EXCEPTIONAL SMOOTH BOL LOOPS

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One new class of smooth Bol loops, exceptional Bol loops, is introduced and studied. The approach to the Campbell-Hausdorff formula is outlined. Bol-Bruck loops and Moufang loops are exceptional which justifies our consideration.

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1. Basic differential equations. We consider a $C^3$-smooth (and, consequently, $C^\omega$-smooth) right Bol loop $Q = \langle Q, \cdot, \varepsilon \rangle$ with the defining identity $y \cdot [(b \cdot x) \cdot b] = [(y \cdot b) \cdot x] \cdot b$ (right Bol identity), or, in another form, $R_b * R_x * R_b = R_{(b \cdot x) \cdot b}$, $R_b z \overset{\text{def}}{=} z \cdot b$.

(Here $(f * g)y = (g \circ f)y = g(f(y))$.)

The differential equation of a right Bol loop is of the form

$$-A^{\alpha}_{\beta}(x) \frac{\partial (x \cdot y)^{\alpha}}{\partial x^{\beta}} + 2\Gamma^\beta_{\alpha}(y) \frac{\partial (x \cdot y)^{\alpha}}{\partial y^{\beta}} = A^\alpha_{\sigma}(x \cdot y),$$

$$A^\mu_{\tau}(z) = \left[ \frac{\partial (z \cdot b)^{\mu}}{\partial b^{\tau}} \right]_{b=\varepsilon}, \quad a^\mu_{\tau}(z) = \left[ \frac{\partial (b \cdot z)^{\mu}}{\partial b^{\tau}} \right]_{b=\varepsilon},$$

$$\Gamma^\mu_{\tau}(z) = \frac{1}{2} \left[ A^\mu_{\tau}(z) + a^\mu_{\tau}(z) \right].$$

(1.1)

See [1, Chapter 5].

According to general theory, the conditions of integrability for (1.1) are

$$[A_{\alpha}, [A_{\beta}, A_{\gamma}]] = \rho_{\alpha,\beta\gamma}^\nu A_{\nu}, \quad A_{\nu}(x) = A^\nu_{\tau}(x) \frac{\partial}{\partial x^{\tau}}, \quad \rho_{\alpha,\beta\gamma}^\nu = \text{const.},$$

(1.2)

where $A_{\nu}$ is the so-called basic left fundamental vector fields. Evidently $A^\mu_{\mu}(\varepsilon) = \delta^\nu_{\mu}$.

Since a $C^3$-smooth right Bol loop is right monoalternative, that is, $(a \cdot t b) \cdot u b = a \cdot (t + u) b$, where $t b = \text{Exp}$

$$[(\text{Exp} t \zeta)^{\alpha}]^{\prime} = A^\alpha_{\beta}(\text{Exp} t \zeta) \zeta^{\beta}, \quad (\text{Exp} t \zeta)|_{t=0} = \varepsilon,$$

(1.3)

we get the differential equation

$$[(a \cdot t b)^{\alpha}]^{\prime} = A^\alpha_{\beta}(a \cdot t b) (\text{Exp}^{-1} b)^{\beta}, \quad a \cdot 0 b = a,$$

(1.4)

(see [1, Propositions 4.12, 4.22]), allowing to restore the law of composition $a \cdot b = \psi(a, b, 1)$, where $\psi(a, b, t) = a \cdot t b$ is the solution of (1.4) with the initial condition $\psi(a, b, 0) = a$. 
2. Conditions of integrability. Now we introduce

\[ [A_\alpha, A_\beta](x) = c^\gamma_{\alpha \delta}(x) A_\gamma(x), \quad c^\gamma_\beta(x) = (\text{Exp}^{-1} x)^\alpha c^\gamma_{\alpha \beta}(x). \]  

(2.1)

Then (1.2) means \( A_\alpha(x)(c^{\nu \beta \gamma}(x) A_\nu(x)) - c^{\nu \beta \gamma}(x) A_\nu(x) A_\alpha(x) = \rho_{\alpha \beta \gamma}^\mu A_\mu(x), \) or

\[ A_\alpha^\gamma(x) \frac{\partial c^{\nu \beta \gamma}(x)}{\partial x^\sigma} + c^{\nu \beta \gamma}(x) c^{\mu \alpha \nu}(x) = \rho_{\alpha \beta \gamma}^\mu. \]  

(2.2)

Taking \( x = tb \) and contracting both parts with \( (\text{Exp}^{-1} b)^\alpha \) and \( (\text{Exp}^{-1} b)^\beta \), in normal coordinates \( ((\text{Exp}^{-1} b)^\nu = b^\nu) \), we get

\[ b^\alpha A_\sigma^\alpha(t b) \frac{\partial \{ b^\beta c^{\mu \nu}(t b) \}}{\partial (t b)^\sigma} + b^\beta c^{\nu \beta}(t b) b^\alpha c^{\mu \alpha \nu}(t b) = \rho_{\gamma}^\mu b^\beta b^\alpha \beta. \]  

