SOME RESULTS ON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN COMPLETE METRIC SPACES

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Received 17 October 2001

Using the concept of w -distance, we improve some well-known fixed point theorems.

2000 Mathematics Subject Classification: 47H10.

1. Introduction. Recently, Ume [3] improved the fixed point theorems in a complete metric space using the concept of w -distance, introduced by Kada, Suzuki, and Takahashi [2], and more general contractive mappings than quasi-contractive mappings.

In this paper, using the concept of w -distance, we first prove common fixed point theorems for multivalued mappings in complete metric spaces, then these theorems are used to improve Ćirić's fixed point theorem [1], Kada-Suzuki-Takahashi's fixed point theorem [2], and Ume's fixed point theorem [3].

2. Preliminaries. Throughout, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

DEFINITION 2.1 (see [2]). Let (X, d) be a metric space, then a function $p : X \times X \rightarrow [0, \infty)$ is called a w -distance on X if the following are satisfied:

- (1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
- (2) for any $x \in X$, $p(x, \cdot) : X \rightarrow [0, \infty)$ is lower semicontinuous;
- (3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

DEFINITION 2.2. Let (X, d) be a metric space with a w -distance p , then

- (1) for any $x \in X$ and $A \subseteq X$, $d(x, A) := \inf\{d(x, y) : y \in A\}$ and $d(A, x) := \inf\{d(y, x) : y \in A\}$;
- (2) for any $x \in X$ and $A \subseteq X$, $p(x, A) := \inf\{p(x, y) : y \in A\}$ and $p(A, x) := \inf\{p(y, x) : y \in A\}$;
- (3) for any $A, B \subseteq X$, $p(A, B) := \inf\{p(x, y) : x \in A, y \in B\}$;
- (4) $CB_p(X) = \{A \mid A \text{ is nonempty closed subset of } X \text{ and } \sup_{x, y \in A} p(x, y) < \infty\}$.

The following lemmas are fundamental.

LEMMA 2.3 (see [2]). Let X be a metric space with a metric d , let p be a w -distance on X . Let $\{x_n\}$ and $\{y_n\}$ be sequences in X , let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:

- (1) if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;
- (2) if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z ;
- (3) if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;
- (4) if $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

LEMMA 2.4 (see [3]). Let X be a metric space with a metric d , let p be a w -distance on X , and let T be a mapping of X into itself satisfying

$$p(Tx, Ty) \leq q \cdot \max \{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\} \quad (2.1)$$

for all $x, y \in X$ and some $q \in [0, 1)$. Then

- (1) for each $x \in X$, $n \in \mathbb{N}$, and $i, j \in \mathbb{N}$ with $i, j \leq n$,

$$p(T^i x, T^j x) \leq q \cdot \delta(O(x, n)); \quad (2.2)$$

- (2) for each $x \in X$ and $n \in \mathbb{N}$, there exist $k, l \in \mathbb{N}$ with $k, l \leq n$ such that

$$\delta(O(x, n)) = \max \{p(x, x), p(x, T^k x), p(T^l x, x)\}; \quad (2.3)$$

- (3) for each $x \in X$,

$$\delta(O(x, \infty)) \leq \frac{1}{1-q} \{p(x, x) + p(x, Tx) + p(Tx, x)\}; \quad (2.4)$$

- (4) for each $x \in X$, $\{T^n x\}_{n=1}^\infty$ is a Cauchy sequence.

3. Main results

THEOREM 3.1. Let X be a complete metric space with a metric d and let p be a w -distance on X . Suppose that S and T are two mappings of X into $CB_p(X)$ and $\varphi : X \times X \rightarrow [0, \infty)$ is a mapping such that

$$\max \{p(u_1, u_2), p(v_1, v_2)\} \leq q \cdot \varphi(x, y) \quad (3.1)$$

for all nonempty subsets A, B of X , $u_1 \in SA$, $u_2 \in S^2A$, $v_1 \in TB$, $v_2 \in T^2B$, $x \in A$, $y \in B$, and some $q \in [0, 1)$,

$$\sup \left\{ \sup \left(\frac{\varphi(x, y)}{\min[p(x, SA), p(y, TB)]} : x \in A, y \in B \right) : A, B \subseteq X \right\} < \frac{1}{q}, \quad (3.2)$$

$$\inf \{p(y, u) + p(x, Sx) + p(y, Ty) : x, y \in X\} > 0, \quad (3.3)$$

for every $u \in X$ with $u \notin Su$ or $u \notin Tu$, where SA means $\bigcup_{a \in A} Sa$. Then S and T have a common fixed point in X .

