ON A SUBCLASS OF α -CONVEX λ -SPIRAL FUNCTIONS

F. M. AL-BOUDI and M. M. HIDAN

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Let H denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the unit disc $\Delta = \{z : |z| < 1\}$. In this paper, we introduce the class $M_{\alpha}^{\lambda}[A, B]$ of functions $f \in H$ with $f(z)f'(z)/z \neq 0$, satisfying for $z \in \Delta : \{(e^{i\lambda} - \alpha \cos \lambda)(zf'(z)/f(z)) + \alpha \cos \lambda(1 + zf''(z)/f'(z))\} \prec \cos \lambda((1 + Az)/(1 + Bz)) + i \sin \lambda$, where $\prec$ denotes subordination, α and λ are real numbers, $|\lambda| < \pi/2$ and $-1 \leq B < A \leq 1$. Functions in $M_{\alpha}^{\lambda}[A, B]$ are shown to be λ -spiral-like and hence univalent. Integral representation, coefficients bounds, and other results are given.

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1. Introduction. Let H denote the class of functions f analytic in the unit disc $\Delta = \{z : |z| < 1\}$ and be given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Let $f \in H$ with $f(z)f'(z)/z \neq 0$ in Δ and α be a real number. Then f is said to be α -convex λ -spiral function, if and only if it satisfies the inequality $\operatorname{Re}\{(e^{i\lambda} - \alpha \cos \lambda)(zf'(z)/f(z)) + \alpha \cos \lambda(1 + zf''(z)/f'(z))\} > 0$ in Δ , for some λ , $|\lambda| < \pi/2$. The class of these functions, which is denoted by $SC(\alpha, \lambda)$ was defined and studied by Umarany [8].

In this paper, we introduce and study a subclass of $SC(\alpha, \lambda)$ defined by using subordination to convex functions.

DEFINITION 1.1. Let $f \in H$ with $f(z)f'(z)/z \neq 0$ in Δ . Then f is said to belong to the class $MS_{\alpha}^{\lambda}[A, B]$ if and only if for $z \in \Delta$,

$$K(\alpha, \lambda, f(z)) = \left\{ (e^{i\lambda} - \alpha \cos \lambda) \frac{zf'(z)}{f(z)} + \alpha \cos \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} \prec \cos \lambda \frac{1 + Az}{1 + Bz} + i \sin \lambda, \quad (1.1)$$

where $\prec$ denotes subordination, α and λ are real numbers, $|\lambda| < \pi/2$ and A and B are arbitrary fixed numbers such that $-1 \leq B < A \leq 1$.

It is clear from Definition 1.1 that a function $f \in MS_{\alpha}^{\lambda}[A, B]$ if and only if there exists a function $w(z)$ analytic in Δ and satisfying $w(0) = 0$ and $|w(z)| < 1$, $z \in \Delta$, such that

$$K(\alpha, \lambda, f(z)) = \cos \lambda \frac{1 + Aw(z)}{1 + Bw(z)} + i \sin \lambda. \quad (1.2)$$

It is also clear from Definition 1.1 that $M_{\alpha}^0[A, B] \equiv M_{\alpha}[A, B]$, the subclass of α -convex functions introduced by Kim and Jung [5] and $M_0^{\lambda}[A, B] \equiv Sp^{\lambda}[A, B]$, the subclass of spiral-like functions introduced by Dashrath and Shukla [2].

In this paper, we show that functions in $M_\alpha^\lambda[A, B]$ are spiral-like and hence univalent in Δ . Integral representation, coefficient bounds, and other results are given.

2. Spiral-likeness. To derive our main result, we prove the following lemma.

LEMMA 2.1. *Let $f \in H$, then $f \in MS_\alpha^\lambda[A, B]$ if and only if*

$$|K(\alpha, \lambda, f(z)) - m| < M, \quad z \in \Delta, \quad (2.1)$$

where

$$m = \cos \lambda \frac{1 - AB}{1 - B^2} + i \sin \lambda, \quad M = \frac{(A - B)}{1 - B^2} \cos \lambda. \quad (2.2)$$

PROOF. Suppose that $f \in MS_\alpha^\lambda[A, B]$. Then from (1.2) we obtain

$$\begin{aligned} K(\alpha, \lambda, f(z)) - m &= \frac{e^{i\lambda} - m + [(A - B) \cos \lambda + B e^{i\lambda} - mB]w(z)}{1 + Bw(z)} \\ &= M \frac{B + w(z)}{1 + Bw(z)}, \quad B \neq -1, \end{aligned} \quad (2.3)$$

using (2.2), hence

$$K(\alpha, \lambda, f(z)) - m = Mq(z). \quad (2.4)$$

It is clear that the function q satisfies $|q(z)| < 1$. Hence (2.1) follows from (2.4).

