\nu}} \\ &= \frac{\hat{t}_{nn} X_n P_n}{p_n} + \sum_{\nu=1}^{n-1} (\hat{t}_{n\nu} P_{\nu} - \hat{t}_{n,\nu+1} P_{\nu-1}) \frac{X_{\nu}}{p_{\nu}}\end{aligned}$$

$$\begin{aligned}
&= \frac{t_{nn}P_nX_n}{p_n} + \sum_{v=1}^{n-1} \left[P_v(\hat{t}_{nv} - \hat{t}_{n,v+1}) + \hat{t}_{n,v+1}(P_v - P_{v-1}) \right] \frac{X_v}{p_v} \\
&= \frac{P_n t_{nn} X_n}{p_n} + \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \Delta_v \hat{t}_{nv} + \hat{t}_{n,v+1} \right) X_v \\
&= T_{n1} + T_{n2} + T_{n3}.
\end{aligned} \tag{27}$$

From Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{s-1} |T_{ni}|^s < \infty, \quad i = 1, 2, 3. \tag{28}$$

Using condition (i) of [Theorem 3](#),

$$\begin{aligned}
J_1 &:= \sum_{n=1}^{\infty} n^{s-1} |T_{n1}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \frac{t_{nn}P_nX_n}{p_n} \right|^s \\
&= O(1) \sum_{n=1}^{\infty} n^{s-1} (n^{1/s-1/k})^s |X_n|^s \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} |X_n|^k (n^{s-s/k-k+1} |X_n|^{s-k}).
\end{aligned} \tag{29}$$

But

$$n^{s-s/k-k+1} |X_n|^{s-k} = (n^{1-1/k} |X_n|)^{s-k} = O((n |X_n|)^{s-k}) = O(1), \tag{30}$$

from (ii) of [Theorem 3](#).

Since $\sum a_n$ is summable, $|\bar{N}, p_n|_k, J_1 = O(1)$.

Using Hölder's inequality and conditions (i), (ii), (iii), and (iv) of [Theorem 3](#).

$$\begin{aligned}
J_2 &:= \sum_{n=1}^{\infty} n^{s-1} |T_{n2}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right) (\Delta_v \hat{t}_{nv}) X_v \right|^s \\
&= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{v=1}^{n-1} v^{1/s-1/k} |t_{vv}|^{-1} |\Delta_v \hat{t}_{nv}| |X_v| \right)^s \\
&= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{v=1}^{n-1} v^{1-s/k} |t_{vv}|^{-s} |\Delta_v \hat{t}_{nv}| |X_v|^s \right) \times \left(\sum_{v=1}^{n-1} |\Delta_v \hat{t}_{nv}| \right)^{s-1} \\
&= O(1) \sum_{n=1}^{\infty} (n |t_{nn}|)^{s-1} \sum_{v=1}^{n-1} v^{1-s/k} |t_{vv}|^{-s} |\Delta_v \hat{t}_{nv}| |X_v|^s \\
&= O(1) \sum_{v=1}^{\infty} v^{1-s/k} |t_{vv}|^{-s} |X_v|^s \sum_{n=v+1}^{\infty} (n |t_{nn}|)^{s-1} |\Delta_v \hat{t}_{nv}|
\end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{\nu=1}^{\infty} \nu^{1-s/k} |t_{\nu\nu}|^{-s} |X_{\nu}|^s \nu^{s-1} |t_{\nu\nu}|^s \\
&= O(1) \sum_{\nu=1}^{\infty} \nu^{s-s/k} |X_{\nu}|^s \\
&= O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_{\nu}|^k \left(\nu^{s-s/k-k+1} |X_{\nu}|^{s-k} \right) \\
&= O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_{\nu}|^k = O(1).
\end{aligned} \tag{31}$$

By Hölder's inequality and conditions (v), (vi), and (iii) of [Theorem 3](#), we have

