COMMON FIXED POINT THEOREMS FOR A WEAK DISTANCE IN COMPLETE METRIC SPACES

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Using the concept of a $w$-distance, we obtain common fixed point theorems on complete metric spaces. Our results generalize the corresponding theorems of Jungck, Fisher, Dien, and Liu.

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1. Introduction. In 1976, Caristi [1] proved a fixed point theorem in a complete metric space which generalizes the Banach contraction principle. This theorem is very useful and has many applications. Later, Dien [3] showed that a pair of mappings satisfying both the Banach contraction principle and Caristi’s condition in a complete metric space has a common fixed point. That is to say, let $(X,d)$ be a complete metric space and let $S$ and $T$ be two orbitally continuous mappings of $X$ into itself. Suppose that there exists a finite number of functions $\{\varphi_i\}_{1 \leq i \leq N_0}$ of $X$ into $\mathbb{R}_+$ such that

$$d(Sx,Ty) \leq q \cdot d(x,y) + \sum_{i=1}^{N_0} [\varphi_i(x) - \varphi_i(Sx) + \varphi_i(y) - \varphi_i(Ty)]$$

(1.1)

for all $x,y \in X$ and some $q \in [0,1)$. Then $S$ and $T$ have a unique common fixed point $z$ in $X$. Further, if $x \in X$ then $S^n x \to z$ and $T^n x \to z$ as $n \to \infty$. In particular, if $S$ is an identity mapping, $q = 0$, and $N_0 = 1$, then this means a Caristi’s fixed point theorem.

Recently, Liu [7] obtained necessary and sufficient conditions for the existence of fixed point of continuous self-mapping by using the ideas of Jungck [5] and Dien [3]: let $f$ be a continuous self-mapping of a metric space $(X,d)$, then $f$ has a fixed point in $X$ if and only if there exist $z \in X$, a mapping $g : X \to X$, and a function $\Phi$ from $X$ into $[0,\infty)$ such that $f$ and $g$ are compatible, $g(X) \subset f(X)$, $g$ is continuous, and

$$d(gx,z) \leq rd(fx,z) + [\Phi(fx) - \Phi(gx)]$$

(1.2)

for all $x \in X$ and some $r \in [0,1)$.

In 1996, Kada et al. [6] introduced the concept of $w$-distance on a metric space as follows: let $X$ be a metric space with metric $d$, then a function $p : X \times X \to [0,\infty)$ is called a $w$-distance on $X$ if the following are satisfied:

1. $p(x,z) \leq p(x,y) + p(y,z)$ for any $x,y,z \in X$;
2. for any $x \in X$, $p(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;
(3) for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( p(z, x) \leq \delta \) and \( p(z, y) \leq \delta \) imply \( d(x, y) \leq \epsilon \).

In this paper, using the concept of a \( w \)-distance, we obtain common fixed point theorems on complete metric spaces. Our results generalize the corresponding theorems of Jungck [5], Fisher [4], Dien [3], and Liu [7].

2. Definitions and preliminaries. Throughout, we denote by \( \mathbb{N} \) the set of positive integers and by \( \mathbb{R}_+ := [0, \infty) \).

**Definition 2.1** (see [3]). A mapping \( T \) of a space \( X \) into itself is said to be orbitally continuous if \( x_0 \in X \) such that \( x_0 = \lim_{i \to \infty} T^{n_i}x \) for some \( x \in X \), then \( Tx_0 = \lim_{i \to \infty} T(T^{n_i}x) \).

**Definition 2.2** (see [2]). Let \( T \) be a mapping of a metric space \( X \) into itself. For each \( x \in X \), let
\[
O(T, x, n) = \{x, Tx, \ldots, T^n x\}, \quad n = 1, 2, \ldots,
\]
and
\[
O(T, x, \infty) = \{x, Tx, \ldots\}.
\]

A space \( X \) is said to be \( T \)-orbitally complete if and only if every Cauchy sequence, which is contained in \( O(T, x, \infty) \) for some \( x \in X \), converges in \( X \).

**Definition 2.3** (see [6]). Let \( X \) be a metric space with metric \( d \). Then a function \( p : X \times X \to \mathbb{R}_+ \) is called a \( w \)-distance on \( X \) if the following properties are satisfied:
1. \( p(x, z) \leq p(x, y) + p(y, z) \) for any \( x, y, z \in X \);
2. for any \( x \in X \), \( p(x, \cdot) : X \to \mathbb{R}_+ \) is lower semicontinuous;
3. for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( p(z, x) \leq \delta \) and \( p(z, y) \leq \delta \) imply \( d(x, y) \leq \epsilon \).