Conversely, suppose that (2.1) holds. Then

$$\left| \frac{K(\alpha, \lambda, f(z))}{M} - \frac{m}{M} \right| < 1, \quad B \neq -1. \quad (2.5)$$

Let

$$g(z) = \frac{K(\alpha, \lambda, f(z))}{M} - \frac{m}{M}, \quad B \neq -1, \quad (2.6)$$

$$w(z) = \frac{g(z) - g(0)}{1 - g(z)g(0)} = \frac{K(\alpha, \lambda, f(z)) - e^{i\lambda}}{(A - B) \cos \lambda + B e^{i\lambda} - BK(\alpha, \lambda, f(z))}. \quad (2.7)$$

Clearly $w(0) = 0$ and $|w(z)| < 1$. Rearranging (2.7) we get (1.2), hence $f \in MS_\alpha^\lambda[A, B]$. We note that condition (2.1) can be written as

$$\left| \frac{K(\alpha, \lambda, f(z)) - i \sin \lambda - (1 - A)/(1 - B) \cos \lambda}{\cos \lambda - \cos \lambda(1 - A)/(1 - B)} - \frac{1}{1 + B} \right| < \frac{1}{1 + B}, \quad z \in \Delta. \quad (2.8)$$

As $B \rightarrow -1$ and $A = 1$, the above condition reduces to the necessary and sufficient condition for f to belong to $MS_\alpha^\lambda[1, -1]$ (see [8]). $\square$

The following lemma is due to Jack [3].

LEMMA 2.2. *Let w be a nonconstant and analytic function in Δ , $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0 we can write*

$$z_0 w'(z_0) = \phi w(z_0), \quad (2.9)$$

where ϕ is a real number such that $\phi \geq 1$.

REMARK 2.3. Throughout, $-1 \leq B < A \leq 1$, unless otherwise indicated, $|\lambda| < \pi/2$.

THEOREM 2.4. *If $f \in MS_\alpha^\lambda[A, B]$, then $f \in \text{Sp}^\lambda[A, B]$ and hence univalent.*

PROOF. Let

$$\frac{zf'(z)}{f(z)} = \frac{1 + [(A-B)\cos\lambda e^{-i\lambda} + B]w(z)}{1 + Bw(z)} = \frac{1 + \eta w(z)}{1 + Bw(z)}, \quad (2.10)$$

where $\eta = (A-B)\cos\lambda e^{-i\lambda} + B$. Differentiating (2.10) logarithmically, we get

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{\eta zw'(z)}{1 + \eta w(z)} - \frac{Bzw'(z)}{1 + Bw(z)}. \quad (2.11)$$

Multiplying both sides of (2.11) by $\alpha \cos \lambda$ and adding $e^{i\lambda}(zf'(z)/f(z))$ to both sides, we obtain

$$\begin{aligned} \alpha \cos \lambda \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) + e^{i\lambda} \frac{zf'(z)}{f(z)} \\ = \left(\frac{\eta zw'(z)}{1 + \eta w(z)} - \frac{Bzw'(z)}{1 + Bw(z)} \right) \alpha \cos \lambda + e^{i\lambda} \frac{1 + \eta w(z)}{1 + Bw(z)}. \end{aligned} \quad (2.12)$$

Let r^* be the distance from the origin to the pole of w nearest to the origin. Then w is analytic in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.2 for $|z| \leq r$ ($r < r_0$), there exists a point z_0 such that

$$z_0 w'(z_0) = \phi w(z_0), \quad \phi \geq 1. \quad (2.13)$$

From (2.11) and (2.12), we obtain

$$K(\alpha, \lambda, f(z)) - m = \frac{N(z_0)}{R(z_0)}, \quad B \neq -1, \quad (2.14)$$

