IJMMS 31:11 (2002) 651–657
PII. S0161171202109033
http://ijmms.hindawi.com

LOCAL COMPLETENESS OF $\ell_p(E)$, $1 \le p < \infty$

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Received 10 September 2001 and in revised form 26 February 2002

We study the heredity of local completeness and the strict Mackey convergence property from the locally convex space E to the space of absolutely p-summable sequences on E, $\ell_p(E)$ for $1 \le p < \infty$.

2000 Mathematics Subject Classification: 46A03, 46A17.

1. Introduction. In 1956, Grothendieck [5], introduced the Banach-valued sequence space $\ell_p(E)$, the space of absolutely p-summable sequences on a Banach space E, where he discussed tensor products of ℓ_p and E, with $1 \le p \le \infty$. Later, in 1969 Pietsch [8] used Banach-valued sequence spaces $\ell_p(E)$, to study p-summing operators between Banach spaces, also see Diestel et al. [2]. In this paper, we discuss how local completeness and the strict Mackey convergence condition of E imply local completeness and the strict Mackey convergence condition in $\ell_p(E)$ in the case $1 \le p < \infty$. The case $p = \infty$ was studied in [1].

2. Definitions and notation. Throughout this paper, (E,t) denotes a Hausdorff locally convex space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\{\rho_j\}_{j\in J}$ denotes the family of continuous seminorms associated with the topology t on E.

Let $D \subset E$ be a bounded, closed, and absolutely convex set. Denote by $E_D = \cup_{k=1}^{\infty} kD$, and for each $x \in E_D$, $\rho_D(x) = \inf\{r > 0 : x \in rD\}$, the Minkowski seminorm associated with D. Now $E_D \subset E$ and the boundedness of D implies that $i : (E_D, \rho_D) \to (E,t)$ is continuous, and ρ_D is a norm so that, for every $j \in J$ there exists $r_j \in \mathbb{R}^+$ such that $\rho_j|_{E_D} \le r_j \rho_D$.

Remark 2.1. For each $D \subset E$ bounded, closed, and absolutely convex, the family of seminorms $\{\rho_j\}_{j\in J}$ can be replaced by an equivalent family $\{\rho'_j\}_{j\in J}$ such that $\rho'_j \le \rho_D$. To construct the family $\{\rho'_j\}_{j\in J}$ we know that there exists $r_j > 0$ such that $\rho_j(x) \le r_j\rho_D(x)$ for every $x \in E_D$ so it suffices to take $\rho'_j = (1/r_j)\rho_j$ if $r_j > 1$, and we will have $\rho'_j \le \rho_D$, for every $j \in J$. For simplicity we will always work with an equivalent family of seminorms, also denoted by $\{\rho_j\}_{j\in J}$ such that $\rho_j(x) \le \rho_D(x)$ holds for every $j \in J$ and $x \in E_D$.

A bounded, closed, and absolutely convex set $D \subset E$, called a disk, is a Banach disk if (E_D, ρ_D) is a Banach space. If every bounded set $A \subset E$ is contained in a Banach disk we say that E is locally complete. Let (E,t) satisfies the strict Mackey convergence condition if for every bounded set $A \subset E$, there exists a disk D that contains A such that the topologies of (E,t) and (E_D, ρ_D) agree on A.

Every metrizable space satisfies the strict Mackey convergence condition, [7]. In addition, the strict Mackey convergence condition is preserved under the formation of closed subspaces, countable products, and countable direct sums, [6]. The strict Mackey convergence condition for webbed spaces is studied in [3, 4].

Remark 2.2. Using the family of seminorms $\{\rho_j\}_{j\in J}$ it is easy to see that the strict Mackey convergence condition is equivalent to: for each D there exists $j_0 \in J$ such that $\rho_{j_0|D} = \rho_D$.

Let p be a real number such that $1 \le p < \infty$. The space $\ell_p(E)$ of absolutely p-summable sequences on E is

$$\ell_p(E) = \left\{(x_n) \subset E : \sum_{n=1}^{\infty} \rho_j^p(x_n) < \infty,\ \forall j \in J\right\}. \tag{2.1}$$

The family of seminorms $\rho_{\rho_j}((x_n)) = (\sum_{n=1}^{\infty} \rho_j^p(x_n))^{1/p}$, $j \in J$, induce a topology of locally convex space in $\ell_p(E)$; we will denote by τ this topology.

