A NEW PROOF OF SOME IDENTITIES OF BRESSOUD

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We provide a new proof of the following two identities due to Bressoud: $\sum_{m=0}^N q^{m^2} \begin{bmatrix} N \\ m \end{bmatrix} = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+1)/2} \begin{bmatrix} 2N \\ N+2m \end{bmatrix}$, $\sum_{m=0}^N q^{m^2+m} \begin{bmatrix} N \\ m \end{bmatrix} = (1/(1-q^{N+1})) \sum_{m=-\infty}^{\infty} (-1)^m \times q^{m(5m+3)/2} \begin{bmatrix} 2N+2 \\ N+2m+2 \end{bmatrix}$, which can be considered as finite versions of the Rogers-Ramanujan identities.

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In [1], Bressoud proves the following theorem, from which the Rogers-Ramanujan identities follow on letting $N \rightarrow \infty$.

THEOREM 1. *For each integer $N \geq 0$,*

$$\begin{aligned} \sum_{m=0}^N q^{m^2} \begin{bmatrix} N \\ m \end{bmatrix} &= \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+1)/2} \begin{bmatrix} 2N \\ N+2m \end{bmatrix}, \\ \sum_{m=0}^N q^{m^2+m} \begin{bmatrix} N \\ m \end{bmatrix} &= \frac{1}{1-q^{N+1}} \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+3)/2} \begin{bmatrix} 2N+2 \\ N+2m+2 \end{bmatrix}. \end{aligned} \quad (1)$$

Here,

$$\begin{bmatrix} N \\ m \end{bmatrix} = \begin{cases} \frac{(q)_N}{(q)_m (q)_{N-m}} & \text{if } 0 \leq m \leq N; \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

denotes a Gaussian binomial coefficient, where we adopt the standard q -series notation

$$(q)_n = \prod_{j=1}^n (1-q^j). \quad (3)$$

We give an alternative proof of [Theorem 1](#) by showing that the left and right sides of (1) satisfy the same recurrence relations.

Define, for integers a and $N \geq 0$,

$$S_a(N) = \sum_{n=0}^N q^{n^2+an} \begin{bmatrix} N \\ n \end{bmatrix}. \quad (4)$$

LEMMA 2. *For each integer $N \geq 1$ and each a ,*

$$S_a(N) = S_a(N-1) + q^{N+a} S_{a+1}(N-1), \quad (5)$$

$$S_a(N) = S_{a+1}(N-1) + q^{a+1} S_{a+2}(N-1). \quad (6)$$

PROOF. Using the identity

$$\begin{bmatrix} N \\ n \end{bmatrix} = q^{N-n} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \begin{bmatrix} N-1 \\ n \end{bmatrix} \quad (7)$$

gives

$$\begin{aligned} S_a(N) &= q^N \sum_{n=1}^N q^{n^2+(a-1)n} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \sum_{n=0}^{N-1} q^{n^2+an} \begin{bmatrix} N-1 \\ n \end{bmatrix} \\ &= q^N \sum_{n=0}^{N-1} q^{(n+1)^2+(a-1)(n+1)} \begin{bmatrix} N-1 \\ n \end{bmatrix} + S_a(N-1) \\ &= q^{N+a} S_{a+1}(N-1) + S_a(N-1). \end{aligned} \quad (8)$$

On the other hand, using the identity

$$\begin{bmatrix} N \\ n \end{bmatrix} = \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} N-1 \\ n \end{bmatrix} \quad (9)$$

gives

$$\begin{aligned} S_a(N) &= \sum_{n=1}^N q^{n^2+an} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \sum_{n=0}^{N-1} q^{n^2+(a+1)n} \begin{bmatrix} N-1 \\ n \end{bmatrix} \\ &= \sum_{n=0}^{N-1} q^{(n+1)^2+a(n+1)} \begin{bmatrix} N-1 \\ n \end{bmatrix} + S_{a+1}(N-1) \\ &= q^{a+1} S_{a+2}(N-1) + S_{a+1}(N-1). \end{aligned} \quad (10)$$

□

We now equate (5) and (6).

