ON THE RELATION BETWEEN INTERIOR CRITICAL POINTS OF POSITIVE SOLUTIONS AND PARAMETERS FOR A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS

G. A. AFROUZI and M. KHALEGHY MOGHADDAM

Received 10 September 2001

We consider the boundary value problem $-u''(x) = \lambda f(u(x))$, $x \in (0, 1)$; $u'(0) = 0$; $u'(1) + \alpha u(1) = 0$, where $\alpha > 0$, $\lambda > 0$ are parameters and $f \in C^2[0, \infty)$ such that $f(0) < 0$. In this paper, we study for the two cases $\rho = 0$ and $\rho = \theta$ (ρ is the value of the solution at $x = 0$ and θ is such that $F(\theta) = 0$ where $F(s) = \int_0^s f(t)dt$) the relation between λ and the number of interior critical points of the nonnegative solutions of the above system.

2000 Mathematics Subject Classification: 34B15.

1. Introduction. We consider the two point boundary value problem with Neumann-Robin boundary conditions

$$-u''(x) = \lambda f(u(x)), \quad x \in (0, 1), \quad (1.1)$$

$$u'(0) = 0, \quad (1.2)$$

$$u'(1) + \alpha u(1) = 0, \quad (1.3)$$

where $\alpha > 0$, $\lambda > 0$ are parameters, $f \in C^2[0, \infty)$ and $f(0) < 0$, and we will assume that there exist $\beta, \theta > 0$ such that $f(s) < 0$ on $[0, \beta)$, $f(\beta) = 0$, $f'(s) \geq 0$, $f''(s) > 0$, $\lim_{s \rightarrow \infty} (f(s)/s) = \infty$, and $F(\theta) = 0$ where $F(s) = \int_0^s f(t)dt$. It is proved in [1, Theorems 3.4.1(a) and 3.4.1(b)] that for any $n = 0, 1, 2, \dots$, $\alpha \in (0, \infty)$, $\rho = \theta$ ($\rho = 0$), (1.1), (1.2), and (1.3) have exactly two nonnegative solutions $u_{2n,i}(u_{2n+1,i})$, $i = 1, 2$ with $2n$ (and $2n + 1$) interior critical points. Also it is shown in [3, Theorem 1.4] that for the following Dirichlet boundary conditions

$$\begin{aligned} -u''(x) &= \lambda f(u(x)), \quad x \in (0, 1), \\ u(0) &= 0 = u(1), \end{aligned} \quad (1.4)$$

where n is a positive integer, there exists $\lambda^* > 0$ such that (1.4) has a unique nonnegative solution with n interior zeros if and only if $\lambda = (n + 1)^2 \lambda^*$. Equation (1.1) in the cases Neumann and Dirichlet-Robin boundary conditions have been studied in [2, 4], respectively. We discuss the relation between interior critical points of nonnegative solutions and λ 's for problem (1.1), (1.2), and (1.3) for the case $\rho = \theta$ in Section 2, and for the case $\rho = 0$ in Section 3. Finally, in Section 4 we compare λ 's in two cases $\rho = \theta$ and $\rho = 0$ for any $n = 0, 1, 2, \dots$

FIGURE 2.1

2. The case $\rho = \theta$. In [1] it has been established that for $\alpha \in (0, \infty)$, $\rho = \theta$, and $n = 0, 1, 2, \dots$, there exists a unique number $m_{2n,1}^* \in (0, \alpha\theta)$ such that

$$G(m_{2n,1}^*) = H(m_{2n,1}^*), \quad (2.1)$$

where

$$\begin{aligned} G(m) &= \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + 2n \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m \in (0, \alpha\theta), \\ H(m) &= \frac{m}{\sqrt{-F(m/\alpha)}}, \quad m \in (0, \alpha\theta). \end{aligned} \quad (2.2)$$

So we obtain $\lambda = \lambda_{2n,1}(\theta, m_{2n,1}^*)$ such that $\sqrt{2\lambda} = G(m_{2n,1}^*) = H(m_{2n,1}^*)$ (see Figure 2.1), that is,

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + \frac{2n}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds \quad m = m_{2n,1}^*. \quad (2.3)$$