The space $\ell_p(E_D)$ is defined by $\ell_p(E_D) = \{(x_n) \subset E_D : \sum_{n=1}^{\infty} \rho_D^p(x_n) < \infty\}$ and endowed with the topology generated by the norm

$$\rho_{\rho_D}((x_n)) = \left[\sum_{n=1}^{\infty} \rho_D^p(x_n)\right]^{1/p}. \tag{2.2}$$

We denote $A_D = \{(x_n) \in \ell_p(E) : (x_n)_{n\in\mathbb{N}} \subset D\}$.

Note that $\rho_{\rho_j}|_{\ell_p(E_D)} \le \rho_{\rho_D}$ for every $j \in J$ since $\rho_j|_{E_D} \le \rho_D$.

3. Bounded sets. In this section, we characterize the bounded sets of $\ell_p(E)$ in terms of the bounded sets of E.

Lemma 3.1. *Let D be a disk in (E,t); then*

(i) $\ell_p(E_D) \subseteq \{(x_n) \in \ell_p(E) : \{x_n\} \subset kD$ *for some* $k \in \mathbb{N}\}$;

(ii) *if there exists $j_0 \in J$, depending on D, such that $\rho_{j_0|D} = \rho_D$ (i.e., the strict Mackey convergence condition holds), then $\{(x_n) \in \ell_p(E) : \{x_n\} \subset kD$ for some $k \in \mathbb{N}\} \subset \ell_p(E_D)$.*

Proof. (i) Let $(x_n) \in \ell_p(E_D)$. Then $\sum_{n=1}^{\infty}[\rho_D(x_n)]^p < \infty$ so that given $\varepsilon = 1$ there exists $n_0 \in \mathbb{N}$, such that for each $n > n_0$, we have $\rho_D(x_n) \le (\sum_{n_0}^{\infty} \rho_D^p(x_n))^{1/p} \le 1$ which means that $x_n \in D$ for every $n > n_0$.

Now for $i = 1,2,\dots,n_0$ there exists $k_i \ge 0$ such that $x_i \in k_iD$. We take $k = \max\{1,k_1,\dots,k_{n_0}\}$. Then $\{x_n\} \subset kD$ and we have $\ell_p(E_D) \subset \{(x_n) \in \ell_p(E) : \{x_n\} \subset kD$ for some $k \in \mathbb{N}\}$.

(ii) Let $(x_n) \in \{(y_n) \in \ell_p(E) : \{y_n\} \subset kD$ for some $k \in \mathbb{N}\}$. Thus $x_n \in E_D$ for every $n \in \mathbb{N}$ since $\{x_n\} \subset kD$.

Now observe that $\sum_{n=1}^{\infty} \rho_D^p(x_n) = \sum_{n=1}^{\infty} \rho_{j_0}^p(x_n) < \infty$ since $(x_n) \in \ell_p(E)$. Hence in this case we have the equality $\ell_p(E_D) = \{(x_n) \in \ell_p(E) : \{x_n\} \subset kD$ for some $k \in \mathbb{N}\}$. □

Remark 3.2. Note that $kA_D = A_{kD}$ for every $k \in \mathbb{N}$.

Corollary 3.3. *If E satisfies the strict Mackey convergence condition, then $\ell_p(E)_{A_D} = \ell_p(E_D)$.*

Proof. It follows from the equality in the proof of Lemma 3.1(ii) that $\ell_p(E)_{A_D} \subset \ell_p(E_D)$. Now let $(x_n) \in \ell_p(E_D)$. Then by Lemma 3.1(i), $(x_n) \subset kD$ for some $k \in \mathbb{N}$ so $\{x_n\} \subset A_{kD} = kA_D$ and $(x_n) \in \ell_p(E)_{A_D}$. □

Remark 3.4. If (E,t) satisfies the strict Mackey convergence condition, then

$$\ell_p(E)_{A_D} = \ell_p(E_D) = \{(x_n) \in \ell_p(E) : \{x_n\} \subset A_{kD} \text{ for some } k \in \mathbb{N}\}. \tag{3.1}$$

Lemma 3.5. (i) $\rho_{A_D}((x_n)) = \sup\{\rho_D(x_n) : n \in \mathbb{N}\}$;
(ii) $\rho_{A_D}((x_n)) \le \rho_{\rho_D}((x_n))$ *for every* $(x_n) \in \ell_p(E_D)$.

