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NORMAL LIE SUBSUPERGROUPS AND NON-ABELIAN SUPERCIRCLES

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We propose and study an appropriate analog of normal Lie subgroups in the supergeometrical context. We prove that the ringed space obtained taking the quotient of a Lie supergroup by a normal Lie subsupergroup, is still a Lie supergroup. We show how one can construct Lie supergroup structures over topologically nontrivial Lie groups and how the previous property of normal Lie subsupergroups can be used, in order to explicitly obtain the coproduct, counit, and antipode of these structures. We illustrate the general theory by carrying out the previous constructions over the circle, which leads to non-abelian super generalizations of the circle.

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1. Introduction. We can associate to any differentiable manifold M the commutative algebra $C^\infty(M)$ of all smooth functions on M. Reversing the emphasis, we may regard $C^\infty(M)$ as the primary object, since all of the fundamental concepts of differential geometry (tangent vectors, vector bundles, differential forms, etc.) involve constructions directly related to $C^\infty(M)$. Seeking for generalizations of the notion of manifold, it is natural to remove the commutativity property of $C^\infty(M)$ and consider appropriate generalizations of the sheaf structure of C^∞-functions, thus preserving a rich geometrical content related to the differentiable structure of the manifold M. In particular, the approach of [2, 3, 9, 11], which led to supermanifold theory, is of interest for us.

Supermanifolds and Lie supergroups are of particular interest in modern physics and mathematics as well. Indeed, they provide the natural mathematical context for supersymmetric field theory and supermechanics (see [1, 4, 5, 7, 9, 12, 17]) and on the other hand, they have interesting applications in geometry, analysis, and representation theory (see [2, 6, 9, 11, 13, 14, 15]). In particular, recognizing the Lie supergroups as symmetry groups for Hamiltonian supermechanics [1, 8, 9] and supergauge theory [7, 12, 17], opens an exciting interaction between the physics of systems with spin and the representation theory of Lie groups. Taking into account that, usual symmetry groups (as, e.g., the circle $\mathbf{S}^1$ of electromagnetism) have no uniquely defined super generalization, as we will explain, the situation becomes even more interesting.

Our aim in this paper is to discuss the supergeometrical analog of the notion of normal subgroup and to apply the results in the construction of Lie supergroups. Normal Lie subsupergroups are introduced in Definition 3.1. This definition is natural and preserves, in the setting of supergeometry, the basic property of normal subgroups: quotients by normal Lie subsupergroups are still Lie supergroups (see Proposition 3.2).

Using this property, we show in Proposition 4.1 that each Lie supergroup $(G,\mathcal{A})$ can be obtained as a quotient of its universal covering supergroup by its fundamental group. The consequences of this statement in supergeometry are immediately visible: if we can find Lie superalgebra structures on $\mathfrak{g} = T_e(G,\mathcal{A})$ other than the one defined by the Lie supergroup $(G,\mathcal{A})$, then we can integrate these Lie superalgebra structures to Lie supergroup structures on the universal covering of $(G,\mathcal{A})$. If, for some of these structures, the fundamental group of G is a normal Lie subsupergroup, then we obtain, by taking the quotient, Lie supergroups isomorphic to $(G,\mathcal{A})$ only as supermanifolds (*not* as Lie supergroups). Thus, given a topologically nontrivial ringed space with a Lie supergroup structure on it, we have a method to construct explicitly other Lie supergroup structures nonequivalent to the given one.

Another consequence of Proposition 4.1 is that a Lie superalgebra $\mathfrak{g}$ does not necessarily integrate to a Lie supergroup $(G,\mathcal{A})$ when the underlying Lie group G is not simply connected. Indeed, the Lie superalgebra integrates uniquely to a Lie supergroup on the corresponding simply connected ringed space $(\hat{G},\hat{\mathcal{A}})$, but if the fundamental group $\pi_1(G)$ is not a normal Lie subsupergroup of $(\hat{G},\hat{\mathcal{A}})$, the quotient $(\hat{G}/\pi_1(G),\hat{\mathcal{A}}/C^\infty)$, which is isomorphic as a supermanifold to $(G,\mathcal{A})$, does not have a Lie supergroup structure with Lie superalgebra equal to $\mathfrak{g}$. More specifically, we propose an algorithm for classifying all the Lie supergroup structures having as base manifold a given Lie group (see the remarks after the proof of Proposition 4.1).

We finally apply this algorithm in order to find all possible supertoruses of dimension $(1,n)$. The results are summarized in Theorem 5.4 and, as one may expect, a Lie superalgebra of dimension $(1,n)$ does not necessarily integrate to a supertorus. Explicit formulas for the coproduct, counit, and antipode on these supertoruses are also presented. Note here that our explicit constructions of Lie supergroups are based on the techniques of [10].

2. Preliminaries. A Lie supergroup is a supermanifold $(G,\mathcal{A})$ for which G is a Lie group and $\mathcal{A}(G)$ a Hopf superalgebra with antipode, such that the projection $\rho:\mathcal{A}(G)\to C^\infty(G)$ is a morphism of Hopf superalgebras [9, 12]. For a Lie supergroup $(G,\mathcal{A})$, the tangent space at the identity $\mathfrak{g} = T_e(G,\mathcal{A})$ inherits a Lie superalgebra structure by the coproduct on $\mathcal{A}(G)$; we call $\mathfrak{g}$ Lie superalgebra of $(G,\mathcal{A})$. The vector space $\mathfrak{g}$ coincides with the set of primitive elements, with respect to the identity, of the finite dual $\mathcal{A}(G)^\circ$. The space $\mathcal{A}(G)^\circ$ is defined [9] as the subspace of the full dual of $\mathcal{A}(G)$ whose elements have kernels containing an ideal of finite codimension. We denote by $\Delta_{\mathcal{A}}$, $\epsilon_{\mathcal{A}}$, and $s_{\mathcal{A}}$ the coproduct, counit, and antipode for $\mathcal{A}(G)$ and by $\Delta^\circ_{\mathcal{A}}$, $\epsilon^\circ_{\mathcal{A}}$, and $s^\circ_{\mathcal{A}}$ the corresponding maps of $\mathcal{A}(G)^\circ$.