Thus (1.1), (1.2), and (1.3) have exactly a nonzero solution $u_{2n,1}$ with $2n$ interior critical points where $u'_{2n,1}(1) = -m_{2n,1}^*$ and $u'_{2n,1}(0) = \theta$ at $\lambda = \lambda_{2n,1}(\theta, m_{2n,1}^*)$. Also, the equation

$$\sqrt{\lambda} = \frac{2n+1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds \quad (2.4)$$

has a unique solution $\lambda = \lambda_{2n,2}(\theta, 0)$ such that for this λ problem, (1.1), (1.2), and (1.3) have exactly a nonnegative solution $u_{2n,2}$ with $2n$ interior critical points such that $u'_{2n,2}(1) = 0$ and $u_{2n,2}(0) = \theta$.

In [1], it is proved that

$$\lambda_{2n,1}(\theta, m_{2n,1}^*) < \lambda_{2n,2}(\theta, 0) < \lambda_{2(n+1),1}(\theta, m_{2(n+1),1}^*). \quad (2.5)$$

Now we are ready to prove the main theorem of this section.

FIGURE 2.2

THEOREM 2.1. *Let $n = 0, 1, 2, \dots$, then*

$$\lambda_{2n,2} - \lambda_{2n,1} < \lambda_{2(n+1),2} - \lambda_{2(n+1),1}, \quad (2.6)$$

that is, $n \mapsto \lambda_{2n,2} - \lambda_{2n,1}$ is a strictly increasing function.

PROOF. Since $G(m)$ is dependent on n , so we write it by $G_n(m)$, that is,

$$G_n(m) = \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + 2n \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m \in (0, \alpha\theta). \quad (2.7)$$

So it is easy to see that $\{G_n(m)\}_{n=0}^{\infty}$ is a strictly increasing sequence of n for every $m \in (0, \alpha\theta)$, that is,

$$G_n(m) < G_{n+1}(m), \quad m \in (0, \alpha\theta) \quad (2.8)$$

and we can easily see that

$$m_{2n,1}^* < m_{2(n+1),1}^*, \quad n = 0, 1, 2, \dots \quad (2.9)$$

(see Figure 2.2). On the other hand, from (2.3) and (2.4) we have

$$\begin{aligned} \sqrt{\lambda_{2n,2}} - \sqrt{\lambda_{2n,1}} &= \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds - \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n,1}^*, \\ \sqrt{\lambda_{2(n+1),2}} - \sqrt{\lambda_{2(n+1),1}} &= \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds - \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2(n+1),1}^*. \end{aligned} \quad (2.10)$$

Thus combining (2.9) and (2.10) we obtain

$$\sqrt{\lambda_{2n,2}} - \sqrt{\lambda_{2n,1}} < \sqrt{\lambda_{2(n+1),2}} - \sqrt{\lambda_{2(n+1),1}}. \quad (2.11)$$

Also combining (2.4) and (2.9) we have

$$\begin{aligned} & \sqrt{\lambda_{2n,2}} + \sqrt{\lambda_{2n,1}} \\ &= \frac{4n+1}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{1}{\sqrt{2}} \int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n,1}^*, \end{aligned} \quad (2.12)$$

$$\begin{aligned} & \sqrt{\lambda_{2(n+1),2}} + \sqrt{\lambda_{2(n+1),1}} \\ &= \frac{4n+5}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{1}{\sqrt{2}} \int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2(n+1),1}^*. \end{aligned} \quad (2.13)$$

Since $0 < m/\alpha < \theta$ for $m = m_{2n,1}^*$, so we have

$$\int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds < \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n,1}^*, \quad (2.14)$$

and then

$$\frac{1}{\sqrt{2}} \int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{4n+1}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds < \frac{1}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{4n+1}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds. \quad (2.15)$$

Now from (2.12) and (2.15) we obtain

$$\sqrt{\lambda_{2n,2}} + \sqrt{\lambda_{2n,1}} < \frac{4n+2}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds. \quad (2.16)$$

On the other hand, by the positivity of

$$\frac{3}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{1}{\sqrt{2}} \int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2(n+1),1}^*, \quad (2.17)$$

and also from (2.13) and (2.16) we obtain

$$\sqrt{\lambda_{2n,2}} + \sqrt{\lambda_{2n,1}} < \sqrt{\lambda_{2(n+1),2}} + \sqrt{\lambda_{2(n+1),1}}. \quad (2.18)$$