LEMMA 3. For integers $N \geq 0$ and each a ,

$$S_a(N) + (q^{N+a+1} - 1)S_{a+1}(N) - q^{a+1}S_{a+2}(N) = 0. \quad (11)$$

PROOF. Equating (5) and (6) gives

$$S_a(N-1) + (q^{N+a} - 1)S_{a+1}(N-1) - q^{a+1}S_{a+2}(N-1) = 0 \quad (12)$$

for $N \geq 1$. Replacing N by $N+1$ gives

$$S_a(N) + (q^{N+a+1} - 1)S_{a+1}(N) - q^{a+1}S_{a+2}(N) = 0. \quad (13)$$

□

We will use the $a = 0$ case of Lemma 3 which is

$$S_0(N) + (q^{N+1} - 1)S_1(N) - qS_2(N) = 0. \quad (14)$$

Clearly, $S_a(0) = 1$ for all a . Also, for $N > 0$, (5) gives

$$S_0(N) = S_0(N-1) + q^N S_1(N-1) \quad (15)$$

and, together with (14), gives

$$\begin{aligned} S_1(N) &= S_1(N-1) + q^{N+1}S_2(N-1) \\ &= S_1(N-1) + q^N[S_0(N-1) + (q^N-1)S_1(N-1)] \\ &= q^N S_0(N-1) + (q^{2N} - q^N + 1)S_1(N-1). \end{aligned} \quad (16)$$

Together with the initial conditions $S_0(0) = S_1(0) = 1$, (15) and (16) completely define $S_0(N)$ and $S_1(N)$ for $N \geq 0$.

We now gather some consequences of these recurrences which will be used later.

LEMMA 4. For $N \geq 2$,

$$S_0(N) = (1 + q^{2N-1})S_0(N-1) + q^N(1 - q^N)S_1(N-2); \quad (17)$$

and for $N \geq 1$,

$$S_1(N) = q^N S_0(N) + (1 - q^N)S_1(N-1). \quad (18)$$

PROOF. First of all, from (15) and (16), we have

$$S_1(N) - q^N S_0(N) = (1 - q^N)S_1(N-1) \quad (19)$$

and so, for $N \geq 2$,

$$S_1(N-1) - q^{N-1}S_0(N-1) = (1 - q^{N-1})S_1(N-1). \quad (20)$$

Hence, by (15) again,

$$\begin{aligned} S_0(N) &= S_0(N-1) + q^N S_1(N-1) \\ &= S_0(N-1) + q^N [q^{N-1}S_0(N-1) + (1 - q^N)S_1(N-2)] \\ &= (1 + q^{2N-1})S_0(N-1) + q^N(1 - q^N)S_1(N-2), \end{aligned} \quad (21)$$

and also by using (16),

$$\begin{aligned} S_1(N) &= q^N S_0(N-1) + (1 - q^N + q^{2N})S_1(N-1) \\ &= q^N [S_0(N) - q^N S_1(N-1)] + (1 - q^N + q^{2N})S_1(N-1) \\ &= q^N S_0(N) + (1 - q^N)S_1(N-1). \end{aligned} \quad (22)$$

□

The recurrences (17) and (18) with the initial conditions $S_0(0) = S_1(0) = 1$, $S_0(1) = 1 + q$ define $S_0(N)$ and $S_1(N)$ uniquely for $N \geq 0$.