By a left action of the Lie supergroup $(G,\mathcal{A})$ on the supermanifold $(Y,\mathcal{B})$ we mean a supermanifold morphism $\Phi:(G,\mathcal{A})\times(Y,\mathcal{B})\to(Y,\mathcal{B})$, such that $\Phi^*:\mathcal{B}(Y)\to\mathcal{A}(G)\otimes\mathcal{B}(Y)$ defines a left $\mathcal{A}(G)$-comodule structure over $\mathcal{B}(Y)$. Similarly, we define a right action. The sheaf $\mathcal{B}/\mathcal{A}$ of the quotient ringed space $(Y/G,\mathcal{B}/\mathcal{A})$ is defined by

$$(\mathcal{B}/\mathcal{A})(U) = \{f\in\mathcal{B}(V)\mid \Phi^* f = 1\otimes f\},\quad V=\pi^{-1}(U), \tag{2.1}$$

where $\pi: Y\to Y/G$ is the projection.

If $\star$ is the algebra product of $\mathcal{A}(G)^\circ$ and $\Delta^\circ_{\mathcal{A}}(a) = \sum_i a'_i \otimes a''_i$, the map $a \otimes b \mapsto \sum_i (-1)^{|b||a''_i|} a'_i \star b \star s^\circ_{\mathcal{A}}(a''_i)$, $a, b \in \mathcal{A}(G)^\circ$, is a smooth morphism of superalgebras in the sense of [9], that is, there exists a morphism of supermanifolds $\mathrm{AD} : (G, \mathcal{A}) \times (G, \mathcal{A}) \to (G, \mathcal{A})$ such that

$$\mathrm{AD}_*(a \otimes b) = \sum_i (-1)^{|b||a''_i|} a'_i \star b \star s^\circ_{\mathcal{A}}(a''_i). \tag{2.2}$$

This morphism has the properties of a left action and it is called adjoint action of $(G, \mathcal{A})$ on itself. We set $\mathrm{AD}_{*a}(b) = \mathrm{AD}_*(a \otimes b)$.

Now let $\mathbb{R}(G)$ be the group algebra of G over the real numbers and $E(\mathfrak{g})$ the universal enveloping algebra of the Lie superalgebra $\mathfrak{g}$. Then, it is a standard fact [9] that $\mathcal{A}(G)^\circ = \mathbb{R}(G) \divideontimes E(\mathfrak{g})$, where $\divideontimes$ is the smash product between $\mathbb{R}(G)$ and $E(\mathfrak{g})$ with respect to the representation $\pi : G \to \mathrm{Aut}(\mathfrak{g})$ given by $\pi(g)v = \delta_g \star v \star \delta_{g^{-1}}$; here, $\delta_g : \mathcal{A}(G) \to \mathbb{R}$ is the superalgebra morphism $\delta_g(f) = \rho(f)(g)$, for all $f \in \mathcal{A}(G)$. Given a Lie group G and a Lie superalgebra $\mathfrak{g}$ with $\mathrm{Lie}\,(G) = \mathfrak{g}_0$, we have a sheaf $F^{\mathfrak{g}}$ of supercommutative superalgebras defined by $F^{\mathfrak{g}}(U) = \mathrm{Hom}_{E(\mathfrak{g}_0)}(E(\mathfrak{g}), C^\infty(U))$, $U \subset G$, the space of $E(\mathfrak{g}_0)$-linear homomorphisms $E(\mathfrak{g}) \to C^\infty(U)$; $(G, F^{\mathfrak{g}})$ satisfies the requirements in the definition of a supermanifold [3, 11], the product and the unit on $F^{\mathfrak{g}}(G)$ being given by

$$\begin{aligned} m(\phi_1 \otimes \phi_2) &= m_{C^\infty} \circ (\phi_1 \otimes \phi_2) \circ \Delta_{E(\mathfrak{g})}, \quad \forall \phi_1, \phi_2 \in F^{\mathfrak{g}}(G), \\ 1(u)(g) &= \epsilon_{E(\mathfrak{g})}(u), \quad \forall u \in E(\mathfrak{g}),\ g \in G, \end{aligned} \tag{2.3}$$

where m_{C^∞} is the multiplication in $C^\infty(G)$, and $\Delta_{E(\mathfrak{g})}$ and $\epsilon_{E(\mathfrak{g})}$ are the coproduct and the counit in $E(\mathfrak{g})$, respectively. We have similar definitions when G is replaced by an open subset U. Here, the $E(\mathfrak{g}_0)$-module structure over $C^\infty(U)$ is given by the infinitesimal generators of the left multiplication on G [10]. The following theorem is very useful in what follows.