The metric \( d \) is a \( w \)-distance on \( X \). Other examples of \( w \)-distance are stated in [6].

**Definition 2.4** (see [5]). Let \( (X, d) \) be a metric space and \( f, g : X \to X \). The mappings \( f \) and \( g \) are called compatible if and only if for every sequence \( \{x_n\}_{n \in \mathbb{N}} \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \), it implies
\[
\lim_{n \to \infty} d(f gx_n, gf x_n) = 0. \tag{2.2}
\]

**Lemma 2.5** (see [6]). Let \( X \) be a metric space with metric \( d \), and \( p \) a \( w \)-distance on \( X \). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences in \( X \), let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be sequences in \( \mathbb{R}_+ \) converging to 0, and let \( x, y, z \in X \). Then the following properties hold:
1. if \( p(x_n, y_n) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( y = z \). In particular, if \( p(x, y) = 0 \) and \( p(x, z) = 0 \), then \( y = z \);
2. if \( p(x_n, y_n) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( \{y_n\} \) converges to \( z \);
3. if \( p(x_n, x_m) \leq \alpha_n \) for any \( n, m \in \mathbb{N} \) with \( m > n \), then \( \{x_n\} \) is a Cauchy sequence;
4. if \( p(y, x_n) \leq \alpha_n \) for any \( n \in \mathbb{N} \), then \( \{x_n\} \) is a Cauchy sequence.
3. Main results

**Theorem 3.1.** Let \((X,d)\) be a complete metric space with a \(w\)-distance \(p\). Suppose that two mappings \(f, g : X \to X\) and a function \(\varphi\) from \(X\) into \(\mathbb{R}_+\) are satisfying the following conditions:

(i) \(g(X) \subseteq f(X)\),

(ii) there exists \(t \in X\) such that \(p(t, gx) \leq r \cdot p(t, fx) + [\varphi(fx) - \varphi(gx)]\) for all \(x \in X\) and some \(r \in [0,1)\),

(iii) for every sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(X\) satisfying

\[
\lim_{n \to \infty} p(t, f x_n) = \lim_{n \to \infty} p(t, g x_n) = 0, \tag{3.1}
\]

it implies that

\[
\lim_{n \to \infty} \max \{p(t, f x_n), p(t, g x_n), p(f g x_n, g f x_n)\} = 0, \tag{3.2}
\]

(iv) for each \(u \in X\) with \(u \neq fu\) or \(u \neq gu\),

\[
\inf \{p(u, f x) + p(u, g x) + p(f g x, g f x) : x \in X\} > 0. \tag{3.3}
\]

Then \(f\) and \(g\) have a unique common fixed point in \(X\).

**Proof.** Let \(x_0\) be a given point of \(X\). By (i), there exists \(x_n \in X\) such that \(g x_n = f x_{n-1}\) for \(n \geq 1\). From Theorem 3.1(ii), we have

\[
p(t, f x_{j+1}) = p(t, g x_j) \leq r \cdot p(t, f x_j) + [\varphi(f x_j) - \varphi(g x_j)], \tag{3.4}
\]

which implies that

\[
\sum_{j=0}^{n-1} p(t, f x_{j+1}) \leq r \cdot \sum_{j=0}^{n-1} p(t, f x_j) + \sum_{j=0}^{n-1} [\varphi(f x_j) - \varphi(g x_j)], \tag{3.5}
\]

that is,

\[
\sum_{j=1}^{n} p(t, f x_j) \leq \frac{r}{1-r} p(t, f x_0) + \frac{1}{1-r} [\varphi(f x_0) - \varphi(f x_n)], \tag{3.6}
\]

\[
\leq \frac{r}{1-r} p(t, f x_0) + \frac{1}{1-r} \varphi(f x_0),
\]

which means that the series \(\sum_{n=1}^{\infty} p(t, f x_n)\) is convergent, so

\[
\lim_{n \to \infty} p(t, f x_n) = \lim_{n \to \infty} p(t, g x_n) = 0. \tag{3.7}
\]
Suppose that \( t \neq ft \) or \( t \neq gt \). Then, from Theorem 3.1(iii) and (iv) we obtain that
\[
0 < \inf \left\{ p(t,fx) + p(t,gx) + p(fgx,gfx) : x \in X \right\}
\leq \inf \left\{ p(t,fx_n) + p(t,gx_n) + p(fgx_n,gfx_n) : n \in \mathbb{N} \right\} \tag{3.8}
= 0.
\]
This is a contradiction. Hence \( t \) is a common fixed point of \( f \) and \( g \).