Proof. (i) Let $s = \sup\{\rho_D(x_n) : n \in \mathbb{N}\}$. Then $\{x_n\} \subset sD$ so $\{x_n\} \subset A_{sD} = sA_D$ and then $\rho_{A_D}((x_n)) \le s$. Now take $r = \rho_{A_D}((x_n))$. Then $\{x_n\} \subset rA_D = A_{rD}$ and then $\{x_n\} \subset rD$ which means that $r \ge s$.

(ii) $\rho_{\rho_D}((x_n)) = (\sum_{n=1}^{\infty} \rho_D^p(x_n))^{1/p} \ge \rho_D(x_n)$ for every $n \in \mathbb{N}$. Using (i) we have $\rho_{\rho_D}((x_n)) \ge \rho_{A_D}((x_n))$. □

Note that A_D is not bounded in $\ell_p(E)$; we need to construct a "smaller" set, in the sense of boundedness.

Define for each $j \in J$ and $m \in \mathbb{N}$ the set $A_D(j,m) = \{(x_n)_n \in A_D : \rho_{\rho_j}((x_n)) \le m\}$ and for each $B \subset \ell_p(E)$, let $B^* = \{x \in E : x \in \{x_n\} \text{ and } (x_n) \in B\}$.

The next proposition gives a way to look at the bounded sets in $\ell_p(E)$.

Proposition 3.6. *If $\beta = \{D_\lambda\}_{\lambda \in \wedge}$ is a fundamental system of bounded disks in E, then $\{C = \cap_{j\in J}\{A_{D_\lambda}(j,m_j)\} : \lambda \in \Lambda,\ (m_j) \in \mathbb{N}^J\}$ is a fundamental system of τ-bounded sets in $\ell_p(E)$.*

Proof. Let $B \subset \ell_p(E)$ be a bounded set. Then B^* is bounded in E so $B^* \subset D_\lambda$ for some λ. For each $x \in B^*$, if $x \in (x_n)$ then given $j \in J$ there is some s_j such that $\rho_j(x) \le \rho_{\rho_j}((x_n)) \le s_j$ so that $\rho_{\rho_j}(B) \le s_j$. Now let $m_j \in \mathbb{N}$ be such that $s_j \le m_j$. We have $B \subset C = \cap_{j\in J} A_{D_\lambda}(j,m_j)$. □

Remark 3.7. (i) If D is bounded in E, then for each $j \in J$, by Remark 2.1 $\rho_{j|E_D} \le \rho_D$.
(ii) If C is bounded in $\ell_p(E)$, then for each $j \in J$, by Remark 2.1 $\rho_{\rho_j} \mid \ell_p(E)_C \le \rho_C$.

4. Main results

Proposition 4.1. *If for some D there exists $j_0 \in J$, such that $\rho_{j_0}|_D = \rho_D$ in E, then $\rho_{\rho_{j_0}|C} = \rho_C$ where $C = \cap_{j\in J} A_D(j,m_j)$ in $\ell_p(E)$. Equivalently, if E satisfies the strict Mackey convergence condition, then $\ell_p(E)$ also satisfies the strict Mackey convergence condition.*

Proof. Let $(x_n) \in C$. Then $s = \rho_{\rho_{j_0}}(x_n) = (\sum_{n=1}^{\infty} \rho_{j_0}^p(x_n))^{1/p} = (\sum_{n=1}^{\infty} \rho_D^p(x_n))^{1/p} \ge \rho_D(x_n) \ge \rho_{\rho_j}(x_n)$ for every $j \in J$ and $n \in \mathbb{N}$. So we have $(x_n) \in \cap_{j\in J} A_D(j,s) = s[\cap_{j\in J} A_D(j,1)] \subset sC$. Thus $\rho_C((x_n)) \le s = \rho_{\rho_{j_0}}(x_n)$ and since C is bounded in $\ell_p(E)$ we have $\rho_{\rho_j} \le \rho_C$ for each $j \in J$; now $\rho_{\rho_j}|_C \le \rho_C$ for every $j \in J$, so for j_0 we have $\rho_{\rho_{j_0}}|_C = \rho_C$.

Notice that if B is a bounded set in $\ell_p(E)$, then $\rho_{\rho_j}(B) \le m_j$ for all $j \in J$ with $m_j \in N$ and then $B \subset \cap_{j\in J} A_{B^*}(j,m_j)$.