Theorem 2.1 (see [10]). *The supermanifold $(G, F^{\mathfrak{g}})$ is a Lie supergroup if there exists a representation $\mathrm{Ad}_1 : G \to \mathrm{Aut}\,\mathfrak{g}_1$ such that $\mathrm{Ad}_1(\exp(ta))b = b + t[a, b] + \cdots$, for all $a \in \mathfrak{g}_0$, $b \in \mathfrak{g}_1$.*

The explicit formulas for the coproduct Δ, antipode s, and counit ϵ of $F^{\mathfrak{g}}(G)$ are the following (see [10, 16]):

$$\begin{aligned} \Delta\phi(u \otimes v)(g, h) &= \phi(u \cdot \mathrm{Ad}(g)v)(gh), \\ s\phi(u)(g) &= \phi(\mathrm{Ad}\,(g^{-1}) s_{E(\mathfrak{g})} u)(g^{-1}), \\ \epsilon(\phi) &= \phi(1)(e), \quad \forall \phi \in F^{\mathfrak{g}}(G), \end{aligned} \tag{2.4}$$

for all $u, v \in E(\mathfrak{g})$, $g, h \in G$, $\phi \in F^{\mathfrak{g}}(G)$. In the previous equations, $s_{E(\mathfrak{g})}$ is the antipode of $E(\mathfrak{g})$, while Ad coincides with the adjoint representation of the Lie group G on $\mathfrak{g}_0$ when applied to even elements, and with Ad_1 when applied to odd elements.

We will say that $(H, \mathcal{B})$ is a Lie subsupergroup of $(G, \mathcal{A})$ if $\mathcal{B}(H)^\circ \subset \mathcal{A}(G)^\circ$ and the inclusion map $\mathcal{B}(H)^\circ \hookrightarrow \mathcal{A}(G)^\circ$ is a smooth morphism of Hopf superalgebras. Then we have a unique morphism $i : (H, \mathcal{B}) \to (G, \mathcal{A})$ such that $i_* : \mathcal{B}(H)^\circ \to \mathcal{A}(G)^\circ$ is the inclusion map.

3. Normal Lie subsupergroups. We propose in this section an appropriate generalization of normal Lie subgroups in the context of supergeometry.

Definition 3.1. The Lie subsupergroup $(H,\mathcal{B})$ of $(G,\mathcal{A})$ is a normal Lie subsupergroup, if the coalgebra $\mathcal{B}(H)^\circ$ is invariant under the adjoint action $\mathrm{AD}:(G,\mathcal{A})\times(G,\mathcal{A})\to(G,\mathcal{A})$.

In other words, for every $a\in\mathcal{A}(G)^\circ$, $b\in\mathcal{B}(H)^\circ$, we must have $\mathrm{AD}_{*a}(b)\in\mathcal{B}(H)^\circ$. Using the fact that $\mathcal{A}(G)^\circ=\mathbb{R}(G)\divideontimes E(\mathfrak{g})$ and $\mathrm{AD}_{*(a_1\star a_2)}=\mathrm{AD}_{*a_1}\circ\mathrm{AD}_{*a_2}$, it is clear that the condition of invariance for $\mathcal{B}(H)^\circ$ is equivalent to the condition $\mathrm{AD}_{*a}(b)\in\mathcal{B}(H)^\circ$ for every $a\in\mathcal{A}(G)^\circ$ group-like or primitive with respect to δ_e.

By (2.2), $(H,\mathcal{B})$ is a normal Lie subsupergroup of $(G,\mathcal{A})$ if and only if the following conditions are satisfied:

$$\forall g\in G,\ b\in\mathcal{B}(H)^\circ,\quad \exists\bar{b}\in\mathcal{B}(H)^\circ:\delta_g\star b=\bar{b}\star\delta_g, \tag{3.1}$$

$$\forall a\in\mathfrak{g},\ b\in\mathcal{B}(H)^\circ,\quad \exists\bar{b}\in\mathcal{B}(H)^\circ:a\star b=(-1)^{|a||b|}b\star a+\bar{b}. \tag{3.2}$$

Note that we have $\epsilon^\circ_{\mathcal{B}}(b)=\epsilon^\circ_{\mathcal{B}}(\bar{b})$ in (3.1) and $\epsilon^\circ_{\mathcal{B}}(\bar{b})=0$ in (3.2).

Proposition 3.2. *If $(H,\mathcal{B})$ is a normal Lie subsupergroup of $(G,\mathcal{A})$, then the quotient $(K,\mathcal{C})=(G/H,\mathcal{A}/\mathcal{B})$ possesses a natural structure of Lie supergroup. In particular, if $\pi:(G,\mathcal{A})\to(K,\mathcal{C})$ is the projection, then the coproduct $\Delta_{\mathcal{C}}$, the counit $\epsilon_{\mathcal{C}}$, and the antipode $s_{\mathcal{C}}$ on $\mathcal{C}(K)$ are given by the following relations: $(\pi^*\otimes\pi^*)\circ\Delta_{\mathcal{C}}=\Delta_{\mathcal{A}}\circ\pi^*$, $\epsilon_{\mathcal{C}}=\epsilon_{\mathcal{A}}\circ\pi^*$, and $\pi^*\circ s_{\mathcal{C}}=s_{\mathcal{A}}\circ\pi^*$.*

Proof. We observe that if $(H,\mathcal{B})$ is a normal Lie subsupergroup of $(G,\mathcal{A})$, then H is a normal Lie subgroup of G, and therefore $K=G/H$ is a Lie group.