where

$$\begin{aligned} N(z_0) &= e^{i\lambda} - m + [(\eta e^{i\lambda} - Bm) + (e^{i\lambda} - m)\eta + \alpha \cos \lambda (\eta - B)\phi] w(z_0) \\ &\quad + (\eta e^{i\lambda} - Bm)\eta w^2(z_0), \end{aligned} \quad (2.15)$$

$$R(z_0) = 1 + (B + \eta)w(z_0) + B\eta w^2(z_0). \quad (2.16)$$

Now suppose that it was possible to have $\max_{|z|=r} |w(z)| = 1$ for some $r, r < r_0 \leq 1$. At the point z_0 where this occurred, we would have $|w(z_0)| = 1$. Then, by using the identities

$$e^{i\lambda} - m = BM, \quad \eta e^{i\lambda} - Bm = M, \quad B \neq -1 \quad (2.17)$$

in (2.15) we have

$$N(z_0) = BM + [\alpha \cos \lambda (\eta - B) \phi + \eta BM + M] w(z_0) + \eta M w^2(z_0). \quad (2.18)$$

From (2.16) and (2.18), we get

$$|N(z_0)|^2 - M^2 |R(z_0)|^2 = \tilde{a} + 2\tilde{b} \operatorname{Re} \{w(z_0)\}, \quad (2.19)$$

where

$$\begin{aligned} \tilde{a} &= \alpha \cos \lambda (\eta - B) \phi \{ \alpha \cos \lambda (\eta - B) \phi + 2M(1 + B\eta) \}, \\ \tilde{b} &= \alpha \cos \lambda (\eta - B) \phi M(B + \eta). \end{aligned} \quad (2.20)$$

From (2.19), we have

$$|N(z_0)|^2 - M^2 |R(z_0)|^2 > 0, \quad \text{provided } \tilde{a} \pm 2\tilde{b} > 0. \quad (2.21)$$

Now

$$\begin{aligned} \tilde{a} + \tilde{b} &= \alpha \cos \lambda (\eta - B) \phi \{ \alpha \cos \lambda (\eta - B) \phi + M(2 + 2B\eta + B + \eta) \} > 0, \\ \tilde{a} - \tilde{b} &= \alpha \cos \lambda (\eta - B) \phi \{ \alpha \cos \lambda (\eta - B) \phi + M(2 + 2B\eta - B - \eta) \} > 0. \end{aligned} \quad (2.22)$$

Thus it follows from (2.14) and (2.21) that

$$|K(\alpha, \lambda, f(z)) - m| > M. \quad (2.23)$$

But in view of Lemma 2.1, this is contrary to our assumption $f \in MS_\alpha^\lambda[A, B]$. So we cannot have $|w(z_0)| = 1$. Thus $|w(z)| \neq 1$ in $|z| < r_0$. Since $w(0) = 0$, $|w(z)|$ is continuous and $|w(z_0)| \neq 1$ in $|z| < r_0$, w cannot have a pole at $|z| = r_0$. Therefore, w is analytic in Δ and satisfies $|w(z)| < 1$ for $z \in \Delta$. Hence $f \in \operatorname{Sp}^\lambda[A, B]$. $\square$

REMARK 2.5. When $A = 1$ and $B = -1$, a result of Umarani [8] follows from Theorem 2.4.

3. Integral representation

THEOREM 3.1. A necessary and sufficient condition for the function f to be in $MS_\alpha^\lambda[A, B]$, $\alpha > 0$, is that f has the integral representation

$$f(z) = \left[\frac{e^{i\lambda}}{\alpha \cos \lambda} \int_0^z (g(t))^{e^{i\lambda}/\alpha \cos \lambda} t^{-1} dt \right]^{\alpha e^{-i\lambda} \cos \lambda}, \quad (3.1)$$

for some $g \in \operatorname{Sp}^\lambda[A, B]$, where the powers are assumed to be principal values.

