A NEW TRIPLE SUM COMBINATORIAL IDENTITY

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We prove a new triple sum combinatorial identity derived from $r_p(x, y, z) = (x + y - z)^p - (x^p + y^p - z^p)$, extending a previous result by Sinyor et al.

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1. Introduction. The following combinatorial identity

$$\begin{aligned} \sum_{i=0}^n \sum_{j=0}^{2l} \sum_{k=0}^j \frac{(-1)^{i+j-k}}{m-l} \binom{m+l-k}{2l-k+1} \binom{m-l+k-1}{k} \binom{2l-j}{n-i} \binom{j}{i} \\ = \begin{cases} \frac{1}{2m+1} \binom{2m+1}{2m-2l+n} \binom{m-l+n/2}{n/2}, & \text{for } n \text{ even,} \\ 0, & \text{for } n \text{ odd,} \end{cases} \end{aligned} \quad (1.1)$$

for $0 \leq l \leq m-1$, $0 \leq n \leq 2l$, and m, l, n nonnegative integers, arose from considering

$$r_p(x, y, z) = (x + y - z)^p - (x^p + y^p - z^p), \quad (1.2)$$

where x, y, z are any nonzero integers, and p is odd and $p > 2$. We will use an identity proved in [2] which also arose from considering (1.2).

2. Proof of the identity. Define $f_p(x, y, z) = r_p(x, y, z) / (p(x + y)(z - x)(z - y))$. It is easily seen that $f_p(x, y, z)$ is a nonzero rational function and can be expressed as a polynomial in $(z - x)(z - y)$. Define coefficients $a_{i,(m-j)}$ such that

$$f_p(x, y, z) = \sum_{j=0}^{m-1} \sum_{i=0}^{2j} a_{i,(m-j)} x^{2j-i} y^i (z - x)^{m-j-1} (z - y)^{m-j-1}, \quad (2.1)$$

where $p = 2m + 1$. In [2] we proved the following combinatorial identity:

$$\begin{aligned} \sum_{l' \leq l} \sum_{j' \leq j} \frac{1}{m-l'} \binom{m+l'-j'}{2l'-j'+1} \binom{m-l'+j'-1}{j'} \binom{m-l'}{2(l-l')-(j-j')} \binom{m-l'}{j-j'} \\ = \frac{1}{2(m-l)} \binom{2m}{2l+1} \binom{2l+1}{j} = \frac{1}{2l+1} \binom{2m}{2l} \binom{2l+1}{j}, \end{aligned} \quad (2.2)$$

and we showed that $a_{i,(m-j)}$ defined above satisfy a recurrence and are of the form

$$a_{i,(m-j)} = \sum_{k=0}^i \frac{(-1)^{i-k}}{m-j} \binom{m+j-k}{2j-k+1} \binom{m-j+k-1}{k}. \quad (2.3)$$

Now substitute (2.3) in (2.1) and change the variables x and y to $r = (x+y)/2$, $s = (x-y)/2$, which gives

$$\begin{aligned} f_p(x, y, z) &= \sum_{j=0}^{m-1} \sum_{i=0}^{2j} a_{i,(m-j)} (r+s)^{2j-i} (r-s)^i (z-x)^{m-j-1} (z-y)^{m-j-1} \\ &= \sum_{j=0}^{m-1} \sum_{i=0}^{2j} \sum_{u=0}^{2j} \sum_{v=0}^u \sum_{k=0}^i \frac{(-1)^{v+i-k}}{m-j} \binom{m+j-k}{2j-k+1} \binom{m-j+k-1}{k} \binom{2j-i}{u-v} \binom{i}{v} \\ &\quad \times r^{2j-u} s^u (z-x)^{m-j-1} (z-y)^{m-j-1}. \end{aligned} \quad (2.4)$$

We now derive another form of $f_p(x, y, z)$ which will establish the identity.

Define $a = z - x$ and $b = z - y$. Then

$$\begin{aligned} r_p(x, y, z) &= (x+y-z)^p + z^p - x^p - y^p \\ &= \left(\frac{2r-a-b}{2} \right)^p + \left(\frac{2r+a+b}{2} \right)^p - \left(\frac{2r-a+b}{2} \right)^p - \left(\frac{2r+a-b}{2} \right)^p \\ &= r^p \left[\left(1 - \frac{a+b}{2r} \right)^p + \left(1 + \frac{a+b}{2r} \right)^p - \left(1 + \frac{b-a}{2r} \right)^p - \left(1 - \frac{b-a}{2r} \right)^p \right]. \end{aligned} \quad (2.5)$$

Using an identity from Gould [1], $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^k = ((1 + \sqrt{x})^n + (1 - \sqrt{x})^n)/2$, (2.5) becomes

$$\begin{aligned} r_p(x, y, z) &= 2r^p \sum_{t=0}^{\lfloor p/2 \rfloor} \binom{p}{2t} \left[\left(\frac{a+b}{2r} \right)^{2t} - \left(\frac{b-a}{2r} \right)^{2t} \right], \\ f_p(x, y, z) &= \frac{r^{p-1}}{pab} \sum_{t=0}^{\lfloor p/2 \rfloor} \binom{p}{2t} \left[\left(\frac{a+b}{2r} \right)^{2t} - \left(\frac{b-a}{2r} \right)^{2t} \right] \\ &= \sum_{t=0}^{\lfloor p/2 \rfloor} \frac{r^{2m-2t}}{pab} \binom{p}{2t} \left[\left(\frac{a+b}{2} \right)^{2t} - \left(\frac{b-a}{2} \right)^{2t} \right] \\ &= \sum_{t=0}^{\lfloor p/2 \rfloor} \frac{r^{2m-2t}}{pab} \binom{p}{2t} [(ab + s^2)^t - s^{2t}] \\ &= \sum_t \frac{r^{2m-2t}}{p} \binom{p}{2t} \sum_i \binom{t}{i} s^{2i} a^{t-i-1} b^{t-i-1}. \end{aligned} \quad (2.6)$$

Substituting $j = m - t + i$, we obtain

$$f_p(x, y, z) = \sum_j \sum_i \frac{1}{2m+1} \binom{2m+1}{2m-2j+2i} \binom{m-j+i}{i} r^{2j-2i} s^{2i} a^{m-j-1} b^{m-j-1}, \quad (2.7)$$

where $0 \leq j \leq m-1$ and $0 \leq i \leq j$.

Note that (2.7) can be considered as an expression in $r^{2j-i} s^i$ where all terms with odd i are zero and $0 \leq i \leq 2j$, that is,

$$f_p(x, y, z) = \begin{cases} \sum_{j=0}^{m-1} \sum_{i=0}^{2j} \frac{1}{2m+1} \binom{2m+1}{2m-2j+i} \binom{m-j+i/2}{i/2} r^{2j-i} s^i (ab)^{m-j-1}, & i \text{ even,} \\ 0, & i \text{ odd.} \end{cases} \quad (2.8)$$

We now equate (2.4) and (2.8), removing the common summation terms, which gives the identity.

REMARK 2.1. By using Wegschaider's MultiSum, a computer algebra package which is available from <http://www.risc.uni-linz.ac.at/research/combinat/risc/software/>, we found recurrence equations satisfied by the summand of our identity but we could not use them to give a short proof for our identity.

REFERENCES

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