Then consider two elements $a,a'\in\mathcal{A}(G)^\circ$ and let $c=\pi_*(a)$, $c'=\pi_*(a')$. We put $c\star c'=\pi_*(a\star a')$. We show that the operation between c and c' defines a superalgebra law on $\mathcal{C}(K)^\circ$ with unit $\epsilon_{\mathcal{C}}=\delta_e$, $e\in K$ being the unit of K. It is sufficient to show that the multiplication on $\mathcal{C}(K)^\circ$ is a well-defined operation. It is therefore necessary to study in detail the projection $\pi_*:\mathcal{A}(G)^\circ\to\mathcal{C}(K)^\circ$ in this particular case. If $f\in\mathcal{C}(K)$, $a\in\mathcal{A}(G)^\circ$, $b\in\mathcal{B}(H)^\circ$, and Φ is the right action of $(H,\mathcal{B})$ on $(G,\mathcal{A})$, we take $\Phi_*(a\otimes b)(\pi^*f)=(-1)^{|b||f|}a(\pi^*f)b(1_{\mathcal{B}})$, so $\pi_*\Phi_*(a\otimes b)=(-1)^{|a||b|}\epsilon^\circ_{\mathcal{B}}(b)\pi_*(a)$. The right action of $(H,\mathcal{B})$ on $(G,\mathcal{A})$ is built up from the coproduct $\Delta_{\mathcal{A}}$ and we have $\Phi^*=(\mathrm{id}\otimes i^*)\circ\Delta_{\mathcal{A}}$, $i:(H,\mathcal{B})\hookrightarrow(G,\mathcal{A})$ being the canonical inclusion; we thus find the following relation:

$$\pi_*(a\star b)=(-1)^{|a||b|}\epsilon^\circ_{\mathcal{B}}(b)\pi_*(a). \tag{3.3}$$

Suppose now that $b,b'\in\mathcal{B}(H)^\circ$ are such that $\epsilon^\circ_{\mathcal{B}}(b)\epsilon^\circ_{\mathcal{B}}(b')=1$, so $|b|=|b'|=0$. In this case $\pi_*(a\star b)\star\pi_*(a'\star b')=\pi_*(a)\star\pi_*(a')=c\star c'$. To have a well-defined operation between c and c', we must also have $\pi_*(a\star b\star a'\star b')=\pi_*(a\star a')$. It is sufficient to examine the case where $a'=\delta_g\star v$, $g\in G$, $v\in\mathfrak{g}$. Using conditions (3.1) and (3.2), we obtain that $b\star a'=b\star\delta_g\star v=\delta_g\star\bar{b}\star v=\delta_g\star(v\star\bar{b}+\hat{b})$,

where $\epsilon^\circ_{\mathcal{B}}(\hat{b}) = 0$ and $\epsilon^\circ_{\mathcal{B}}(\bar{b}) = \epsilon^\circ_{\mathcal{B}}(b)$. Using (3.3) and the fact that $\epsilon^\circ_{\mathcal{B}} : \mathcal{B}(H)^\circ \to \mathbb{R}$ is a morphism of superalgebras, we find that

$$\pi_*(a \star b \star a' \star b') = \pi_*(a \star a' \star \bar{b} \star b') + \pi_*(a \star \delta_g \star \hat{b} \star b') = \pi_*(a \star a'). \tag{3.4}$$

This means that indeed $\mathcal{C}(K)^\circ$ possesses a superalgebra structure.

We finally define the antipode $s^\circ_{\mathcal{C}} : \mathcal{C}(K)^\circ \to \mathcal{C}(K)^\circ$ by $s^\circ_{\mathcal{C}}(\pi_*(a)) = \pi_*(s^\circ_{\mathcal{A}}(a))$. Using the previous argument as well as the identity $\epsilon^\circ_{\mathcal{B}} \circ s^\circ_{\mathcal{B}} = \epsilon^\circ_{\mathcal{B}}$, we can easily prove that $s^\circ_{\mathcal{C}}$ is well defined. We observe now that with the previous structure on it, $\mathcal{C}(K)^\circ$ is a Hopf superalgebra. Furthermore, the morphisms $\Delta_{\mathcal{C}} : \mathcal{C}(K) \to \mathcal{C}(K) \otimes \mathcal{C}(K)$ and $s_{\mathcal{C}} : \mathcal{C}(K) \to \mathcal{C}(K)$, defined by the relations

$$(\pi^* \otimes \pi^*) \circ \Delta_{\mathcal{C}} = \Delta_{\mathcal{A}} \circ \pi^*, \qquad \pi^* \circ s_{\mathcal{C}} = s_{\mathcal{A}} \circ \pi^*, \tag{3.5}$$

are such that the restrictions of their duals (transpose maps) on $\mathcal{C}(K)^\circ \otimes \mathcal{C}(K)^\circ$ and $\mathcal{C}(K)^\circ$, respectively, coincide with the algebra product and the antipode on $\mathcal{C}(K)^\circ$. We conclude that $(K, \mathcal{C})$ is a Lie supergroup. □

Finally we consider two examples of normal Lie subsupergroups.

Kernel of a morphism. Let $\phi : (G, \mathcal{A}) \to (K, \mathcal{C})$ be a morphism of Lie supergroups, that is, a morphism of supermanifolds such that $\phi_* : \mathcal{A}(G)^\circ \to \mathcal{C}(K)^\circ$ is a morphism of superalgebras [9]. First we define an appropriate notion for the kernel and the image of the morphism ϕ. To this end, we set $H = \ker(\phi_*|_G)$ and $\mathfrak{h} = \ker(\phi_*|_{\mathfrak{g}})$. Then, H is a Lie subgroup of G and $\mathfrak{h}$ a Lie subsuperalgebra of $\mathfrak{g}$, whose even part coincides with the Lie algebra of H, $\mathfrak{h}_0 = T_eH$. Furthermore, H and $\mathfrak{h}$ satisfy the hypothesis of Theorem 2.1 (indeed, it is sufficient to choose $\mathrm{Ad}_1(h)b = \delta_h \star b \star \delta_{h^{-1}}$, where $\star$ is the product in $\mathcal{A}(G)^\circ$; obviously, Ad_1 preserves $\mathfrak{h}$ and in particular $\mathfrak{h}_1$ because $h \in \ker(\phi_*|_G)$) and one can construct a Lie subsupergroup $(H, \mathcal{B})$ of $(G, \mathcal{A})$ with $\mathfrak{h} = T_e(H, \mathcal{B})$; $(H, \mathcal{B})$ is the kernel of the morphism ϕ. Similarly, the image of ϕ is the Lie supergroup defined by the subsupergroup $\mathrm{im}(\phi_*|_G)$ of K and the Lie subsuperalgebra $\mathrm{im}(\phi_*|_{\mathfrak{g}})$ of $\mathfrak{k} = T_e(K, \mathcal{C})$.