PROOF. Let

$$\beta = \sup \left\{ \sup \left(\frac{\varphi(x, y)}{\min[p(x, SA), p(y, TB)]} : x \in A, y \in B \right) : A, B \subseteq X \right\}, \quad (3.4)$$

and $k = \beta q$. Define $x_{n+1} \in Sx_n$ and $y_{n+1} \in Ty_n$ for all $n \in \mathbb{N}$. Then $x_n \in Sx_{n-1}$, $x_{n+1} \in S^2x_{n-1}$, $y_n \in Ty_{n-1}$, and $y_{n+1} \in T^2y_{n-1}$. From (3.1) and (3.2), we have

$$p(x_n, x_{n+1}) \leq kp(x_{n-1}, x_n) \leq \cdots \leq k^{n-1}p(x_1, x_2), \quad (3.5)$$

$$p(y_n, y_{n+1}) \leq kp(y_{n-1}, y_n) \leq \cdots \leq k^{n-1}p(y_1, y_2), \quad (3.6)$$

for all $n \in \mathbb{N}$ and some $k \in [0, 1)$. Let n and m be any positive integers such that $n < m$. Then, from (3.6), we obtain

$$\begin{aligned} p(y_n, y_m) &\leq p(y_n, y_{n+1}) + \cdots + p(y_{m-1}, y_m) \\ &= \sum_{i=0}^{m-n-1} p(y_{n+i}, y_{n+i+1}) \\ &\leq \sum_{i=0}^{m-n-1} k^{n+i-1} p(y_1, y_2) \\ &\leq \frac{k^{n-1}}{(1-k)} p(y_1, y_2). \end{aligned} \quad (3.7)$$

By Lemma 2.3, $\{y_n\}$ is a Cauchy sequence. Since X is complete, $\{y_n\}$ converges to $u \in X$. Then, since $p(y_n, \cdot)$ is lower semicontinuous, from (3.7) we have

$$p(y_n, u) \leq \liminf_{m \rightarrow \infty} p(y_n, y_m) \leq \frac{k^{n-1}}{(1-k)} p(y_1, y_2). \quad (3.8)$$

Suppose that $u \notin Su$ or $u \notin Tu$. Then, by (3.3), (3.5), (3.6), and (3.8), we have

$$\begin{aligned} 0 &< \inf \{p(y, u) + p(x, Sx) + p(y, Ty) : x, y \in X\} \\ &\leq \inf \{p(y_n, u) + p(x_n, x_{n+1}) + p(y_n, y_{n+1}) : n \in \mathbb{N}\} \\ &\leq \inf \left\{ \frac{k^{n-1}}{(1-k)} p(y_1, y_2) + k^{n-1} p(x_1, x_2) + k^{n-1} p(y_1, y_2) : n \in \mathbb{N} \right\} \\ &= \left\{ \frac{2-k}{(1-k)} p(y_1, y_2) + p(x_1, x_2) \right\} \inf \{k^{n-1} : n \in \mathbb{N}\} \\ &= 0. \end{aligned} \quad (3.9)$$

This is a contradiction. Therefore we have $u \in Su$ and $u \in Tu$. □

THEOREM 3.2. *Let X be a complete metric space with a metric d and let p be a w -distance on X . Suppose that S and T are two mappings of X into $CB_p(X)$ and $\varphi : X \times X \rightarrow [0, \infty)$ is a mapping such that*

$$\max \{p(u_1, u_2), p(v_1, v_2)\} \leq q \cdot \varphi(x, y) \quad (3.10)$$

for all $x, y \in X$, $u_1 \in Sx$, $u_2 \in S^2x$, $v_1 \in Ty$, $v_2 \in T^2y$, and some $q \in [0, 1)$,

$$\sup \left\{ \sup \left(\frac{\varphi(x, y)}{\min [p(x, Sx), p(y, Ty)]} : x \in A, y \in B \right) : A, B \subseteq X \right\} < \frac{1}{q}, \quad (3.11)$$

and (3.3) is satisfied. Then S and T have a common fixed point in X .