This gives the property we need to characterize the bounded sets in $\ell_p(E)$. □

Theorem 4.2. *If E is locally complete and satisfies the strict Mackey convergence condition, then $(\ell_p(E)_C,\rho_C)$ where $C = \cap_{j\in J} A_D(j,m_j)$ in $\ell_p(E)$, is a Banach space so $\ell_p(E)$ is locally complete.*

Proof. Let D be a bounded closed disk such that (E_D,ρ_D) is a Banach space and let $C = \cap_{j\in J} A_D(j,m_j)$. By Remark 2.1 there is a $j_0 \in J$ such that $\rho_{j_0}|_D = \rho_D$. We will show that $(\ell_p(E)_C,\rho_C)$ is a Banach space. By Corollary 3.3 we have $\ell_p(E)_{A_D} = \ell_p(E_D)$ and since $C \subset A_D$, $\ell_p(E)_C \subset \ell_p(E)_{A_D}$. Let $(x_n^k)_{k\in\mathbb{N}} \subset \ell_p(E)_C$ be a ρ_C-Cauchy sequence. Thus for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n,m \ge N$ we have $\rho_C((x_n^k)-(x_m^k)) < \varepsilon$. Using Remark 3.7(ii) we have that $\rho_{\rho_j} \mid \ell(E)_C \le \rho_C$. Hence (x_n^k) is also a ρ_{ρ_j}-Cauchy sequence and then a $\rho_{\rho_{j_0}}$-Cauchy sequence. Thus $\rho_D(x_n^k - x_m^k) = \rho_{j_0}(x_n^k - x_m^k) \le \rho_{\rho_{j_0}}((x_n^k)-(x_m^k))$, then the sequence $(x_n^k)_{k\in\mathbb{N}}$ for every $n \in \mathbb{N}$ is also a ρ_D-Cauchy sequence in (E_D,ρ_D) which is a Banach space, so there exists z^k in E_D such that (x_n^k) converges to z^k with respect to the norm ρ_D. Using Remark 3.7(i) we have $\rho_{j|E_D} \le \rho_D$. Hence, we have the following claims.

Claim 1. We have that (x_n^k) converges to z^k with respect to the seminorm ρ_j for every $j \in J$.

Claim 2. Consider the sequence formed by the $(z^k)_{k\in\mathbb{N}} \in \ell_p(E_D)$. We compute

$$\begin{aligned}
\sum_{k=1}^{\infty} (\rho_D(z^k))^p &= \lim_{m\to\infty} \sum_{k=1}^{m} (\rho_D(z^k))^p \\
&= \lim_{m\to\infty} \sum_{k=1}^{m} (\rho_{j_0}(z^k))^p \\
&= \lim_{m\to\infty} \sum_{k=1}^{m} \rho_{j_0}\Big(\lim_{n\to\infty} x_n^k\Big)^p \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{k=1}^{m} \rho_{j_0}(x_n^k)^p \\
&\le \lim_{m\to\infty} \lim_{n\to\infty} \sum_{k=1}^{\infty} \rho_{j_0}(x_n^k)^p \\
&= \lim_{n\to\infty} \sum_{k=1}^{\infty} \rho_{j_0}(x_n^k)^p \\
&\le \lim_{n\to\infty} \rho_{\rho_{j_0}}((x_n)) \\
&\le \varepsilon + \rho_{\rho_{j_0}}((x_N)) < \infty, \quad \text{for some } N \in \mathbb{N}.
\end{aligned} \tag{4.1}$$

In this last inequality we used $x_n = (x_n^k)_{k\in\mathbb{N}}$ and since it is a $\rho_{\rho_{j_0}}$-Cauchy sequence, given $\varepsilon > 0$, $\rho_{\rho_{j_0}}(x_n^k) - \rho_{\rho_{j_0}}(x_m^k) \le \rho_{\rho_{j_0}}((x_n^k)-(x_m^k)) < \varepsilon$ for every $n,m > N$, so $\rho_{\rho_{j_0}}((x_n)) \le \varepsilon + \rho_{\rho_{j_0}}((x_N))$. Notice that (x_n) is a ρ_{ρ_j}-Cauchy sequence for every $j \in J$.

