AN APPLICATION TO KATO'S SQUARE ROOT PROBLEM

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We find all complex potentials Q such that the general Schrödinger operator on $\mathbb{R}^n$, given by $L = -\Delta + Q$, where Δ is the Laplace differential operator, verifies the well-known Kato's square problem. As an application, we will consider the case where $Q \in L^1_{\text{loc}}(\Omega)$.

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1. Introduction. Let $\Omega \subseteq \mathbb{R}^n$, an open set and let's stand in the Hilbert space $H = L^2(\Omega, C) (= L^2(\Omega))$. Consider Q , a measurable complex function and let Φ and Ψ be the sesquilinear forms given by,

$$\begin{aligned}\Phi(u, v) &= \int_{\Omega} \nabla u \overline{\nabla v} dx \quad \forall u, v \in D(\Phi) = H^1_0(\Omega), \\ \Psi(u, v) &= \int_{\Omega} Qu \bar{v} dx \quad \forall u, v \in D(\Psi),\end{aligned}\tag{1.1}$$

where $D(\Psi) = \{u \in L^2(\Omega) : Q|u|^2 \in L^1(\Omega)\}$. Assume that the potential Q verifies that there exists $\beta > 0$ and there exists $\theta \in (0, \pi/2)$, such that

$$|\arg(Q - \beta)| \leq \frac{\pi}{2} - \theta.\tag{1.2}$$

The sesquilinear forms Φ and Ψ are both closed, densely defined, and sectorial. According to Kato's first representation theorem (see [2]), we can associate to Φ and Ψ , m -sectorial linear operators defined, respectively, by

$$\begin{aligned}Au &= -\Delta u \quad \text{with} \quad D(A) = \{u \in H^1_0(\Omega) : \Delta u \in L^2(\Omega)\}, \\ Bu &= Qu \quad \text{with} \quad D(B) = \{u \in L^2(\Omega) : Qu \in L^2(\Omega)\}.\end{aligned}\tag{1.3}$$

By Schrödinger operator, we mean a partial differential operator on $\mathbb{R}^n$ of the form

$$L = A + B; \quad A = -\Delta; \quad B = Q = Q(x),\tag{1.4}$$

where Δ is the n -dimensional Laplace operator $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$. The name comes from the form of Schrödinger's equation which, in units with $\hbar = m = 1$ reads

$$i \frac{\partial u}{\partial t} = Lu.\tag{1.5}$$

Our aim here is to find all potentials Q such that

$$D(L^{1/2}) = D(A^{1/2}) \cap D(B^{1/2}) = D(L^{*1/2}).\tag{1.6}$$

For that we use the author results [1] related to the sum of linear operators connected to Kato's square root problem. The case where $\Omega = \mathbb{R}^n$ will be studied later as a consequence of the general case.

2. Schrödinger operators and Kato's condition

DEFINITION 2.1. A linear operator C is said to verify Kato's square root problem (or Kato's condition) if

$$D(C^{1/2}) = D(Y) = D(C^{*1/2}), \quad (2.1)$$

where Y is the sesquilinear form associated to C .

HYPOTHESIS ON Q . Suppose that Q is chosen such as,

$$\overline{D(\Phi) \cap D(\Psi)} = L^2(\Omega). \quad (2.2)$$

PROPOSITION 2.2. Let A and B be the linear operators given by (1.3). Assume that the potential Q verifies (2.2). Then there exists a unique operator sum $A \oplus B$, which is m -sectorial, verifying Kato's condition and

- (i) $A \oplus B = \overline{A+B}$ if $\overline{A+B}$ is a maximal operator,
- (ii) $|\operatorname{Im}\langle (A \oplus B)u, u \rangle| \leq \operatorname{Re}\langle (A \oplus B)u, u \rangle$, for all $u \in D(A \oplus B)$.

PROOF. Assume that Q verifies hypothesis (2.2). So, the sesquilinear form given by, $Y = \Phi + \Psi$, is a closed, sectorial, and densely defined. By Kato's first representation theorem (see [2]), there exists a unique m -sectorial sum operator, $A \oplus B$, associated to Y , verifying

$$Y(u, v) = \langle (A \oplus B)u, v \rangle \quad \forall u \in D(A \oplus B), v \in D(\Phi) \cap D(\Psi) \quad (2.3)$$

since A and B both verify Kato's condition, according to author's theorem (see [1, Theorem 2, page 462]), the operator $A \oplus B$ verifies the same condition, that is,

$$D((A \oplus B)^{1/2}) = D(\Phi) \cap D(\Psi) = D((A \oplus B)^{*1/2}) \quad (2.4)$$

and (ii) is satisfied, where β is given by (1.2). The operator $A \oplus B$ is defined as

$$\begin{aligned} (A \oplus B)u &= -\Delta u + Qu, \quad \forall u \in D(A \oplus B), \\ D(A \oplus B) &= \{u \in H_0^1(\Omega) : Q|u|^2 \in L^1(\Omega), -\Delta u + Qu \in L^2(\Omega)\}. \end{aligned} \quad (2.5)$$

Using also author's theorem (see [1, Theorem 2, page 462]), it follows that (i) is satisfied. $\square$

3. Some applications. Consider the same operators, that is, $A = -\Delta$ and $B = Q$ in $L^2(\Omega)$. Assume that $Q \in L_{\text{loc}}^1(\Omega)$, in this case (2.2) is satisfied. According to Brézis and Kato, the operator $\overline{A+B}$ is maximal in $L^2(\Omega)$ (then $A \oplus B = \overline{A+B}$ and Kato's condition is satisfied) and is given by (2.5).

CASE $\Omega = \mathbb{R}^n$. We always assume $Q \in L^1_{\text{loc}}(\mathbb{R}^n)$, it follows that

$$\begin{aligned} Au &= -\Delta u \quad \text{with} \quad D(A) = H^2(\mathbb{R}^n); \quad D(A^{1/2}) = H^1(\mathbb{R}^n), \\ Bu &= Qu \quad \text{with} \quad D(B^{1/2}) = \{u \in L^2(\mathbb{R}^n) : Q|u|^2 \in L^1(\mathbb{R}^n)\}, \end{aligned} \quad (3.1)$$

and $D(A^{1/2}) \cap D(B^{1/2})$ is dense in $L^2(\mathbb{R}^n)$ because,

$$C_0^\infty(\mathbb{R}^n) \subseteq D(\Phi) \cap D(\Psi). \quad (3.2)$$

In conclusion, Kato's condition is satisfied by $\overline{A+B}$, that is,

$$D\left(\sqrt{A+B}\right) = D(\sqrt{A}) \cap D(\sqrt{B}) = D\left(\sqrt{A+B^*}\right). \quad (3.3)$$

For example when $n = 1$, then

$$D\left(\sqrt{A+B}\right) = H^1(\mathbb{R}) = D\left(\sqrt{A+B^*}\right). \quad (3.4)$$

REMARK 3.1. Condition (2.2) could be weakened as

$$\overline{D(A)} \cap \overline{D(B)} = L^2(\Omega). \quad (3.5)$$

But in general the algebraic sum of two operators is not always defined (because this concept is not well adapted to problems arising in mathematical analysis).

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