PROOF. By a method similar to that in the proof of [Theorem 3.1](#), the result follows. $\square$

THEOREM 3.3. *Let X be a complete metric space with a metric d and let p be a w -distance on X . Suppose that T is a mapping of X into $CB_p(X)$ and $\psi : X \rightarrow [0, \infty)$ is a mapping such that*

$$p(u_1, u_2) \leq q \cdot \psi(x) \quad (3.12)$$

for all $x \in X$, $u_1 \in Tx$, $u_2 \in T^2x$ and some $q \in [0, 1)$,

$$\sup \left\{ \frac{\psi(x)}{p(x, Tx)} : x \in X \right\} < \frac{1}{q}, \quad (3.13)$$

$$\inf \{p(x, u) + p(x, Tx) : x \in X\} > 0,$$

for every $u \in X$ with $u \notin Tu$. Then T has a fixed point in X .

PROOF. By a method similar to that in the proof of [Theorem 3.1](#), the result follows. $\square$

THEOREM 3.4. *Let X be a complete metric space with a metric d and let p be a w -distance on X . Suppose that S and T are self-mapping of X and $\varphi : X \times X \rightarrow [0, \infty)$ is a mapping such that*

$$\max \{p(Sx, S^2x), p(Ty, T^2y)\} \leq q \cdot \varphi(x, y) \quad (3.14)$$

for all $x, y \in X$ and some $q \in [0, 1)$,

$$\sup \left\{ \frac{\varphi(x, y)}{\min[p(x, Sx), p(y, Ty)]} : x, y \in X \right\} < \frac{1}{q}, \quad (3.15)$$

$$\inf \{p(y, u) + p(x, Sx) + p(y, Ty) : x, y \in X\} > 0,$$

for every $u \in X$ with $u \neq Su$ or $u \neq Tu$. Then S and T have a common fixed point in X .

PROOF. By a method similar to that in the proof of [Theorem 3.1](#), the result follows. $\square$

From [Theorem 3.1](#), we have the following corollary.

COROLLARY 3.5. *Let X be a complete metric space with a metric d and let p be a w -distance on X . Suppose that S and T are two mappings of X into $CB_p(X)$ and $\varphi : X \times X \rightarrow [0, \infty)$ is a mapping such that*

$$\begin{aligned} & \max \left\{ \sup [p(u_1, u_2) : u_1 \in Sx, u_2 \in S^2x], \right. \\ & \left. \sup [p(v_1, v_2) : v_1 \in Tx, v_2 \in T^2x] \right\} \leq q \cdot \varphi(x, y) \end{aligned} \quad (3.16)$$

for all $x, y \in X$ and some $q \in [0, 1)$, and that (3.3) and (3.11) are satisfied. Then S and T have a common fixed point in X .

From [Theorem 3.3](#), we have the following corollaries.

COROLLARY 3.6. *Let X be a complete metric space with a metric d and let p be a w -distance on X . Suppose that T is a mapping of X into $CB_p(X)$ and $\psi : X \rightarrow [0, \infty)$ is a mapping such that*

$$\sup [p(u_1, u_2) : u_1 \in Tx, u_2 \in T^2x] \leq q \cdot \psi(x) \quad (3.17)$$

for all $x \in X$ and some $q \in [0, 1)$, and that (3.13) is satisfied. Then T has a fixed point in X .