Proposition 3.3. *The kernel $(H, \mathcal{B})$ of a morphism of Lie supergroups $\phi : (G, \mathcal{A}) \to (K, \mathcal{C})$ is a normal Lie subsupergroup of $(G, \mathcal{A})$ and the quotient $(G/H, \mathcal{A}/\mathcal{B})$ is isomorphic to the image of ϕ.*

Proof. We have already explained why the kernel of a morphism of Lie supergroups is a Lie subsupergroup. Now, if we take $b = \delta_h$, $h \in H$, or $b = v \in \mathfrak{h}$, then (3.1) holds for $\bar{b} = \delta_g \star b \star \delta_{g^{-1}}$. Note that $\bar{b} \in \mathcal{B}(H)^\circ$ because $\phi_*(\delta_h) = \delta_e$ and $\phi_*(v) = 0$. A similar argument applies to (3.2): if $b = \delta_h$, $h \in H$, then the difference $\bar{b} = a \star \delta_h - \delta_h \star a$ is a primitive element with respect to δ_h and $\phi_*(\bar{b}) = 0$ because $\phi_*(\delta_h) = \delta_e$, so $\bar{b} \in \mathcal{B}(H)^\circ$. On the other hand, if $b = v \in \mathfrak{h}$, then the element $\bar{b} = [a, v] = a \star v - (-1)^{|a||v|} v \star a$ belongs to $\mathfrak{g}$ and $\phi_*(\bar{b}) = 0$ (since $\phi_*(v) = 0$), so again $\bar{b} \in \mathcal{B}(H)^\circ$. Consequently, by the decomposition $\mathcal{B}(H)^\circ = \mathbb{R}(H) \ltimes E(\mathfrak{h})$, (3.1) and (3.2) hold for the subsupergroup $(H, \mathcal{B})$.

Now the isomorphism between the image of ϕ and the quotient $(G/H, \mathcal{A}/\mathcal{B})$ follows easily from the results of [9] since G/H is isomorphic to $\mathrm{im}(\phi_*|_G)$ and $\mathfrak{g}/\mathfrak{h}$ is isomorphic to $\mathrm{im}(\phi_*|_{\mathfrak{g}})$. □

General and special linear supergroups. We consider a vector superspace $V = V_0 \oplus V_1$. The space $\mathfrak{gl}(V) = \mathrm{End}(V)$ has the structure of a Lie superalgebra with even and odd parts given by $\mathfrak{gl}(V)_0 = \mathfrak{gl}(V_0) \oplus \mathfrak{gl}(V_1)$ and $\mathfrak{gl}(V)_1 = L(V_0,V_1) \oplus L(V_1,V_0)$, where $L(V_i,V_j)$ is the space of linear maps of V_i into V_j, $i,j \in \{0,1\}$. We have the following matrix representation for the elements of $\mathfrak{gl}(V)$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A \in \mathfrak{gl}(V_0),\ B \in L(V_1,V_0),\ C \in L(V_0,V_1),\ D \in \mathfrak{gl}(V_1). \tag{3.6}$$

Now the Lie group $G = \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$ has Lie algebra $\mathfrak{gl}(V)_0$ and the representation $\mathrm{Ad}_1 : G \to \mathrm{Aut}\,\mathfrak{gl}(V)_1$ defined by

$$\left[\mathrm{Ad}_1 \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}\right] \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & MBN^{-1} \\ NCM^{-1} & 0 \end{pmatrix} \tag{3.7}$$

satisfies the assumptions of Theorem 2.1. The corresponding Lie supergroup $(G,F^{\mathfrak{g}})$ is called general linear supergroup. The Lie subsuperalgebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(V) \mid \mathrm{tr}\,A = \mathrm{tr}\,D \right\} \tag{3.8}$$

of $\mathfrak{gl}(V)$ has even part equal to the Lie algebra of the Lie group

$$H = \left\{ \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \in \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \mid \det M = \det N \right\}. \tag{3.9}$$

Using again Theorem 2.1 with the representation Ad_1 previously defined, (3.7), we obtain a Lie supergroup $(H,F^{\mathfrak{h}})$, the special linear supergroup. This is a normal Lie subsupergroup of $(G,F^{\mathfrak{g}})$ and the quotient $(G/H,F^{\mathfrak{g}}/F^{\mathfrak{h}})$ is a one-dimensional (ordinary) Lie group isomorphic to the multiplicative group of nonzero real numbers.

4. A criterion for the construction of Lie supergroups. We observe that if $(H,\mathcal{B})$ is a normal Lie subsupergroup of $(G,\mathcal{A})$ of dimension $(0,0)$, then $(K,\mathcal{C}) = (G/H,\mathcal{A}/\mathcal{B})$ has dimension equal to $\dim(G,\mathcal{A})$ and the Lie superalgebras of $(G,\mathcal{A})$ and $(K,\mathcal{C})$ coincide. In particular, we prove the following proposition.