PROOF. Let $f, g \in H$ and f be given as in (3.1). Differentiating both sides and simplifying we get

$$g(z) = f(z) \left(\frac{zf'(z)}{f(z)} \right)^{\alpha e^{-i\lambda} \cos \lambda}. \quad (3.2)$$

Differentiating (3.2) logarithmically and multiplying both sides by $ze^{i\lambda}$ we obtain

$$e^{i\lambda} \frac{zg'(z)}{g(z)} = (e^{i\lambda} - \alpha \cos \lambda) \frac{zf'(z)}{f(z)} + \alpha \cos \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right). \quad (3.3)$$

Hence $f \in MS_\alpha^\lambda[A, B]$ if and only if $g \in Sp^\lambda[A, B]$. $\square$

REMARK 3.2. Using the integral representation and the external function of the class $Sp^\lambda[A, B]$ (see [2]) we get the external function of the class $MS_\alpha^\lambda[A, B]$ as

$$f(z) = \begin{cases} \left(\frac{e^{i\lambda}}{\alpha \cos \lambda} \int_0^z t^{e^{i\lambda}/\alpha \cos \lambda - 1} (1+Bt)^{(A-B)/\alpha B} dt \right)^{\alpha e^{-i\lambda} \cos \lambda} & \text{if } B \neq 0, \\ \left(\frac{e^{i\lambda}}{\alpha \cos \lambda} \int_0^z t^{e^{i\lambda}/\alpha \cos \lambda} (\exp(Ate^{-i\lambda} \cos \lambda))^{e^{i\lambda}/\alpha \cos \lambda} dt \right)^{\alpha e^{-i\lambda} \cos \lambda} & \text{if } B = 0. \end{cases} \quad (3.4)$$

4. Coefficients bounds. To derive our next result, we need the following lemma (see [4]).

LEMMA 4.1. Let $w(z) = c_1 z + c_2 z^2 + \dots$ be an analytic function with $|w(z)| < 1$ in Δ . If v is any complex number, then

$$|c_2 - vc_1^2| \leq \max\{1, |v|\}. \quad (4.1)$$

The equality may be attained with the functions $w(z) = z^2$ and $w(z) = z$.

THEOREM 4.2. Let $f \in MS_\alpha^\lambda[A, B]$ given by $f(z) = z + \sum_{k=2}^\infty a_k z^k$, and let μ be any complex number. Then

$$|a_3 - \mu a_2^2| \leq \frac{(A-B) \cos \lambda}{2|e^{i\lambda} + 2\alpha \cos \lambda|} \max\{1, |v|\}, \quad (4.2)$$

where

$$v = \frac{2\mu(e^{i\lambda} + 2\alpha \cos \lambda)(A-B) \cos \lambda - (A-B) \cos \lambda (e^{i\lambda} + 3\alpha \cos \lambda)}{(e^{i\lambda} + \alpha \cos \lambda)^2} + \frac{(e^{i\lambda} + \alpha \cos \lambda)^2 B}{(e^{i\lambda} + \alpha \cos \lambda)^2}. \quad (4.3)$$

This result is sharp.

PROOF. Let $f \in MS_\alpha^\lambda[A, B]$ and let $w(z) = c_1 z + c_2 z^2 + \dots$ be an analytic function with $|w(z)| < 1$ in Δ . Then

$$(e^{i\lambda} - \alpha \cos \lambda) \frac{zf'(z)}{f(z)} + \alpha \cos \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) = \frac{e^{i\lambda} + e^{i\lambda} \eta w(z)}{1 + Bw(z)}, \quad (4.4)$$

where $\eta = (A - B) \cos \lambda e^{-i\lambda} + B$, from (1.2). Equating the coefficients in both sides of (4.4) we get

$$c_1 = \frac{e^{i\lambda} + \alpha \cos \lambda}{(A - B) \cos \lambda} a_2, \quad (4.5)$$

$$c_2 = \frac{2(e^{i\lambda} + 2\alpha \cos \lambda)}{(A - B) \cos \lambda} a_3 - \frac{(A - B) \cos \lambda (e^{i\lambda} + 3\alpha \cos \lambda) - (e^{i\lambda} + \alpha \cos \lambda)^2 B}{(A - B)^2 \cos^2 \lambda} a_2^2. \quad (4.6)$$

From (4.5) and (4.6), we obtain

$$c_2 - \nu c_1^2 = \frac{2(e^{i\lambda} + 2\alpha \cos \lambda)}{(A - B) \cos \lambda} \{a_3 - \mu a_2^2\}, \quad (4.7)$$

where

$$\begin{aligned} \mu = \frac{(A - B) \cos \lambda}{2(e^{i\lambda} + 2\alpha \cos \lambda)} \left\{ \frac{(e^{i\lambda} + 3\alpha \cos \lambda)(A - B) \cos \lambda - (e^{i\lambda} + \alpha \cos \lambda)^2 B}{(A - B)^2 \cos^2 \lambda} \right. \\ \left. + \nu \frac{(e^{i\lambda} + \alpha \cos \lambda)^2}{(A - B)^2 \cos^2 \lambda} \right\}. \end{aligned} \quad (4.8)$$

