A NOTE ON RUSCHEWEYH TYPE OF INTEGRAL OPERATORS FOR UNIFORMLY $\alpha$-CONVEX FUNCTIONS

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We prove that the class of uniformly $\alpha$-convex functions introduced by Kanas is closed under the generalized Ruscheweyh integral operator for $0 < \alpha \leq 1$.

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We denote by $\mathcal{A}$ the class of functions $f(z) = z + a_2 z^2 + \cdots$ which are analytic in $\Delta = \{z \in \mathbb{C}: |z| < 1\}$. Let $S$ denote the class of functions in $\mathcal{A}$ that are univalent in $\Delta$. The subclasses of $S$ containing functions which are uniformly convex and uniformly starlike, introduced by Goodman [1, 2], are denoted by $UCV$ and $UST$, respectively.

The class of uniformly $\alpha$-convex functions was introduced by Kanas [3] and she gave an analytic condition for such functions as follows: $f(z)$ is a uniformly $\alpha$-convex function if and only if

$$\text{Re} \left\{ (1 - \alpha) \frac{(z - \zeta)f''(z)}{f(z) - f(\zeta)} + \alpha \left( 1 + \frac{(z - \zeta)f''(z)}{f'(z)} \right) \right\} > 0$$

for all $z, \zeta \in \Delta$ and $0 \leq \alpha \leq 1$. For $\zeta = 0$, this class of functions reduces to Mocanu’s class $M(\alpha)$ of $\alpha$-convex functions [4].

In this note, for $\alpha > 0$, we consider the integral operator

$$F(z) = \frac{F_\alpha(z, \zeta) - F_\alpha(0, \zeta)}{F_\alpha(0, \zeta)},$$

where

$$F_\alpha(z, \zeta) = \left\{ \frac{c + 1}{(z - \zeta)^c} \int_{z}^{\zeta} (t - \zeta)^{c-1} \left( f(t) - f(\zeta) \right)^{1/\alpha} dt \right\}^\alpha$$

for all $z \in \Delta$ and for fixed $\zeta \in \Delta$ with $z \neq \zeta$. We prove that this normalized function $F(z)$ is a uniformly $\alpha$-convex function when $f(z)$ is a uniformly $\alpha$-convex function in the sense of Kanas [3].

For $\zeta = 0$ the operator $F(z)$ reduces to Ruscheweyh’s integral operator [5]. It is well known that Mocanu’s class $M(\alpha)$ of $\alpha$-convex functions is closed under Ruscheweyh’s integral operator for $\alpha > 0$. 
**Theorem 1.** Let \( f(z) = z + a_2 z^2 + \cdots \) be a uniformly \( \alpha \)-convex function in \( \Delta \) and let \( c > 0 \). Then, for \( 0 < \alpha \leq 1 \), the function

\[
F(z) = \frac{F_\alpha(z, \zeta) - F_\alpha(0, \zeta)}{F_\alpha(0, \zeta)}, \quad z, \zeta \in \Delta,
\]

(4)
is uniformly \( \alpha \)-convex where \( F_\alpha(z, \zeta) \) is defined as in (3).

**Proof.** We have from (3) that

\[
F^{1/\alpha}_\alpha(z, \zeta) = \frac{c + 1/\alpha}{(z - \zeta)^c} \int_\zeta^z (t - \zeta)^{c-1} (f(t) - f(\zeta))^{1/\alpha} \, dt. \quad (5)
\]

Differentiating with respect to \( z \), we have

\[
(z - \zeta) c^{-1} F^{1/\alpha-1}_\alpha(z, \zeta) F'_{\alpha}(z, \zeta) + c (z - \zeta) c^{-1} F^{1/\alpha}_\alpha(z, \zeta)
\]

\[
= \left( c + \frac{1}{\alpha} \right) (z - \zeta)^{c-1} (f(z) - f(\zeta))^{1/\alpha} \quad (6)
\]

and again differentiating with respect to \( z \) we get

\[
\frac{1}{\alpha} \left\{ (z - \zeta) F^{1/\alpha-1}_\alpha(z, \zeta) F''_{\alpha}(z, \zeta) + (z - \zeta) \left( \frac{1}{\alpha} - 1 \right) F^{1/\alpha-2}_\alpha(z, \zeta) (F'_{\alpha}(z, \zeta))^2 \right\}
\]