COROLLARY 3.7. *Let X be a complete metric space with a metric d and let p be a w -distance on X . Suppose that T is a self-mapping of X and $\psi : X \rightarrow [0, \infty)$ is a mapping such that*

$$p(Tx, T^2x) \leq q \cdot \psi(x) \quad (3.18)$$

for all $x \in X$ and some $q \in [0, 1)$,

$$\sup \left\{ \frac{\psi(x)}{p(x, Tx)} : x \in X \right\} < \frac{1}{q}, \quad (3.19)$$

$$\inf \{p(x, u) + p(x, Tx) : x \in X\} > 0,$$

for every $u \in X$ with $u \neq Tu$. Then T has a fixed point in X .

From [Corollary 3.7](#), we have the following corollaries.

COROLLARY 3.8 (see [\[3\]](#)). *Let X be a complete metric space with a metric d and let p be a w -distance on X . Suppose that T is a self-mapping of X such that*

$$p(Tx, Ty) \leq q \cdot \max \{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\} \quad (3.20)$$

for all $x, y \in X$ and some $q \in [0, 1)$, and that

$$\inf \{p(x, u) + p(x, Tx) : x \in X\} > 0 \quad (3.21)$$

for every $u \in X$ with $u \neq Tu$. Then T has a unique fixed point in X .

PROOF. By (3.20) and [Lemma 2.4\(3\)](#), we have

$$\sup \{p(T^i x, T^j x) \mid i, j \in \mathbb{N} \cup \{0\}\} < \infty \quad (3.22)$$

for every $x \in X$. Thus we may define a function $r : X \times X \rightarrow [0, \infty)$ by

$$r(x, y) = \max \{ \sup [p(T^i x, T^j x) \mid i, j \in \mathbb{N} \cup \{0\}], p(x, y) \} \quad (3.23)$$

for every $x, y \in X$. Clearly, r is a w -distance on X . Let x be a given element of X , then, by using [Lemma 2.4\(1\)](#), (3.20), and (3.23), we have

$$\begin{aligned} r(Tx, T^2x) &= \sup \{p(T^i x, T^j x) \mid i, j \in \mathbb{N}\} \\ &\leq q \cdot \sup \{p(T^i x, T^j x) \mid i, j \in \mathbb{N} \cup \{0\}\} \\ &= q \cdot r(x, Tx). \end{aligned} \quad (3.24)$$

By (3.21) and (3.23), we obtain

$$\inf \{r(x, u) + r(x, Tx) : x \in X\} > 0 \quad (3.25)$$

for every $u \in X$ with $u \neq Tu$. From (3.24), (3.25), and Corollary 3.7, T has a fixed point in X . By (3.20) and Lemma 2.4, it is clear that the fixed point of T is unique. $\square$

COROLLARY 3.9 (see [2]). *Let X be a complete metric space, let p be a w -distance on X , and let T be a mapping from X into itself. Suppose that there exists $q \in [0, 1)$ such that*

$$p(Tx, T^2x) \leq q \cdot p(x, Tx) \quad (3.26)$$

for every $x \in X$ and that

$$\inf \{p(x, y) + p(x, Tx) : x \in X\} > 0 \quad (3.27)$$

for every $y \in X$ with $y \neq Ty$. Then T has a fixed point in X .

PROOF. Define $\psi : X \rightarrow [0, \infty)$ by

$$\psi(x) = p(x, Tx) \quad (3.28)$$

for all $x \in X$. Thus the conditions of Corollary 3.7 are satisfied. Hence T has a fixed point in X . $\square$

From Corollary 3.8, we have the following corollary.

COROLLARY 3.10 (see [1]). *Let X be a complete metric space with a metric d and let T be a mapping from X into itself. Suppose that T is a quasicontraction, that is, there exists $q \in [0, 1)$ such that*

$$d(Tx, Ty) \leq q \cdot \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \quad (3.29)$$

for every $x, y \in X$. Then T has a unique fixed point in X .

PROOF. It is clear that the metric d is a w -distance and

$$\inf \{d(x, y) + d(x, Tx) : x \in X\} > 0 \quad (3.30)$$

for every $y \in X$ with $y \neq Ty$. Thus, by Corollary 3.8 or 3.9, T has a unique fixed point in X . $\square$

ACKNOWLEDGMENT. This work was supported by grant No. 2001-1-10100-005-2 from the Basic Research Program of the Korea Science & Engineering Foundation.

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