Now combining (2.11) and (2.18) we obtain

$$\begin{aligned} & (\sqrt{\lambda_{2n,2}} - \sqrt{\lambda_{2n,1}}) (\sqrt{\lambda_{2n,2}} + \sqrt{\lambda_{2n,1}}) < (\sqrt{\lambda_{2n,2}} + \sqrt{\lambda_{2n,1}}) (\sqrt{\lambda_{2(n+1),2}} - \sqrt{\lambda_{2(n+1),1}}) \\ & < (\sqrt{\lambda_{2(n+1),2}} - \sqrt{\lambda_{2(n+1),1}}) (\sqrt{\lambda_{2(n+1),2}} + \sqrt{\lambda_{2(n+1),1}}) \end{aligned} \quad (2.19)$$

and so,

$$\lambda_{2n,2} - \lambda_{2n,1} < \lambda_{2(n+1),2} - \lambda_{2(n+1),1}, \quad (2.20)$$

thus, the proof is complete. $\square$

FIGURE 3.1

3. The case $\rho = 0$. Also in [1] it has been established that for $\alpha \in (0, \infty)$, $\rho = 0$, and $n = 0, 1, 2, \dots$, there exists a unique number $m_{2n+1,1}^* \in (0, \alpha\theta)$ such that

$$\tilde{G}(m_{2n+1,1}^*) = H(m_{2n+1,1}^*), \quad (3.1)$$

where

$$\begin{aligned} \tilde{G}(m) &= \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + (2n+1) \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m \in (0, \alpha\theta), \\ H(m) &= \frac{m}{\sqrt{-F(m/\alpha)}}, \quad m \in (0, \alpha\theta). \end{aligned} \quad (3.2)$$

So we obtain $\lambda = \lambda_{2n+1,1}(0, m_{2n+1,1}^*)$ such that $\sqrt{2\lambda} = \tilde{G}(m_{2n+1,1}^*) = H(m_{2n+1,1}^*)$ (see Figure 3.1), that is,

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + \frac{2n+1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n+1,1}^*. \quad (3.3)$$

Thus (1.1), (1.2), and (1.3) have exactly a nonzero solution $u_{2n+1,1}$ with $2n+1$ interior critical points where $u'_{2n+1,1}(1) = -m_{2n+1,1}^*$ and $u'_{2n+1,1}(0) = 0$ at $\lambda = \lambda_{2n+1,1}(0, m_{2n+1,1}^*)$. Also, the equation

$$\sqrt{\lambda} = \frac{2(n+1)}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds \quad (3.4)$$

has a unique solution $\lambda = \lambda_{2n+1,2}(0, 0)$ such that for this λ problem, (1.1), (1.2), and (1.3) have exactly a nonnegative solution $u_{2n+1,2}$ with $2n+1$ interior critical points such that $u'_{2n+1,2}(1) = 0$ and $u_{2n+1,2}(0) = 0$.

In [1], it is proved that

$$\lambda_{2n+1,1}(0, m_{2n+1,1}^*) < \lambda_{2n+1,2}(0, 0) < \lambda_{2(n+1)+1,1}(0, m_{2(n+1)+1,1}^*). \quad (3.5)$$

Now we are ready to prove the main theorem of this section.

FIGURE 3.2

THEOREM 3.1. *Let $n = 0, 1, 2, \dots$, then*

$$\lambda_{2n+1,2} - \lambda_{2n+1,1} < \lambda_{2(n+1)+1,2} - \lambda_{2(n+1)+1,1}, \quad (3.6)$$

that is, $n \mapsto \lambda_{2n+1,2} - \lambda_{2n+1,1}$ is a strictly increasing function.