$$\begin{aligned}
J_3 &:= \sum_{n=1}^{\infty} n^{s-1} |T_{n3}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{\nu=1}^{n-1} \hat{t}_{n,\nu+1} X_{\nu} \right|^s \\
&\leq \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=1}^{n-1} |\hat{t}_{n,\nu+1}| |X_{\nu}| \right)^s \\
&\leq \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=1}^{n-1} |t_{\nu\nu}|^{1-s} |\hat{t}_{n,\nu+1}| |X_{\nu}|^s \right) \\
&\quad \times \left(\sum_{\nu=1}^{n-1} |t_{\nu\nu}| |\hat{t}_{n,\nu+1}| \right)^{s-1} \\
&= O(1) \sum_{n=1}^{\infty} (n |t_{nn}|)^{s-1} \sum_{\nu=1}^{n-1} |t_{\nu\nu}|^{1-s} |\hat{t}_{n,\nu+1}| |X_{\nu}|^s \\
&= O(1) \sum_{\nu=1}^{\infty} |t_{\nu\nu}|^{1-s} |X_{\nu}|^s \sum_{n=\nu+1}^{\infty} (n |t_{nn}|)^{s-1} |\hat{t}_{n,\nu+1}| \\
&= O(1) \sum_{\nu=1}^{\infty} |t_{\nu\nu}|^{1-s} |X_{\nu}|^s (\nu |t_{\nu\nu}|)^{s-1} \\
&= O(1) \sum_{\nu=1}^{\infty} \nu^{s-1} |X_{\nu}|^s \\
&= O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_{\nu}|^k (\nu |X_{\nu}|)^{s-k} \\
&= O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_{\nu}|^k = O(1).
\end{aligned} \tag{32}$$

□

COROLLARY 4 (see [\[5\]](#)). *Let T be a nonnegative lower triangular matrix, $\{p_n\}$ a positive sequence satisfying*

- (i) $t_{ni} \geq t_{n+1,i}$, $n \geq i$, $i = 0, 1, 2, \dots$;
- (ii) $P_n t_{nn} = O(p_n)$;
- (iii) $\bar{t}_{n0} = \bar{t}_{n-1,0}$, $n = 1, 2, \dots$;

- (iv) $\sum_{i=1}^{n-1} |t_{ii}| |\hat{t}_{n,i+1}| = O(t_{nn})$;
- (v) $\sum_{n=i+1}^{\infty} (nt_{nn})^{k-1} |\Delta_i \hat{t}_{ni}| = O(i^{k-1} t_{ii}^k)$;
- (vi) $\sum_{n=i+1}^{\infty} (nt_{nn})^{k-1} |\hat{t}_{n,i+1}| = O((it_{ii})^{k-1})$.

Then $\sum a_n$ summable $|\tilde{N}, p_n|_k$ implies $\sum a_n$ summable $|T|_k$, $k \geq 1$.

PROOF. Since $s = k$ and T is nonnegative, condition (ii) of Theorem 3 is automatically satisfied, and conditions (ii), (iv), (v), and (vi) of Corollary 4 are equivalent to conditions (i), (v), (iv), and (vi) of Theorem 3, respectively

$$\Delta_v \hat{t}_{nv} = \hat{t}_{nv} - \hat{t}_{n,v+1} = \bar{t}_{nv} - \bar{t}_{n-1,v} - \bar{t}_{n,v+1} + \bar{t}_{n-1,v+1} = t_{nv} - t_{n-1,v}. \quad (33)$$

Therefore, using conditions (i) and (iii) of Corollary 4,

$$\sum_{v=1}^{n-1} |\Delta_v \hat{t}_{nv}| = \sum_{v=1}^{n-1} (t_{n-1,v} - t_{nv}) = 1 - t_{n-1,0} - 1 + t_{nn} + t_{n0} \leq t_{nn}, \quad (34)$$

and condition (iii) of Theorem 3 is satisfied. $\square$

REMARK 5. For $1 < k \leq s < \infty$, necessary and sufficient conditions for a triangle $A : \ell^k \rightarrow \ell^s$ are known only for factorable matrices (see Bennett [1]), which include weighted mean matrices. Therefore, we should not expect to obtain a set of necessary and sufficient conditions when an arbitrary triangle is involved.

However, necessary and sufficient conditions for a matrix $A : \ell \rightarrow \ell^s$, $1 \leq s < \infty$ are known. The following result comes from Theorem 2.1 of Rhoades and Savaş [5] by setting each $\lambda_n = 1$.

THEOREM 6. Let T be a lower triangular matrix. Then $\sum a_n$ summable $|\tilde{N}, p_n|_k$ implies $\sum a_n$ summable $|T|_s$, $s \geq 1$ if and only if

- (i) $P_v |t_{vv}| / p_v = O(v^{1/s-1})$,
- (ii) $\sum_{n=v+1}^{\infty} n^{s-1} |\Delta_v \hat{t}_{nv}|^s = O((p_v / P_v)^s)$,
- (iii) $\sum_{n=v+1}^{\infty} n^{s-1} |\hat{t}_{n,v+1}|^s = O(1)$.

REMARK 7. In [5], it is assumed that T has nonnegative entries and row sums one, but these restrictions are not used in the proofs.

Finally, we state necessary and sufficient conditions when $k = s = 1$.