Therefore for j_0 and consequently for ρ_{ρ_D}, then for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\rho_D(x_n^k - z^k) = \rho_D(x_n^k - \lim_{m\to\infty} x_m^k) = \lim_{m\to\infty} \rho_D(x_n^k - x_m^k) < \varepsilon$.

Claim 3. The sequence (x_n^k) converges to $(z^k)_{k\in\mathbb{N}}$ in $\ell_p(E_D)$. Since

$$\begin{aligned}
\rho_{\rho_D}\left(x_n^k - (z^k)_k\right) &= \left[\sum_{k=1}^{\infty} \rho_D^p(x_n^k - z^k)\right]^{1/p} \\
&\le \left[\sum_{k=1}^{N} \rho_D^p(x_n^k - z^k) + \frac{\varepsilon^p}{2}\right]^{1/p} \\
&\le \left(\underbrace{\frac{\varepsilon^p}{2N} + \cdots + \frac{\varepsilon^p}{2N}}_{N \text{ factors}} + \frac{\varepsilon^p}{2}\right)^{1/p} \\
&= \varepsilon, \quad \text{for } n > N.
\end{aligned} \tag{4.2}$$

In the first inequality we used Claim 2. This completes the proof of the convergence.

Claim 4. We have $(z^k)_{k\in\mathbb{N}} \in \ell_p(E)_C$. $(x_n^k)_{k\in\mathbb{N}}$ is a ρ_C-Cauchy sequence so it is bounded and there is an $s \in \mathbb{N}$ such that $(x_n^k) \subset sC$. Using Claim 3, (x_n^k) converges to (z^k) in $\ell_p(E)_C$ with respect to ρ_{ρ_D} and since $\rho_{\rho_j}|_{\ell_p(E_D)} \le \rho_{\rho_D}$ for every $j \in J$ the sequence (x_n^k) is τ-convergent to (z^k), it is convergent for each ρ_{ρ_j}. Now for each $\varepsilon > 0$ there exists N_j such that $\rho_{\rho_j}((z^k)) \le \rho_{\rho_j}((z^k) - (x_n^k)) + \rho_{\rho_j}((x_n^k)) < \varepsilon + sm_j$ for every $j \in J$ and $n \ge N_j$, this means that $(z^k) \in sC \subset \ell_p(E)_C$.

Claim 5. The sequence (x_n^k) converges to $(z^k)_{k\in\mathbb{N}}$ in $\ell_p(E)_C$. Let $\varepsilon > 0$, since (x_n^k) is a ρ_C-Cauchy sequence, there is $N \in \mathbb{N}$ such that $(x_n^k) - (x_m^k) \in \varepsilon C$ for every $n, m \ge N$. C is τ-closed so $(x_n^k) - (\tau - \lim(x_m^k)) \in \varepsilon C$; then $(x_n^k) - (z^k) \in \varepsilon C$ for every $n \ge N$ which means $\rho_C((x_n^k) - (z^k)) \le \varepsilon$ for every $n \ge N$.

Notice that this is true for every $1 \le p < \infty$. The case $p = \infty$ also follows from this and we get the characterization given in [1], although under a stronger hypothesis. Here we need E to satisfy the strict Mackey convergence condition. □

Lemma 4.3. *If $D \subset E$ is t-complete and the net $\{x_\lambda\}_\Lambda$ is a τ-Cauchy net bounded with respect to ρ_C, that is if there exists $s \in \mathbb{N}$ such that $\{x_\lambda\}_\Lambda \subset sC$ then there exists $z \in 2sC$ such that x_λ converges to z with respect to the τ topology in $\ell_p(E)$.*