Hence applying Lemma 4.1, we get

$$|a_3 - \mu a_2^2| = \left| \frac{(A - B) \cos \lambda}{2(e^{i\lambda} + 2\alpha \cos \lambda)} \right| |c_2 - \nu c_1^2| \leq \frac{(A - B) \cos \lambda}{2|e^{i\lambda} + 2\alpha \cos \lambda|} \max\{1, |\nu|\}. \quad (4.9)$$

The sharpness of (4.9) follows from the sharpness of inequality (4.1). $\square$

5. Some radius problems. In this section, we discuss the covering theorem of the class $MS_\alpha^\lambda[A, B]$, that is, we find the radius of the largest disk covered by the image of the unit disk Δ under the mapping $f \in MS_\alpha^\lambda[A, B]$. We also find the α -convex β -spiral radius of functions in $MS_\alpha^\lambda[A, B]$.

THEOREM 5.1. *Let $f \in MS_\alpha^\lambda[A, B]$. Then the disk Δ is mapped onto a domain that contains the disk*

$$|w| < \frac{|e^{i\lambda} + \alpha \cos \lambda|}{2|e^{i\lambda} + \alpha \cos \lambda| + (A - B) \cos \lambda}. \quad (5.1)$$

PROOF. Let $w = f(z) \in MS_\alpha^\lambda[A, B]$ given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and let w_0 be any complex number such that $f(z) \neq w_0$ for $z \in \Delta$. Then

$$\frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + \dots \quad (5.2)$$

belongs to the class S of univalent functions. Hence (see [1])

$$\left| a_2 + \frac{1}{w_0} \right| \leq 2. \quad (5.3)$$

Substituting $|c_1| \leq 1$ in (4.5) (see [7]), we get

$$|a_2| \leq \frac{(A - B) \cos \lambda}{|e^{i\lambda} + \alpha \cos \lambda|}. \quad (5.4)$$

From (5.3) and (5.4), we obtain

$$|w_0| \geq \frac{|e^{i\lambda} + \alpha \cos \lambda|}{2|e^{i\lambda} + \alpha \cos \lambda| + (A-B)\cos \lambda}, \quad (5.5)$$

which is the required result. $\square$

To derive our next theorem we need the following lemma (see [6]).

LEMMA 5.2. *Let $f \in \text{Sp}^\lambda[A, B]$, then $f \in \text{Sp}^\beta[A, B]$, $|\beta| < \pi/2$, on the disc $|z| < r^{**}$ where r^{**} is the smallest positive root of the equation*

$$B[A \cos(\lambda + \beta) + B \sin \lambda \sin(\lambda - \beta)]r^2 + (A - B)r \cos \lambda - \cos \beta = 0. \quad (5.6)$$

This result is sharp.

THEOREM 5.3. *Let $f \in \text{MS}_\alpha^\lambda[A, B]$. Then $f \in \text{MS}_\alpha^\beta[A, B]$, $|\beta| < \pi/2$ on the disk $|z| < r^{**}$, where r^{**} is the smallest positive root of (5.6). This result is sharp.*

PROOF. Let $f \in \text{MS}_\alpha^\lambda[A, B]$. Then from Theorem 3.1, there exists a function $g \in \text{Sp}^\lambda[A, B]$ such that (3.1) is satisfied. From Lemma 5.2, $g \in \text{Sp}^\beta[A, B]$, $|\beta| < \pi/2$, on $|z| < r^{**}$. Applying Theorem 3.1 again, we find that $f \in \text{MS}_\alpha^\beta[A, B]$ on $|z| < r^{**}$. The radius r^{**} is best possible as shown by the function f given by (3.4). $\square$

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F. M. AL-BOUDI: DEPARTMENT OF MATHEMATICS, GIRL'S COLLEGE OF EDUCATION, SCIENCE SECTIONS, SITTEEN STREET, MALAZ, RIYADH, SAUDI ARABIA
E-mail address: fma34@yahoo.com

M. M. HIDAN: DEPARTMENT OF MATHEMATICS, GIRL'S COLLEGE OF EDUCATION, SCIENCE SECTIONS, KABEL, ABHA, SAUDI ARABIA