Proposition 4.1. *Let $(G,\mathcal{A})$ be a Lie supergroup, where G is a connected Lie group, and let $\mathfrak{g} = T_e(G,\mathcal{A}) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. If $\pi_1(G)$ is the fundamental group of G and $\hat{G}$ its universal covering group, then there exists a Lie supergroup structure on $(\hat{G},F^{\mathfrak{g}})$ for which $(\pi_1(G),C^\infty)$ is a normal Lie subsupergroup of $(\hat{G},F^{\mathfrak{g}})$, such that*

$$(\hat{G}/\pi_1(G),F^{\mathfrak{g}}/C^\infty) \cong (G,\mathcal{A}). \tag{4.1}$$

Proof. If $(G,\mathcal{A})$ is a Lie supergroup, then there exists a representation $\mathrm{Ad}_1 : G \to \mathrm{Aut}\,\mathfrak{g}_1$, such that $\mathrm{Ad}_1(\exp ta)(b) = b + t[a,b] + \cdots$, for all $a \in \mathfrak{g}_0$, $b \in \mathfrak{g}_1$. If $\pi : \hat{G} \to G$ is the projection, then $\widehat{\mathrm{Ad}}_1 = \mathrm{Ad}_1 \circ \pi : \hat{G} \to \mathrm{Aut}\,\mathfrak{g}_1$ is a representation of $\hat{G}$ with the same property. By Theorem 2.1, the supermanifold $(\hat{G},F^{\mathfrak{g}})$ is a Lie supergroup. It remains to show that $(\pi_1(G),C^\infty)$ is a normal Lie subsupergroup of $(\hat{G},F^{\mathfrak{g}})$ (then it is not

difficult to see that $(\hat{G}/\pi_1(G), F^{\mathfrak{g}}/C^\infty) \cong (G,\mathcal{A})$). Taking into account the fact that $\dim(\pi_1(G), C^\infty) = (0,0)$ and that $\pi_1(G)$ is a normal subgroup of $\hat{G}$, it is sufficient to prove that

$$\forall a \in \mathfrak{g},\ h \in \pi_1(G), \quad \exists \bar{b} \in C^\infty(\pi_1(G))^\circ : a \star \delta_h = \delta_h \star a + \bar{b}. \tag{4.2}$$

We first observe that this condition is satisfied for $\bar{b} = 0$ when $a \in \mathfrak{g}_0$. For the case $a \in \mathfrak{g}_1$ we proceed as follows. If $\phi \in \mathrm{Hom}_{E(\mathfrak{g}_0)}(E(\mathfrak{g}), C^\infty(\hat{G}))$ is an odd element, let $\hat{\phi} \in C^\infty(\hat{G}) \otimes \Lambda\mathfrak{g}_1^*$ be the corresponding element under the isomorphism $\mathrm{Hom}_{E(\mathfrak{g}_0)}(E(\mathfrak{g}), C^\infty(\hat{G})) \cong C^\infty(\hat{G}) \otimes \Lambda\mathfrak{g}_1^*$. We obtain $(a \star \delta_h)(\hat{\phi}) = -\Delta\hat{\phi}(a \otimes \delta_h) = -\Delta\phi(a,1)(e,h) = -\phi(a)(h)$ and $(\delta_h \star a)(\hat{\phi}) = -\Delta\hat{\phi}(\delta_h \otimes a) = -\Delta\phi(1,a)(h,e) = -\phi(\widehat{\mathrm{Ad}}_1(h)a)(h) = -\phi(a)(h)$, because $\pi(h) = e$, for all $h \in \pi_1(G)$. This completes the proof of the proposition. □

Proposition 4.1 provides a constructive method to obtain all the possible Lie supergroups from a given Lie group G and a vector superspace $\mathfrak{g}$, because in combination with Proposition 3.2 we have explicit formulas for the coproduct, counit, and antipode of the superalgebras $\mathcal{A}(G)$ corresponding to these Lie supergroups. We proceed through the following steps:

(1) Find first all possible Lie superalgebra structures on $\mathfrak{g}$ such that $\mathfrak{g}_0 = \mathrm{Lie}(G)$.

(2) Construct the Lie supergroup $(\hat{G}, F^{\mathfrak{g}})$ for each of the previous Lie superalgebras $\mathfrak{g}$ (recall that, for each Lie superalgebra structure on $\mathfrak{g}$, this Lie supergroup is unique).

(3) Find for which $(\hat{G}, F^{\mathfrak{g}})$, the group $(\pi_1(G), C^\infty)$ is a normal Lie subsupergroup. The following observation simplifies very much the normal subgroup test: $(\pi_1(G), C^\infty)$ *is a normal Lie subsupergroup of* $(\hat{G}, F^{\mathfrak{g}})$ *if and only if the representation* $\widehat{\mathrm{Ad}}_1$ *is* $\pi_1(G)$-*invariant*; this easily results from the proof of Proposition 4.1. For these cases, the quotient $(\hat{G}/\pi_1(G), F^{\mathfrak{g}}/C^\infty)$ is a Lie supergroup whose underlying Lie group is G. In this way, we obtain all Lie supergroups of dimension $(\dim G, \dim \mathfrak{g}_1)$ with underlying Lie group equal to G.

It is clear that the main difficulty is to find all the Lie superalgebras $\mathfrak{g}$ with $\mathfrak{g}_0 = \mathrm{Lie}(G)$. But even in the case where it is impossible to accomplish this task, the previous technique provides a method to construct explicitly certain Lie supergroup structures over topologically nontrivial Lie groups.

5. Non-abelian supercircles. We are interested in applying the method of the previous section to the study of the Lie supergroups $(G,\mathcal{A})$, with the circle as underlying Lie group and of odd dimension equal to n. We first have the following classification scheme.