We prove that \( t \) is a unique common fixed point of \( f \) and \( g \). Let \( u \) be a common fixed point of \( f \) and \( g \). Then, by Theorem 3.1(ii),
\[
p(t,t) = p(t,gt) \leq r \cdot p(t,ft) + [\varphi(ft) - \varphi(gt)] = r \cdot p(t,t),
p(t,u) = p(t,gu) \leq r \cdot p(t,fu) + [\varphi(fu) - \varphi(gu)] = r \cdot p(t,u). \tag{3.9}
\]
Thus \( p(t,t) = p(t,u) = 0 \). From Lemma 2.5, we obtain \( t = u \). Therefore \( t \) is a unique common fixed point of \( f \) and \( g \).

**Remark 3.2.** Theorem 3.1 generalizes and improves Dien [3, Theorem 2.2] and Liu [7, Theorem 3.2].

**Theorem 3.3.** Let \( f \) be a continuous self-mapping of metric space \((X,d)\). Assume that \( f \) has a fixed point in \( X \). Then there exists a \( w \)-distance \( p,t \in X \), a continuous mapping \( g : X \rightarrow X \), and a function \( \varphi \) from \( X \) into \( \mathbb{R}_+ \) satisfying Theorem 3.1(i), (ii), (iii), and (iv).

**Proof.** Let \( z \) be a fixed point of \( f \), \( r = 1/2 \), \( gx = t = z \), and \( \varphi(x) = 1 \) for all \( x \in X \). Define \( p : X \times X \rightarrow \mathbb{R}_+ \) by
\[
p(x,y) = \max \{d(fx,x),d(fx,y),d(fx,fy)\} \quad \forall x,y \in X. \tag{3.10}
\]
Suppose that
\[
\lim_{n \to \infty} p(t,fx_n) = \lim_{n \to \infty} p(t,gx_n) = 0. \tag{3.11}
\]
Then it is easy to verify that the results of Theorem 3.3 follow.

**Theorem 3.4.** Let \( f \) and \( g \) be a continuous compatible self-mappings of the metric space \((X,d)\). There exists \( t \in X \) satisfying
\[
d(t,gx) \leq r \cdot d(t,fx) + [\varphi(fx) - \varphi(gx)] \tag{3.12}
\]
for all \( x \in X \) and some \( r \in [0,1) \). Then

(i) for every sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( X \) such that
\[
\lim_{n \to \infty} d(t,fx_n) = \lim_{n \to \infty} d(t,gx_n) = 0 \tag{3.13}
\]
for some \( t \in X \), it implies that
\[
\lim_{n \to \infty} \max \{d(t,fx_n),d(t,gx_n),d(fgx_n,gfx_n)\} = 0; \tag{3.14}
\]
(ii) for each \( u \in X \) with \( u \neq fu \) or \( u \neq gu \),
\[
\inf \{ d(u, fx) + d(u, gx) + d(fgx, gfx) : x \in X \} > 0.
\] (3.15)

**Proof.** The results follow by elementary calculation. \( \square \)

**Remark 3.5.** Since the metric \( d \) is \( w \)-distance, from Theorems 3.1, 3.3, and 3.4, we obtain Liu [7, Theorem 3.1].

**Theorem 3.6.** Let \((X, d)\) be a complete metric space with a \( w \)-distance \( p \), two mappings \( f, g : X \rightarrow X \), and two functions \( \varphi, \psi \) from \( X \) into \( \mathbb{R}_+ \) such that Theorem 3.1(i), (iv) are satisfied,

(i) for every sequence \( \{x_n\} \in X \) in \( X \) such that
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t
\] (3.16)
for some \( t \in X \), it implies that
\[
\lim_{n \to \infty} \max \{ p(t, fx_n), p(t, gx_n), p(fgx, gfx) \} = 0,
\] (3.17)

(ii) \[
p(gx, gy) \leq a_1 p(fx, fy) + a_2 p(fx, gx) + a_3 p(fy, gy) + a_4 p(fx, gy) + a_5 [p(gx, fy) d(fy, gx)]^{1/2} \]
\[
+ [\varphi(fx) - \varphi(gx)] + [\psi(fy) - \psi(gy)]
\] (3.18)
for all \( x, y \in X \), where \( a_1, a_2, a_3, a_4, \) and \( a_5 \) are in \([0,1]\) with \( a_1 + a_4 + a_5 < 1 \) and \( a_1 + a_2 + a_3 + 2a_4 < 1 \).