Let

$$\begin{aligned} B_0(N) &= \sum_m (-1)^m q^{m(5m+1)/2} \begin{bmatrix} 2N \\ N+2m \end{bmatrix}, \\ B_1(N) &= \sum_m (-1)^m q^{m(5m+3)/2} \begin{bmatrix} 2N+2 \\ N+2m+2 \end{bmatrix} \end{aligned} \quad (23)$$

denote the sums appearing on the right sides of the identities in [Theorem 1](#). Setting $r = N + 2m$ in the definition of $B_0(N)$ gives

$$\begin{aligned} B_0(N) &= \sum_{r \equiv N \pmod{4}} q^{(5/8)(r-N)^2 + (1/4)(r-N)} \begin{bmatrix} 2N \\ r \end{bmatrix} - \sum_{r \equiv N+2 \pmod{4}} q^{(5/8)(r-N)^2 + (1/4)(r-N)} \begin{bmatrix} 2N \\ r \end{bmatrix} \\ &= q^{-1/40} \left[\sum_{r \equiv N \pmod{4}} q^{(5/8)(r-N+1/5)^2} \begin{bmatrix} 2N \\ r \end{bmatrix} - \sum_{r \equiv N+2 \pmod{4}} q^{(5/8)(r-N+1/5)^2} \begin{bmatrix} 2N \\ r \end{bmatrix} \right]. \end{aligned} \quad (24)$$

This suggests the notation

$$A(M, k, b) = \sum_{2r \equiv M+k \pmod{8}} q^{(5/8)(r-M/2+b)^2} \begin{bmatrix} M \\ r \end{bmatrix} \quad (25)$$

so that

$$q^{1/40} B_0(N) = A\left(2N, 0, \frac{1}{5}\right) - A\left(2N, 4, \frac{1}{5}\right). \quad (26)$$

Of course, $A(M, k, b) = 0$ if $M+k$ is odd, and $A(M, k, b)$ depends only on M, b and the congruence class of k modulo 8. A similar computation yields

$$q^{9/40} B_1(N) = A\left(2N+2, 2, -\frac{2}{5}\right) - A\left(2N+2, -2, -\frac{2}{5}\right). \quad (27)$$

We aim at showing that $B_0(N)$ and $(1 - q^{N+1})B_1(N)$ satisfy the same system of recurrences as $S_0(N)$ and $S_1(N)$.

LEMMA 5. *The following holds*

$$A(M, k, b) = A(M, -k, -b) \quad (28)$$

for each M, k , and b .

PROOF. Replacing r by $M - r$ in the sum for $A(M, k, b)$ yields

$$\begin{aligned} A(M, k, b) &= \sum_{2M-2r \equiv M+k \pmod{8}} q^{(5/8)(M/2-r+b)^2} \begin{bmatrix} M \\ M-r \end{bmatrix} \\ &= \sum_{2r \equiv M-k \pmod{8}} q^{(5/8)(r-M/2-b)^2} \begin{bmatrix} M \\ r \end{bmatrix} \\ &= A(M, -k, -b). \end{aligned} \quad (29)$$

□

We now wish to produce recurrences for the $A(M, k, b)$.

LEMMA 6. *The following holds*

$$\begin{aligned} A(M+1, k, b) &= A\left(M, k-1, b + \frac{1}{2}\right) + q^{M/2+1/10-b} A\left(M, k+1, b + \frac{3}{10}\right), \\ A(M+1, k, b) &= A\left(M, k+1, b - \frac{1}{2}\right) + q^{M/2+1/10+b} A\left(M, k-1, b - \frac{3}{10}\right) \end{aligned} \quad (30)$$

for each M, k , and b .

PROOF. Using the formula

$$\begin{bmatrix} M+1 \\ r \end{bmatrix} = \begin{bmatrix} M \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} M \\ r \end{bmatrix} \quad (31)$$

in the definition of $A(M+1, k, b)$ gives $A(M+1, k, b) = S_1 + S_2$, where

$$\begin{aligned} S_1 &= \sum_{2r \equiv M+k+1 \pmod{8}} q^{(5/8)(r-M/2-1/2+b)^2} \begin{bmatrix} M \\ r-1 \end{bmatrix} \\ &= \sum_{2s \equiv M+k-1 \pmod{8}} q^{(5/8)(s-M/2+1/2+b)^2} \begin{bmatrix} M \\ s \end{bmatrix} \\ &= A\left(M, k-1, b + \frac{1}{2}\right), \\ S_2 &= \sum_{2r \equiv M+k+1 \pmod{8}} q^{r+(5/8)(r-M/2-1/2+b)^2} \begin{bmatrix} M \\ r \end{bmatrix}. \end{aligned} \quad (32)$$