Lemma 5.1. *Let* $\mathfrak{g} = \mathbb{R} \oplus E$, $\mathfrak{g}_0 = \mathbb{R}$, $\mathfrak{g}_1 = E$, *where* E *is a vector space of dimension* n. *Then, all possible Lie superalgebra structures on* $\mathfrak{g}$ *are classified as follows:*

(1) *abelian structure:* $[u,v] = 0$, *for all* $u,v \in \mathfrak{g}$;

(2) *symmetric structure:* $[r,s] = [r,u] = 0$, *for all* $r,s \in \mathbb{R}$, $u \in E$ *and* $[u,v] = g(u,v)$, *where* $g : E \times E \to \mathbb{R}$ *is a symmetric bilinear form;*

(3) *exponential structure:* $[r,s] = [u,v] = 0$, *for all* $r,s \in \mathbb{R}$, $u,v \in E$ *and* $[r,u] = r(\lambda \cdot u)$, *for all* $r \in \mathbb{R}$, $u \in E$, $\lambda \in \mathrm{End}\,E$.

The proof of the above lemma is based on simple arguments of linear algebra and is therefore omitted.

We want now to classify all the Lie supergroups $(G,\mathcal{A})$ with $G=\mathbb{S}^1$ and $\dim(G,\mathcal{A})=(1,n)$. Thus, it is sufficient to take $G=\mathbb{S}^1$ and $\mathfrak{g}=\mathbb{R}\oplus E$ with $E=(\mathbb{R}^n)^*$ and apply the three steps of the previous section (all the information concerning Lie superalgebra structures on $\mathfrak{g}$ is now gathered in Lemma 5.1).

Now, let $(\mathbb{R},F^{\mathfrak{g}})$ be the supermanifold constructed out by the Lie group $\mathbb{R}$ and the Lie superalgebra $\mathfrak{g}$ with $E=(\mathbb{R}^n)^*$, and let $(t,\tau_1,\dots,\tau_n)$ be the canonical supercoordinates on $(\mathbb{R},F^{\mathfrak{g}})$. In other words, $t:\mathbb{R}\to\mathbb{R}$ is the canonical coordinate on $\mathbb{R}$ and $\tau_i\in\mathbb{R}$ is the basis of $\mathbb{R}$ dual to t, $i=1,\dots,n$. We have the following proposition.

Proposition 5.2. *There exist three families of Lie supergroup structures on $(\mathbb{R},F^{\mathfrak{g}})$ which correspond to the three families of Lie superalgebra structures on $\mathfrak{g}$. More precisely,*

(1) *abelian structure:*

$$\begin{gathered}\Delta t=t\otimes 1+1\otimes t,\qquad \Delta\tau_i=\tau_i\otimes 1+1\otimes\tau_i,\qquad \epsilon(t)=\epsilon(\tau_i)=0,\\ s(t)=-t,\qquad s(\tau_i)=-\tau_i;\end{gathered}\tag{5.1}$$

(2) *symmetric structure:*

$$\begin{gathered}\Delta t=t\otimes 1+1\otimes t-\frac{1}{2}\sum_{i,j}g_{ij}\tau_i\otimes\tau_j,\qquad \Delta\tau_i=\tau_i\otimes 1+1\otimes\tau_i,\qquad \epsilon(t)=\epsilon(\tau_i)=0,\\ s(t)=-t,\qquad s(\tau_i)=-\tau_i;\end{gathered}\tag{5.2}$$

(3) *exponential structure:*

$$\begin{gathered}\Delta t=t\otimes 1+1\otimes t,\qquad \Delta\tau_i=\tau_i\otimes 1+\sum_k\left(e^{\lambda t}\right)_{ki}\otimes\tau_k,\qquad \epsilon(t)=\epsilon(\tau_i)=0,\\ s(t)=-t,\qquad s(\tau_i)=-\sum_k\left(e^{-\lambda t}\right)_{ki}\tau_k.\end{gathered}\tag{5.3}$$

Proof. For simplicity, we will consider only the case $n=1$. We construct on $F^{\mathfrak{g}}(\mathbb{R})$ coproduct, counit, and antipode using Theorem 2.1 with $\mathrm{Ad}_1(r)=\mathrm{id}$ for the abelian and symmetric structures and $\mathrm{Ad}_1(r)=e^{\lambda r}$ for the exponential one (now the endomorphism λ and the symmetric bilinear form g reduce to real numbers). In what follows, we set $a=\delta_0\circ\partial/\partial t$ and $b=\delta_0\circ\partial/\partial\tau$.

Consider the symmetric structure. If $\phi\in\mathrm{Hom}_{E(\mathfrak{g}_0)}(E(\mathfrak{g}),C^\infty(\hat{G}))$, we have the following relations for the coproduct and the antipode:

$$\Delta\phi(u\otimes v)(r_1,r_2)=\phi(uv)(r_1+r_2),\qquad s\phi(u)(r)=\phi(s_{E(\mathfrak{g})}u)(-r).\tag{5.4}$$

If $\phi_t,\phi_\tau\in\mathrm{Hom}_{E(\mathfrak{g}_0)}(E(\mathfrak{g}),C^\infty(\hat{G}))$ correspond to $t,\tau\in C^\infty(\mathbb{R})\otimes\Lambda\mathbb{R}$, we find that $s\phi_t(1)(r)=\phi_t(1)(-r)=-t(r)$ and if $v\in\mathfrak{g}_1$, $s\phi_\tau(v)(r)=-\tau(v)1_{C^\infty}(r)$. The formulas for the antipode in the symmetric structure are now immediate from the isomorphism $\mathrm{Hom}_{E(\mathfrak{g}_0)}(E(\mathfrak{g}),C^\infty(\hat{G}))\cong C^\infty(\hat{G})\otimes\Lambda\mathfrak{g}_1^*$. As for the coproduct, an easy calculation shows that $\Delta\phi_\tau(1\otimes v)=\Delta\phi_\tau(v\otimes 1)=\tau(v)$, for all $v\in\mathfrak{g}_1=E$. But these relations are sufficient to completely determine $\Delta\phi_\tau$ and the result is $\Delta\phi_\tau=\phi_\tau\otimes 1+1\otimes\phi_\tau$.