Then \( f \) and \( g \) have a unique common fixed point in \( X \).

**Proof.** Let \( x_0 \) be an arbitrary point of \( X \). By Theorem 3.1(i), we obtain a sequence \( \{x_n\} \) in \( X \) such that \( gx_{n-1} = fx_n \) for \( n \geq 1 \). Let \( y_n = p(fx_n, fx_{n+1}) \) for \( n \geq 0 \). It follows from Theorem 3.6(ii) that
\[
y_{j+1} = p(gx_j, gx_{j+1})
\]
\[
\leq a_1 p(fx_j, fx_{j+1}) + a_2 p(fx_j, gx_j) + a_3 p(fx_{j+1}, gx_{j+1}) + a_4 p(fx_j, gy) + a_5 [p(gx_j, fx_{j+1}) d(fx_{j+1}, gx_j)]^{1/2} \]
\[
+ [\varphi(fx_j) - \varphi(gx_j)] + [\psi(fx_{j+1}) - \psi(gx_{j+1})]
\] (3.19)
\[
\leq (a_1 + a_2 + a_4) y_j + (a_3 + a_4) y_{j+1} + [\varphi(fx_j) - \varphi(fx_{j+1})] + [\psi(fx_{j+1}) - \psi(fx_{j+2})],
\]
which implies that
\[
y_{j+1} \leq L_1 y_j + L_2 [\varphi(fx_j) - \varphi(fx_{j+1}) + \psi(fx_{j+1}) - \psi(fx_{j+2})],
\] (3.20)
where
\[
L_1 = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}, \quad L_2 = \frac{1}{1 - a_3 - a_4}.
\] (3.21)
Thus
\[ \sum_{j=1}^{n} y_j \leq \frac{L_1}{1-L_1} y_0 + \frac{L_2}{1-L_1} [\varphi(fx_0) + \psi(fx_1)] \]  
(3.22)

for all \( n \geq 1 \). Hence, the series \( \sum_{n=1}^{\infty} y_n \) is convergent. For any \( n, r \geq 1 \), we have
\[ p(fx_n, fx_{n+r}) \leq \sum_{i=n}^{n+r-1} y_i. \]  
(3.23)

By Lemma 2.5, this implies that \( \{fx_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( X \). Since \( X \) is a complete metric space, there exists \( t \in X \) such that \( fx_n \to t \) as \( n \to \infty \). From Theorem 3.6(i), we have
\[ \lim_{n \to \infty} \max \{p(t,fx), p(t,gx), p(fgx,gfx)\} = 0. \]  
(3.24)

Suppose that \( t \neq ft \) or \( t \neq gt \), then from Theorem 3.1(iv) we obtain that
\[ 0 < \inf \{p(t,fx) + p(t,gx) + p(fgx,gfx) : x \in X\} \leq \inf \{p(t,fx_n) + p(t,gx_n) + p(fgx_n,gfx_n) : n \in \mathbb{N}\} = 0, \]  
(3.25)

which is a contradiction. Therefore \( t \) is a common fixed point of \( f \) and \( g \). It follows from Lemma 2.5 and Theorem 3.6(ii) that \( t \) is a unique common fixed point of \( f \) and \( g \).

\[ \square \]

**Theorem 3.7.** Let \( f \) be a continuous self-mapping of a metric space \( (X,d) \). Assume that \( f \) has a fixed point in \( X \). Then there exist a \( w \)-distance \( p,t \in X \), a continuous mapping \( g : X \to X \), and functions \( \varphi, \psi \) from \( X \) into \( \mathbb{R}_+ \) satisfying Theorem 3.1(i), (iv) and Theorem 3.6(i), (ii).

**Proof.** By a method similar to that in the proof of Theorem 3.3, the results follow. \[ \square \]

**Remark 3.8.** Since the metric \( d \) is \( w \)-distance, from Theorems 3.4, 3.6, and 3.7, we obtain Jungck [5, Theorem], Fisher [4, Theorem 2], and Liu [7, Theorem 3.3].

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