\[
+ \frac{z - \zeta}{\alpha} F^{1/\alpha-1}_\alpha(z, \zeta) F'_{\alpha}(z, \zeta) + \frac{c}{\alpha} F^{1/\alpha-1}_\alpha(z, \zeta) F'_{\alpha}(z, \zeta)
\]

\[
= \left( c + \frac{1}{\alpha} \right) \frac{1}{\alpha} f''(z) (f(z) - f(\zeta))^{1/\alpha-1}; \quad (7)
\]

\[
F^{1/\alpha-1}_\alpha(z, \zeta) F'_{\alpha}(z, \zeta) \left\{ \frac{\alpha (z - \zeta) F''_{\alpha}(z, \zeta)}{F'_{\alpha}(z, \zeta)} + (1 - \alpha) (z - \zeta) \frac{F'_{\alpha}(z, \zeta)}{F_{\alpha}(z, \zeta)} + \alpha (1 + c) \right\}
\]

\[
= (\alpha c + 1) f''(z) (f(z) - f(\zeta))^{1/\alpha-1}.
\]

Thus we get

\[
F^{1/\alpha-1}_\alpha(z, \zeta) F'_{\alpha}(z, \zeta) \left\{ \frac{\alpha (z - \zeta) F''_{\alpha}(z, \zeta)}{F'_{\alpha}(z, \zeta)} + \frac{(z - \zeta) F'_{\alpha}(z, \zeta)}{F_{\alpha}(z, \zeta)} + c \alpha \right\}
\]

\[
= (c \alpha + 1) f''(z) (f(z) - f(\zeta))^{1/\alpha-1}. \quad (8)
\]

From (2) we have

\[
F'(z) = \frac{F'_{\alpha}(z, \zeta)}{F'_{\alpha}(0, \zeta)}, \quad (9)
\]

showing that \( F(0) = 0 \) and \( F'(0) = 1 \).

Considering

\[
\frac{(z - \zeta) F'(z)}{F(z) - F(\zeta)} = \frac{(z - \zeta) F'_{\alpha}(z, \zeta)}{F_{\alpha}(z, \zeta)} \quad (10)
\]
and differentiating with respect to $z$, we have

$$\frac{F''(z)}{F'(z)} + \frac{1}{z - \zeta} - \frac{F'(z)}{F(z) - F(\zeta)} = \frac{1}{z - \zeta} + \frac{F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)} + \frac{F'_\alpha(z,\zeta)}{F(\zeta)};$$ (11)

$$\frac{(z - \zeta)F''(z)}{F'(z)} + 1 - \frac{(z - \zeta)F'(z)}{F(z) - F(\zeta)} = 1 + \frac{(z - \zeta)F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)} - \frac{F'_\alpha(z,\zeta)(z - \zeta)}{F(\zeta)}. $$ (12)

Substituting (10) and (12) in (8), we obtain

$$F^{1/\alpha - 1}_\alpha(z,\zeta)F'_\alpha(z,\zeta)\left\{\alpha\left[\frac{(z - \zeta)F''(z)}{F'(z)} + 1 - \frac{(z - \zeta)F'(z)}{F(z) - F(\zeta)}\right] + \alpha\left(1 + \frac{(z - \zeta)F''(z)}{F'(z)}\right) + c\alpha\right\} = (c\alpha + 1)\frac{f''(z)}{f'(z)}(f(z) - f(\zeta))^{1/\alpha - 1};$$ (13)

$$F^{1/\alpha - 1}_\alpha(z,\zeta)F'_\alpha(z,\zeta)\left\{(1 - \alpha)\frac{(z - \zeta)F'(z)}{F(z) - F(\zeta)} + \alpha\left(1 + \frac{(z - \zeta)F''(z)}{F'(z)}\right) + c\alpha\right\} = (c\alpha + 1)\frac{f''(z)}{f'(z)}(f(z) - f(\zeta))^{1/\alpha - 1}. $$ (14)

Setting

$$P(z,\zeta) = (1 - \alpha)\frac{(z - \zeta)F'(z)}{F(z) - F(\zeta)} + \alpha\left(1 + \frac{(z - \zeta)F''(z)}{F'(z)}\right) + c\alpha,$$

equation (13) becomes

$$F^{1/\alpha - 1}_\alpha(z,\zeta)F'_\alpha(z,\zeta)\left\{P(z,\zeta) + c\alpha\right\} = (c\alpha + 1)\frac{f''(z)}{f'(z)}(f(z) - f(\zeta))^{1/\alpha - 1}. $$ (15)