(2.3)

According to (1.4), \( (\text{Exp}^{-1} b)^\alpha A_\sigma^\alpha(t b) = d(t b)^\sigma / dt \), and, instead of (2.3), we have

\[ (c^{\mu \nu})' + c^{\nu \beta} c^{\mu \gamma} = \rho_{\beta}^\mu, \quad c^{\nu \beta}_{\mid t=0} = c^{\mu \nu}_{0}, \]  

(4.2)

\[ c^{\mu \nu} = b^\beta c^{\mu \beta \gamma}(t b), \quad \rho_{\beta}^\mu = b^\alpha b^\beta \rho_{\alpha \beta \gamma}^\mu, \quad c^{\mu \gamma}_{\gamma} = b^\beta c^{\mu \beta \gamma}(\epsilon). \]  

(2.5)

Thus \( c^{\mu \nu} \) may be obtained as the unique solution of the differential equation (2.4).

3. Conditions on fundamental vector fields. In accordance with the general theory, [1, Chapter 4], the fields \( A_\nu \) \((\nu = 1, \ldots, r)\), or matrix \( A_\mu^\nu(x) \), can be uniquely restored by \( c^{\mu \nu} \) in normal coordinates \( ((\text{Exp} b) = b) \). For this we should consider \( B^\nu_\sigma(x) \), the inverse of \( A_\mu^\nu(x) \), and, further, solve the equation

\[ [tB^\nu_\mu(t b)]' = \delta^\nu_\mu - c^{\nu \beta}_{\mu} [tB^\nu_\beta(t b)], \quad tB^\nu_\beta \mid _{t=0} = 0. \]  

(3.1)

It gives us \( B^\nu_\mu(t b) \) in normal coordinates, as well as its inverse \( A_\mu^\nu(t b) \).

4. Matrix form of differential equations. As a result, regarding

\[ \bar{c} = \| b^\beta c^{\nu \beta}(\epsilon) \|_{\mu, \nu = 1, 2, \ldots, r}, \quad \bar{\rho} = \| b^\alpha b^\beta \rho_{\alpha \beta \gamma}^\mu \|_{\mu, \nu = 1, 2, \ldots, r} \]  

(4.1)

as given, we should solve, first, the equation

\[ \bar{c}' + \bar{c}^2 = \bar{\rho}, \quad \bar{c} \mid _{t=0} = \bar{c}_0. \]  

(4.2)

which is the matrix form of (2.4) and, further, solve the equation

\[ [t\tilde{B}(t b)]' = \tilde{\Pi} - [t\tilde{B}(t b)]\tilde{c}, \quad t\tilde{B} \mid _{t=0} = \tilde{c}_0. \]  

(4.3)

which is the matrix form of (3.1).

The solution of (4.2) may be presented as a series decomposition depending on \( \bar{c} \) and \( \bar{\rho} \), but the formulas are rather complicated, since \( \bar{c} \) and \( \bar{\rho} \) do not commute in general.
5. Bol loops of special type. If \( \mathfrak{B} \) is a Bol-Bruck loop, we have \( \overline{c} = 0 \), due to the automorphic inverse property \( (x \cdot y)^{-1} = x^{-1} \cdot y^{-1} \), valid for any Bol-Bruck loop.

Indeed, differentiating the above relation by \( x \) and \( y \) at \( x = y = \varepsilon \), we get \( c_{\rho y}^\mu(\varepsilon) = 0 \) and, further, \( \overline{c} = 0 \).

In the Moufang case we have, see [1, Chapter 6, (6.12)], \( \overline{\rho} = \overline{c}^2 \). Thus for Bol-Bruck and Moufang cases, in normal coordinates, we have

\[
\overline{\rho} = \overline{c} = \overline{c}^2 = c_{\rho y}^\mu(\varepsilon), \quad \overline{\rho} = |\overline{\rho}^\mu_{\alpha, \beta y} b^\alpha b^\beta|.
\]

**Definition 5.1.** A right Bol loop satisfying (5.1) is called exceptional.

**Remark 5.2.** The question if the exceptional Bol loops may be described in the terms of identities depending on loop (or module) operations only, is open.

**Remark 5.3.** Of course, (5.1) may be rewritten as identity on \( c_{\rho y}^\mu(\varepsilon) \) and \( \rho^\mu_{\alpha, \beta y} \) which, together with the identities of a right Bol algebra, defines an exceptional Bol algebra.