THEOREM 8. The series $\sum a_n$ summable $|\tilde{N}, p_n|$ implies $\sum a_n$ summable T if and only if

- (i) $P_v |t_{vv}| / p_v = O(1)$;
- (ii) $\sum_{n=v+1}^{\infty} |\Delta_v \hat{t}_{nv}| = O(p_v / P_v)$;
- (iii) $\sum_{n=v+1}^{\infty} |\hat{t}_{n,v+1}| = O(1)$.

PROOF. Note that, with $k = 1$, (12) is automatically satisfied. Therefore, the necessity of the conditions follows from Theorem 1.

To prove the conditions sufficient, use [5, Corollary 4.1] by setting each $\lambda_n = 1$. $\square$

COROLLARY 9. $\sum a_n$ summable $|C, 1|$ implies $\sum a_n |\tilde{N}, q_n|$ if and only if

- (i) $nq_n / Q_n = O(1)$.

PROOF. With each $p_n = 1$, $T = (\tilde{N}, q_n)$, condition (i) of [Theorem 8](#) reduces to condition (i) of [Corollary 9](#).

Using (33),

$$\begin{aligned} \sum_{n=\nu+1}^{\infty} |\Delta_{\nu} \hat{t}_{n\nu}| &= \sum_{n=\nu+1}^{\infty} |t_{n\nu} - t_{n-1,\nu}| = \sum_{n=\nu+1}^{\infty} \left| \frac{p_{\nu}}{P_n} - \frac{p_{\nu}}{P_{n-1}} \right| \\ &= p_{\nu} \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = \frac{p_{\nu}}{P_{\nu}}, \end{aligned} \quad (35)$$

and condition (ii) of [Theorem 8](#) is satisfied. Since $(\tilde{N}, p_n)$ has row sums one,

$$\begin{aligned} \hat{t}_{n,\nu+1} &= \bar{t}_{n,\nu+1} - \bar{t}_{n-1,\nu+1} = \sum_{i=\nu+1}^n t_{ni} - \sum_{i=\nu+1}^n t_{n-1,i} \\ &= 1 - \sum_{i=0}^{\nu} t_{ni} - 1 + \sum_{i=0}^{\nu} t_{n-1,i} \\ &= \sum_{i=0}^{\nu} (t_{n-1,i} - t_{ni}) = \sum_{i=0}^{\nu} \left(\frac{p_i}{P_{n-1}} - \frac{p_i}{P_n} \right) \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{i=0}^{\nu} p_i = \frac{p_n p_{\nu}}{P_n P_{n-1}}. \end{aligned} \quad (36)$$

Therefore

$$\sum_{n=\nu+1}^{\infty} |\hat{t}_{n,\nu+1}| = P_{\nu} \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = 1, \quad (37)$$

and condition (iii) of [Theorem 8](#) is satisfied. $\square$

COROLLARY 10. *The series $\sum a_n$ summable $|\tilde{N}, p_n|_k$ implies $\sum a_n$ summable $|C, 1|_k$ if and only if*

$$(i) \quad P_n / (n p_n) = O(1).$$

PROOF. Using $T = (C, 1)$ in [Theorem 8](#), condition (i) of [Theorem 8](#) reduces to condition (i) of [Corollary 10](#).

From (33) and (i) of [Corollary 10](#),

$$\begin{aligned} \sum_{n=\nu+1}^{\infty} |\Delta_{\nu} \hat{t}_{n\nu}| &= \sum_{n=\nu+1}^{\infty} |t_{n-1,\nu} - t_{n\nu}| = \sum_{n=\nu+1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{\nu+1} = \frac{P_{\nu}}{\nu p_{\nu}} \left(\frac{\nu}{\nu+1} \right) \left(\frac{p_{\nu}}{P_{\nu}} \right) = O\left(\frac{p_{\nu}}{P_{\nu}} \right), \end{aligned} \quad (38)$$

and condition (ii) of [Theorem 8](#) is satisfied.

Using (36),

$$\begin{aligned}
 \sum_{n=v+1}^{\infty} |\hat{t}_{n,v+1}| &= \sum_{n=v+1}^{\infty} \left| \sum_{i=0}^v (t_{n-1,i} - t_{ni}) \right| \\
 &= \sum_{n=v+1}^{\infty} \left| \sum_{i=0}^v \left(\frac{1}{n} - \frac{1}{n+1} \right) \right| \\
 &= \sum_{n=v+1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) (v+1) = (v+1) \left(\frac{1}{v+1} \right) = 1,
 \end{aligned} \tag{39}$$

and condition (iii) of Theorem 8 is satisfied. $\square$

Combining Corollaries 9 and 10, we have the following corollary.

COROLLARY 11. $|\tilde{N}, p_n|$ and $|C, 1|$ are equivalent if and only if

- (i) $np_n/P_n = O(1)$;
- (ii) $P_n/(np_n) = O(1)$.

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