But

$$r + \frac{5(r-M/2-1/2+b)^2}{8} = \frac{5(r-M/2+3/10+b)^2}{8} + \frac{M}{2} + \frac{1}{10} - b. \quad (33)$$

Hence,

$$A(M+1, k, b) = A\left(M, k-1, b + \frac{1}{2}\right) + q^{M/2+1/10-b} A\left(M, k+1, b + \frac{3}{10}\right). \quad (34)$$

Consequently, by [Lemma 5](#) also,

$$\begin{aligned} A(M+1, k, b) &= A(M+1, k, -b) \\ &= A\left(M, -k-1, -b + \frac{1}{2}\right) + q^{M/2+1/10+b} A\left(M, -k+1, -b + \frac{3}{10}\right) \\ &= A\left(M, k+1, b - \frac{1}{2}\right) + q^{M/2+1/10+b} A\left(M, k-1, b - \frac{3}{10}\right). \end{aligned} \quad (35)$$

□

It is convenient to note that replacing M by $M-1$ in these identities gives

$$\begin{aligned} A(M, k, b) &= A\left(M-1, k-1, b + \frac{1}{2}\right) + q^{M/2-2/5-b} A\left(M-1, k+1, b + \frac{3}{10}\right) \\ &= A\left(M-1, k+1, b - \frac{1}{2}\right) + q^{M/2-2/5+b} A\left(M-1, k-1, b - \frac{3}{10}\right). \end{aligned} \quad (36)$$

LEMMA 7. *The sums $B_0(N)$ and $B_1(N)$ obey the recurrences*

$$B_0(N) = (1 + q^{2N-1})B_0(N-1) + q^N B_1(N-2) \quad (37)$$

for $N \geq 2$ and

$$B_1(N) = (1 - q^{N+1})B_1(N-1) + q^N (1 - q^{N+1})B_0(N) \quad (38)$$

for $N \geq 1$.

PROOF. We compute

$$\begin{aligned}
 A\left(2N, k, \frac{1}{5}\right) &= A\left(2N-1, k+1, -\frac{3}{10}\right) + q^{N-1/5} A\left(2N-1, k-1, -\frac{1}{10}\right) \\
 &= A\left(2N-2, k, \frac{1}{5}\right) + q^{N-3/5} A(2N-2, k+2, 0) \\
 &\quad + q^{N-1/5} A\left(2N-2, k-2, \frac{2}{5}\right) + q^{2N-1} A\left(2N-2, k, \frac{1}{5}\right) \\
 &= (1+q^{2N-1}) A\left(2N-2, k, \frac{1}{5}\right) + q^{N-3/5} A(2N-2, k+2, 0) \\
 &\quad + q^{N-1/5} A\left(2N-2, k-2, \frac{2}{5}\right).
 \end{aligned} \tag{39}$$

In particular,

$$\begin{aligned}
 A\left(2N, 0, \frac{1}{5}\right) &= (1+q^{2N-1}) A\left(2N-2, 0, \frac{1}{5}\right) \\
 &\quad + q^{N-3/5} A(2N-2, 2, 0) + q^{N-1/5} A\left(2N-2, -2, \frac{2}{5}\right), \\
 A\left(2N, 4, \frac{1}{5}\right) &= (1+q^{2N-1}) A\left(2N-2, 4, \frac{1}{5}\right) \\
 &\quad + q^{N-3/5} A(2N-2, 6, 0) + q^{N-1/5} A\left(2N-2, 2, \frac{2}{5}\right) \\
 &\quad + q^{N-3/5} A(2N-2, -2, 0) + q^{N-1/5} A\left(2N-2, 2, \frac{2}{5}\right).
 \end{aligned} \tag{40}$$