Taking logarithmic differentiation with respect to $z$, we get

$$(1 - \alpha)(z - \zeta)\frac{F'_\alpha'(z,\zeta)}{F'_\alpha(z,\zeta)} + \alpha(z - \zeta)\frac{F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)} + \frac{\alpha(z - \zeta)P'(z,\zeta)}{P(z,\zeta)} + c\alpha + \alpha$$

$$= \alpha + \alpha \frac{(z - \zeta)f''(z)}{f'(z)} + (1 - \alpha)\frac{(z - \zeta)f'(z)}{f(z) - f(\zeta)};$$

$$\alpha\left[\frac{(z - \zeta)F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)} + 1 - \frac{(z - \zeta)F'_\alpha(z,\zeta)}{F(\zeta)}\right] + \alpha\left(1 + \frac{(z - \zeta)F''(z)}{F'(z)}\right) + \alpha\left(1 + \frac{(z - \zeta)f''(z)}{f'(z)}\right).$$ (16)

Equations (10) and (12) give

$$\alpha\left[\frac{(z - \zeta)F''(z)}{F'(z)} + 1 - \frac{(z - \zeta)F'(z)}{F(z) - F(\zeta)}\right] + \alpha\left(1 + \frac{(z - \zeta)f''(z)}{f'(z)}\right) = \alpha\left(1 + \frac{(z - \zeta)f''(z)}{f'(z)}\right) + (1 - \alpha)\frac{(z - \zeta)f'(z)}{f(z) - f(\zeta)}.$$ (17)
That is

\[
\alpha \left( 1 + \frac{(z - \zeta)F''(z)}{F'(z)} \right) + (1 - \alpha) \frac{(z - \zeta)F'(z)}{F(z) - F(\zeta)} + \alpha \frac{(z - \zeta)P'(z, \zeta)}{P(z, \zeta)} + c\alpha = \alpha \left( 1 + \frac{(z - \zeta)P''(z, \zeta)}{P'(z, \zeta)} \right) + (1 - \alpha) \frac{(z - \zeta)P'(z, \zeta)}{P(z, \zeta) - P(\zeta)}.
\]

(18)

Hence, we have

\[
P(z, \zeta) + \frac{\alpha(z - \zeta)P'(z, \zeta)}{P(z, \zeta) + c\alpha} = \alpha \left( 1 + \frac{(z - \zeta)P''(z, \zeta)}{P'(z, \zeta)} \right) + (1 - \alpha) \frac{(z - \zeta)P'(z, \zeta)}{P(z, \zeta) - P(\zeta)}.
\]

(19)

Since \( f(z) \) is uniformly \( \alpha \)-convex, we have

\[
\text{Re} \left\{ P(z, \zeta) + \frac{\alpha(z - \zeta)P'(z, \zeta)}{P(z, \zeta) + c\alpha} \right\} > 0
\]

for all \( z, \zeta \in \Delta, 0 \leq \alpha \leq 1 \).

We show that \( \text{Re} P(z, \zeta) > 0 \). Suppose that there exists a point \( \zeta_0 \in \Delta \) such that the image of the arc \( \Gamma : z(t) = \zeta_0 + re^{it} \) is tangent to the imaginary axis. Let \( w_0 \) be the point of contact and let \( z_0 \in \Delta \) such that \( w_0 = P(z_0, \zeta_0) \). Then \( \text{Re} P(z_0, \zeta_0) = 0 \) and therefore \( \text{Re} P(z_0, \zeta_0) = ix, \) where \( x \in \mathbb{R} \). Hence the outer normal to \( F(\Gamma) \) is

\[
(z_0 - \zeta_0)P'(z_0, \zeta_0) = y < 0.
\]

(21)

For such \( \zeta_0 \), we have

\[
\text{Re} \left\{ P(z_0, \zeta_0) + \frac{\alpha(z_0 - \zeta_0)P'(z_0, \zeta_0)}{P(z_0, \zeta_0) + c\alpha} \right\} = \text{Re} \left\{ ix + \frac{\alpha y}{c\alpha + ix} \right\} = \text{Re} \left\{ ix + \frac{\alpha y(c\alpha - ix)}{c^2\alpha^2 + x^2} \right\} = \frac{c\alpha y}{(c\alpha)^2 + x^2} < 0 \quad \text{for } c > 0
\]

(22)

which contradicts (20) and hence \( \text{Re} P(z, \zeta) > 0 \) in \( \Delta \) showing that \( F(z) \) is a uniformly \( \alpha \)-convex function.

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\section*{References}


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