PROOF. Since $\tilde{G}(m)$ is dependent on n , so we write it by $\tilde{G}_n(m)$, that is,

$$\tilde{G}_n(m) = \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + (2n+1) \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m \in (0, \alpha\theta). \quad (3.7)$$

It is easy to see that $\{\tilde{G}_n(m)\}_{n=0}^{\infty}$ is a strictly increasing sequence of n for every $m \in (0, \alpha\theta)$, that is,

$$\tilde{G}_n(m) < \tilde{G}_{n+1}(m), \quad m \in (0, \alpha\theta) \quad (3.8)$$

(see Figure 3.2) and we can easily see that

$$m_{2n+1,1}^* < m_{2(n+1)+1,1}^*, \quad n = 0, 1, 2, \dots \quad (3.9)$$

(see Figure 3.2). On the other hand, from (3.3) and (3.4) we have

$$\begin{aligned} \sqrt{\lambda_{2n+1,2}} - \sqrt{\lambda_{2n+1,1}} &= \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds - \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n+1,1}^*, \\ \sqrt{\lambda_{2(n+1)+1,2}} - \sqrt{\lambda_{2(n+1)+1,1}} &= \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds - \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2(n+1)+1,1}^*. \end{aligned} \quad (3.10)$$

Thus, combining (3.9) and (3.10) we obtain

$$\sqrt{\lambda_{2n+1,2}} - \sqrt{\lambda_{2n+1,1}} < \sqrt{\lambda_{2(n+1)+1,2}} - \sqrt{\lambda_{2(n+1)+1,1}}. \quad (3.11)$$

Also combining (3.3) and (3.4) we have

$$\begin{aligned} & \sqrt{\lambda_{2n+1,2}} + \sqrt{\lambda_{2n+1,1}} \\ &= \frac{4n+3}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{1}{\sqrt{2}} \int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n+1,1}^*, \end{aligned} \quad (3.12)$$

$$\begin{aligned} & \sqrt{\lambda_{2(n+1)+1,2}} + \sqrt{\lambda_{2(n+1)+1,1}} \\ &= \frac{4n+7}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{1}{\sqrt{2}} \int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2(n+1)+1,1}^*. \end{aligned} \quad (3.13)$$

Since $0 < m/\alpha < \theta$ for $m = m_{2n+1,1}^*$, so we have

$$\int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds < \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n+1,1}^*, \quad (3.14)$$

and then

$$\begin{aligned} & \frac{1}{\sqrt{2}} \int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{4n+3}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds \\ & < \frac{1}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{4n+3}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds. \end{aligned} \quad (3.15)$$

Now from (3.12) and (3.15) we obtain

$$\sqrt{\lambda_{2n+1,2}} + \sqrt{\lambda_{2n+1,1}} < \frac{4n+4}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds. \quad (3.16)$$

On the other hand, by the positivity of

$$\frac{3}{\sqrt{2}} \int_0^\theta \frac{1}{\sqrt{-F(s)}} ds + \frac{1}{\sqrt{2}} \int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2(n+1)+1,1}^*, \quad (3.17)$$

and also from (3.13) and (3.16) we obtain

$$\sqrt{\lambda_{2n+1,2}} + \sqrt{\lambda_{2n+1,1}} < \sqrt{\lambda_{2(n+1)+1,2}} + \sqrt{\lambda_{2(n+1)+1,1}}. \quad (3.18)$$

Now combining (3.11) and (3.18) we have

$$\lambda_{2n+1,2} - \lambda_{2n+1,1} < \lambda_{2(n+1)+1,2} - \lambda_{2(n+1)+1,1}, \quad (3.19)$$

thus, the proof is complete. $\square$

4. Comparing the two cases $\rho = 0$ and $\rho = \theta$. Now we compare λ 's in the two cases $\rho = 0$ and $\rho = \theta$ for any $n = 0, 1, 2, \dots$, and we are ready to prove the main theorem of this section.

THEOREM 4.1. *Let $n = 0, 1, 2, \dots$, then*

$$\lambda_{2n,2} - \lambda_{2n,1} < \lambda_{2n+1,2} - \lambda_{2n+1,1}, \quad (4.1)$$

that is, the distance between $\lambda_{2n,1}(\theta, m_{2n,1}^*)$ and $\lambda_{2n,2}(\theta, 0)$ is less than the distance between $\lambda_{2n+1,1}(0, m_{2n+1,1}^*)$ and $\lambda_{2n+1,2}(0, 0)$.