Proof. Let $\{x_\lambda\}_\Lambda$ be a τ-Cauchy net, $x_\lambda = (x_\lambda^1, x_\lambda^2, \ldots)$, then for every $\varepsilon > 0$ there exists $\lambda_j \in \Lambda$ such that for every $j \in J$, $\rho_j(x_\lambda^k - x_{\lambda'}^k) \le \rho_{\rho_j}(x_\lambda - x_{\lambda'}) < \varepsilon$ for every $\lambda, \lambda' \ge \lambda_j$ and $k \in \mathbb{N}$. So $\{x_\lambda^k\}_\Lambda \subset D$ is t-Cauchy for each $k \in \mathbb{N}$, and since D is complete there is a z^k such that x_λ^k converges to z^k with respect to the topology t for each $k \in \mathbb{N}$. Let $z = \{z^1, z^2, \ldots\}$. Then $z \subset D$, and for each $j \in J$ and $k \in \mathbb{N}$ we have $\rho_j(x_\lambda^k - z^k) = \rho_j(x_\lambda^k - (\rho_j - \lim_{\lambda'} x_{\lambda'}^k)) = \lim_{\lambda'} \rho_j(x_\lambda^k - x_{\lambda'}^k)$, so raising to the pth power and adding with respect to k we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \rho_j(x_\lambda^k - z^k)^p &= \lim_{n\to\infty} \sum_{k=1}^{n} \rho_j(x_\lambda^k - z^k)^p \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \lim_{\lambda'} \rho_j(x_\lambda^k - x_{\lambda'}^k)^p
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \lim_{\lambda'} \sum_{k=1}^{n} \rho_j(x_\lambda^k - z^k)^p \\
&\le \lim_{\lambda'} \sum_{k=1}^{\infty} \rho_j(x_\lambda^k - z^k)^p \\
&= \lim_{\lambda'} \rho_{\rho_j}(x_\lambda - x_{\lambda'}) < \varepsilon^p,
\end{aligned}
\tag{4.3}
$$

for every $\lambda \ge \lambda_j$.

So we have $\rho_{\rho_j}(x_\lambda - z)^p = \sum_{k=1}^{\infty} \rho_j(x_\lambda^k - z^k)^p < \varepsilon^p$, for every $\lambda \ge \lambda_j$. This means that x_λ converges to z with respect to the topology τ. We still need to prove that $z \in \ell_p(E)$

$$
\begin{aligned}
\rho_{\rho_j}(z)^p &= \sum_{k=1}^{\infty} \rho_j(z^k)^p \\
&= \sum_{k=1}^{\infty} \rho_j(z^k + x_\lambda^k - x_\lambda^k)^p \\
&\le \sum_{k=1}^{\infty} 2^p \left[\rho_j(z^k - x_\lambda^k)^p + \rho_j(x_\lambda^k)^p\right] \\
&= 2^p \sum_{k=1}^{\infty} \rho_j(z^k - x_\lambda^k)^p + 2^p \sum_{k=1}^{\infty} \rho_j(x_\lambda^k)^p \\
&< 2^p \varepsilon^p + 2^p \rho_{\rho_j}(x_\lambda)^p \\
&\le 2^p \varepsilon^p + 2^p m_j
\end{aligned}
\tag{4.4}
$$

$(x_\lambda \in C = \cap_{j\in J} A_D(j, m_j))$, then if we let $\varepsilon \to 0$ we get $\rho_{\rho_j}(z) \le 2m_j$, and finally $z \in 2C \subset \ell_p(E)$. □

Theorem 4.4. *If D is t-complete, then $\ell_p(E)_C$ is ρ_C-complete.*

Proof. Let (x_n^k) be a ρ_C-Cauchy sequence; it is clearly ρ_C-bounded and τ-Cauchy, so $(x_n^k) \subset sC$ for some $s \in \mathbb{N}$. Then by Lemma 4.3, there is a $z = (z^k) \in 2sC \subset \ell_p(E)_C$ such that the sequence (x_n^k) converges to z with respect to the topology τ. Note that A_D is τ-closed so $A_D(j, m)$ is also τ-closed for every $j \in J$ and $m \in \mathbb{N}$; then $C = \cap_{j\in J} A_D(j, m_j)$ is τ-closed. For $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $(x_n^k) - (x_m^k) \in \varepsilon C$ for every $n, m \ge N$, and since C is τ-closed $(x_n^k) - (\tau - \lim(x_m^k)) \in \varepsilon C$ then $(x_n^k) - (z^k) \in \varepsilon C$ for every $n \ge N$. This means that (x_n^k) converges to (z^k) with respect to ρ_C. □

Theorem 4.5. *If E is t-complete, then $\ell_p(E)$ is τ-complete.*

Proof. The proof of Lemma 4.3 can be repeated here to get the τ-completeness of $\ell_p(E)$. □

Acknowledgments. C. Bosch and C. Gómez were partially supported by Asociación Mexicana de Cultura A.C. T. Gilsdorf was partially supported by a Fulbright scholarship. R. Vera was supported by Coordinación de la Investigación Científica de la Universidad Michoacana.

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