6. Explicit form of structure functions. For an exceptional Bol algebra, due to (5.1), the solution of (4.2) is, evidently,

\[
\overline{c} = \overline{\rho}^{1/2} \tanh \left( \overline{\rho}^{1/2} t + \overline{a} \right), \quad \overline{a} = \tanh^{-1} \left( \overline{\rho}^{-1/2} \overline{c} \right),
\]

and may be expressed as

\[
\overline{c} = \overline{\rho}^{1/2} \tanh \left( \overline{\rho}^{1/2} t + \overline{a} \right) + \overline{c} \left[ \overline{\rho} + \overline{c} \left( \overline{\rho}^{-1/2} \tanh \left( \overline{\rho}^{1/2} t \right) \right) \right]^{-1}.
\]

Of course, (6.1) or (6.2) should be understood in the right way: there is no need for the existence of \( \overline{\rho}^{1/2} \), since the final result in the form of the series depends only on \( \overline{\rho} \).

7. Explicit form of basic vector fields. Now we solve (4.3), where \( \overline{c} \) is given by (6.1),

\[
\overline{t B(t b)} = \overline{t B(t b)} \overline{\rho}^{1/2} \tanh \left( \overline{\rho}^{1/2} t + \overline{a} \right), \quad \overline{t B(t b)} \big|_{t = 0} = \overline{c},
\]

or

\[
\overline{t B(t b) \cosh \left( \overline{\rho}^{1/2} t + \overline{a} \right)'} = \cosh \left( \overline{\rho}^{1/2} t + \overline{a} \right), \quad \overline{t B(t b)} \big|_{t = 0} = \overline{c}.
\]

Thus

\[
t B(t b) \cosh \left( \overline{\rho}^{1/2} t + \overline{a} \right) = \overline{\rho}^{-1/2} \sinh \left( \overline{\rho}^{1/2} t + \overline{a} \right) + D.
\]
Since \( [tB(tb)]_{t=0} = \overline{\rho} \rho^{-1/2} \sinh \overline{a} \), we get \( \overline{\rho} = \rho^{-1/2} \frac{\sinh (\rho^{1/2} + \overline{a}) - \sinh \overline{a}}{\cosh (\rho^{1/2} + \overline{a})} \), \( \overline{a} = \tanh^{-1} \left( \rho^{-1/2} \overline{c} \right) \). (7.4)

Or, using the formulas of hyperbolic trigonometry, we get

\[
\overline{B}(b) = \rho^{-1/2} \left( \cosh \rho^{1/2} - \overline{\rho} \right) \tanh \overline{a} + \sinh \rho^{1/2} \overline{c} \cosh \rho^{1/2} \right),
\]
and, by (7.4),

\[
\overline{B}(b) = \rho^{-1} \left( \cosh \rho^{1/2} - \overline{\rho} \right) \frac{\overline{c} + \rho^{-1} \sinh \rho^{1/2}}{\cosh \rho^{1/2} + \left( \rho^{-1/2} \sinh \rho^{1/2} \right) \overline{c} + \rho^{-1} \sinh \rho \overline{v}^{1/2}}, \quad \overline{\rho} \rho^\alpha = \rho^\alpha_{\mu,\nu} b^\mu b^\nu. \quad (7.6)
\]

As a result,

\[
\overline{A}(b) = \overline{B}^{-1}(b) = \left( \cosh \rho^{1/2} + \left( \rho^{-1/2} \sinh \rho^{1/2} \right) \overline{c} \right) \rho^{-1} \left( \cosh \rho^{1/2} - \overline{\rho} \right), \quad (7.7)
\]

meaning that \( \overline{A}(b) = \text{Id} + \sum_{k=1}^{\infty} \overline{a}_k (\overline{\rho})^k \), where \( \overline{a}_k \) depends on \( \overline{\rho} \) only.

8. Analogue of the Campbell-Hausdroff decomposition. Now, using the differential equation (1.4) in normal coordinates (i.e., \( (\text{Exp}^{-1} b)^\mu = b^\mu \)), we may obtain by successive differentiation with respect to \( t \):

\[
\frac{d^k (a \cdot t b)^\alpha}{dt^k} = \{ A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_k} (a \cdot t b)^\alpha \} b^{\sigma_1} b^{\sigma_2} \cdots b^{\sigma_k}. \quad (8.1)
\]

As a result the Taylor decomposition

\[
(a \cdot b)^\alpha = a^\alpha + \sum_{m=1}^{\infty} \frac{1}{m!} (A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_m} a)^\alpha b^{\sigma_1} b^{\sigma_2} \cdots b^{\sigma_m} \quad (8.2)
\]
is valid for an appropriate neighbourhood of \((0, \ldots, 0)\).

Taking \( A_\alpha(a) = A^\alpha_\nu(a) \partial \overline{a}^\nu \), where the matrix \( A^\alpha_\nu(a) \) is defined by (7.7), we get the analogue of the Campbell-Hausdorff series for a smooth exceptional right Bol loop. Certain technical efforts would allow us to obtain an explicit expression of the coefficients \( (A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_m} a)^\alpha \) for an exceptional right Bol loop which would lead to the analogue of the Dynkin formula from the Lie group theory.
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REFERENCES


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