PROOF. Since $0 < m/\alpha < \theta$ for $m = m_{2n,1}^*$, we have

$$\int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds < \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n,1}^*, \quad (4.2)$$

and then

$$\frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + \frac{2n}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds < \frac{2n+1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n,1}^*. \quad (4.3)$$

Now, from (4.3) and (2.3) we obtain

$$\sqrt{\lambda_{2n,1}} < \frac{2n+1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds. \quad (4.4)$$

On the other hand, since $0 < m/\alpha < \theta$ for $m = m_{2n+1,1}^*$, then we have

$$0 < \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n+1,1}^*, \quad (4.5)$$

and then from (3.3) we have

$$\sqrt{\lambda_{2n,1}} < \sqrt{\lambda_{2n+1,1}}. \quad (4.6)$$

Also, we know that

$$\begin{aligned} \sqrt{2\lambda_{2n,1}} &= H(m) = \frac{m}{\sqrt{-F(m/\alpha)}}, \quad m = m_{2n,1}^*, \\ \sqrt{2\lambda_{2n+1,1}} &= H(m) = \frac{m}{\sqrt{-F(m/\alpha)}}, \quad m = m_{2n+1,1}^*. \end{aligned} \quad (4.7)$$

Now since function H is one to one on interval $(0, \alpha\theta)$ (see Figure 4.1), we have

$$m_{2n,1}^* < m_{2n+1,1}^*, \quad (4.8)$$

FIGURE 4.1

so we obtain

$$\int_{m_{2n+1,1}^*/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds < \int_{m_{2n,1}^*/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds. \quad (4.9)$$

Thus combining (2.10), (3.10), and (4.9) we obtain

$$\sqrt{\lambda_{2n,2}} - \sqrt{\lambda_{2n,1}} < \sqrt{\lambda_{2n+1,2}} - \sqrt{\lambda_{2n+1,1}}. \quad (4.10)$$

On the other hand, since $0 < m/\alpha < \theta$ for $m = m_{2n,1}^*$, we have

$$\int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds < \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n,1}^*, \quad (4.11)$$

and then

$$\frac{4n+1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds + \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds < \frac{4n+2}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n,1}^*, \quad (4.12)$$

and thus from (2.12) we obtain

$$\sqrt{\lambda_{2n,2}} - \sqrt{\lambda_{2n,1}} < \frac{4n+2}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad (4.13)$$

and also from (4.13) we have

$$\sqrt{\lambda_{2n,2}} + \sqrt{\lambda_{2n,1}} < \frac{4n+3}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds + \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m_{2n+1,1}^*. \quad (4.14)$$

So from (4.13) and (3.12), we obtain

$$\sqrt{\lambda_{2n,2}} + \sqrt{\lambda_{2n,1}} < \sqrt{\lambda_{2n+1,2}} + \sqrt{\lambda_{2n+1,1}}, \quad (4.15)$$

and also from (4.15) and (4.10), we have

$$\lambda_{2n,2} - \lambda_{2n,1} < \lambda_{2n+1,2} - \lambda_{2n+1,1} \quad (4.16)$$

thus, the proof is complete. $\square$

REFERENCES

- [1] V. Anuradha, C. Maya, and R. Shivaji, *Positive solutions for a class of nonlinear boundary value problems with Neumann-Robin boundary conditions*, J. Math. Anal. Appl. **236** (1999), no. 1, 94–124.
- [2] V. Anuradha and R. Shivaji, *Sign changing solutions for a class of superlinear multi-parameter semi-positone problems*, Nonlinear Anal. **24** (1995), no. 11, 1581–1596.
- [3] A. Castro and R. Shivaji, *Nonnegative solutions for a class of nonpositone problems*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), no. 3-4, 291–302.
- [4] A. R. Miciano and R. Shivaji, *Multiple positive solutions for a class of semipositone Neumann two-point boundary value problems*, J. Math. Anal. Appl. **178** (1993), no. 1, 102–115.

G. A. AFROUZI: DEPARTMENT OF MATHEMATICS, FACULTY OF BASIC SCIENCES, MAZANDARAN UNIVERSITY, BABOLSAR, IRAN

E-mail address: afrouzi@umz.ac.ir

M. KHALEGHY MOGHADDAM: DEPARTMENT OF MATHEMATICS, FACULTY OF BASIC SCIENCES, MAZANDARAN UNIVERSITY, BABOLSAR, IRAN