Noting that

$$\begin{aligned}
 A(2N-2, 2, 0) &= A(2N-2, -2, 0), \\
 A\left(2N-2, 2, \frac{2}{5}\right) &= A\left(2N-2, -2, -\frac{2}{5}\right),
 \end{aligned} \tag{41}$$

subtracting gives

$$\begin{aligned}
 q^{1/40} B_0(N) &= A\left(2N, 0, \frac{1}{5}\right) - A\left(2N, 4, \frac{1}{5}\right) \\
 &= (1+q^{2N-1}) \left[A\left(2N-2, 0, \frac{1}{5}\right) - A\left(2N-2, 4, \frac{1}{5}\right) \right] \\
 &\quad + q^{N-1/5} \left[A\left(2N-2, 2, -\frac{2}{5}\right) - A\left(2N-2, -2, -\frac{2}{5}\right) \right] \\
 &= (1+q^{2N-1}) q^{1/40} B_0(N-1) + q^{N-1/5} q^{9/40} B_1(N-2)
 \end{aligned} \tag{42}$$

and so

$$B_0(N) = (1+q^{2N-1}) B_0(N-1) + q^N B_1(N-2). \tag{43}$$

Also,

$$\begin{aligned}
 A\left(2N+2, k, -\frac{2}{5}\right) &= A\left(2N+1, k-1, \frac{1}{10}\right) + q^{N+1} A\left(2N+1, k+1, -\frac{1}{10}\right) \\
 &= A\left(2N, k, -\frac{2}{5}\right) + q^{N+1/5} A\left(2N, k-2, -\frac{1}{5}\right) \\
 &\quad + q^{N+1} A\left(2N, k, \frac{2}{5}\right) + q^{2N+6/5} A\left(2N, k+2, \frac{1}{5}\right) \\
 &= A\left(2N, k, -\frac{2}{5}\right) + q^{N+1} A\left(2N, -k, -\frac{2}{5}\right) \\
 &\quad + q^{N+1/5} A\left(2N, 2-k, \frac{1}{5}\right) + q^{2N+6/5} A\left(2N, k+2, \frac{1}{5}\right).
 \end{aligned} \tag{44}$$

Consequently,

$$\begin{aligned}
 q^{9/40} B_1(N) &= A\left(2N+2, 2, -\frac{2}{5}\right) - A\left(2N+2, -2, -\frac{2}{5}\right) \\
 &= A\left(2N, 2, -\frac{2}{5}\right) + q^{N+1} A\left(2N, -2, -\frac{2}{5}\right) \\
 &\quad - A\left(2N, -2, -\frac{2}{5}\right) - q^{N+1} A\left(2N, 2, -\frac{2}{5}\right) \\
 &\quad + q^{N+1/5} \left[A\left(2N, 0, \frac{1}{5}\right) - A\left(2N, 4, \frac{1}{5}\right) \right] \\
 &\quad + q^{2N+6/5} \left[A\left(2N, 4, \frac{1}{5}\right) - A\left(2N, 0, \frac{1}{5}\right) \right] \\
 &= (1 - q^{N+1}) [q^{9/40} B_1(N-1) + q^{N+1/5} q^{1/40} B_0(N)]
 \end{aligned} \tag{45}$$

and so

$$B_1(N) = (1 - q^{N+1}) B_1(N-1) + q^N (1 - q^{N+1}) B_0(N). \tag{46}$$

□

By [Lemma 4](#), $S_0(N)$ and $(1 - q^{N+1})S_1(N)$ satisfy the same recurrences as $B_0(N)$ and $B_1(N)$. Also, $S_0(0) = 1 = B_0(0)$, $S_0(1) = 1 + q = B_0(1)$, and $(1 - q)S_1(0) = 1 - q = B_1(0)$. Consequently, we deduce [Theorem 1](#): $S_0(N) = B_0(N)$ and $(1 - q^{N+1})S_1(N) = B_1(N)$.

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