Now, in order to calculate the element $\Delta\phi_t$, we first observe that it is sufficient to evaluate it on $1\otimes 1$ and $b\otimes b$. We easily see that $\Delta\phi_t(1\otimes 1) = t\otimes 1 + 1\otimes t$. On the other hand, $\Delta\phi_t(b\otimes b)(r_1,r_2) = (g/2)[(L^*)_a\phi_t(1)](r_1+r_2)$, where $(L^*)_a$ is the induced derivation, for $a\in\mathfrak{g}_0$, through the left action of $\mathbb{R}$ on itself given by left translations. Now, it is not difficult to see that $\Delta\phi_t = \phi_t\otimes 1 + 1\otimes\phi_t - (g/2)\phi_\tau\otimes\phi_\tau$ which proves our assertion for the symmetric structure.

The verification of the formulas for the abelian and the exponential structure is based on similar arguments and it is left to the reader. □

Remark 5.3. The subcase $n=1$ of the previous proposition appears also in [13], but here we discuss the method by which one constructs explicitly such structures. In the present case $G=\mathbb{R}$, this method can be directly adapted for $n>1$ and shows at the same time clearly how one should proceed for more general connected and simply connected Lie groups.

We first turn our attention to Lie supergroups $(G,\mathcal{A})$ with $G=\mathbf{S}^1$ and $\dim(G,\mathcal{A}) = (1,1)$. We will use Propositions 4.1 and 5.2 with $(G,\mathcal{A}) = (\mathbf{S}^1,F^{\mathfrak{g}})$. In this case we have $\pi_1(G)=\mathbb{Z}$ and $\hat{G}=\mathbb{R}$ and the representation Ad_1 is trivial for the abelian and symmetric structure and $\mathrm{Ad}_1(r)=\exp(r\lambda)$, for all $r\in\mathbb{R}$ for the exponential structure. It is now clear that, for arbitrary odd dimension, the abelian and symmetric structures pass to the circle through the procedure described in Proposition 4.1. Finally, the condition for $(\mathbb{Z},C^\infty)$ being a normal Lie subsupergroup of $(\mathbb{R},F^{\mathfrak{g}})$ in the exponential case, takes the following form: $\exp(z\lambda)=1_n$, for all $z\in\mathbb{Z}$, where 1_n is the unit $n\times n$ matrix.

We have thus proved the following theorem.

Theorem 5.4. *On the supermanifold $(\mathbf{S}^1,F^{\mathfrak{g}})$ there exist three families of Lie supergroup structures:*

(1) *the abelian structure, obtained as in Proposition 4.1 taking the representation Ad_1 trivial;*

(2) *the symmetric structure, obtained as in Proposition 4.1 taking the representation Ad_1 trivial and parametrized by the space of symmetric bilinear forms on $E=\mathfrak{g}_1$;*

(3) *the exponential structure, obtained as in Proposition 4.1 taking the representation Ad_1 as $\mathrm{Ad}_1(r)=\exp(r\lambda)$ and parametrized by the space of endomorphisms $\lambda\in\mathrm{End}\,E$ for which $\exp(\lambda)=1_n$. In particular, there is no exponential structure in odd dimension 1.*

More specifically, combining the formulas of Propositions 5.2 and 3.2, we find the following expressions for the coproduct, counit, and antipode on $F^{\mathfrak{g}}(\mathbf{S}^1)$. Note that in the expressions below we have $f\in C^\infty(\mathbf{S}^1)$ viewed as a periodic function on $\mathbb{R}$ with period 1, $\tau_i\in\mathbb{R}$ are the same as in Proposition 5.2, while Δ_∞, ϵ_∞, and s_∞ are the coproduct, counit, and antipode for the differentiable Lie group structure on $\mathbf{S}^1$.

(1) Abelian structure:

$$\begin{aligned} \Delta f &= \Delta_\infty f, & \Delta\tau_i &= \tau_i\otimes 1 + 1\otimes\tau_i,\\ \epsilon(f) &= \epsilon_\infty(f), & \epsilon(\tau_i) &= 0,\\ s(f) &= s_\infty(f), & s(\tau_i) &= -\tau_i. \end{aligned} \tag{5.5}$$

(2) Symmetric structure:

$$\begin{gathered}\Delta f = \Delta_\infty f + \sum_{i,j,p=1}^{n} (-1)^p \frac{1}{2^p p!} (g_{ij}\tau_i \otimes \tau_j)^p \Delta_\infty \left(\frac{d^p f}{dt^p}\right), \qquad \Delta\tau_i = \tau_i \otimes 1 + 1 \otimes \tau_i, \\ \epsilon(f) = \epsilon_\infty(f), \qquad \epsilon(\tau_i) = 0 \\ s(f) = s_\infty(f), \qquad s(\tau_i) = -\tau_i.\end{gathered} \tag{5.6}$$

(3) Exponential structure:

$$\begin{gathered}\Delta f = \Delta_\infty f, \qquad \Delta\tau_i = \tau_i \otimes 1 + \sum_k ([e^{\lambda t}])_{ki} \otimes \tau_k, \\ \epsilon(f) = \epsilon_\infty(f), \qquad \epsilon(\tau_i) = 0, \\ s(f) = s_\infty(f), \qquad s(\tau_i) = -\sum_k ([e^{-\lambda t}])_{ki}\tau_k.\end{gathered} \tag{5.7}$$

Here, $[e^{\lambda t}] : \mathbf{S}^1 \to \mathrm{Aut}(E)$ is the automorphism-valued function on $\mathbf{S}^1$ defined by $e^{\lambda t}$ when $\exp(\lambda) = 1_n$.

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