ASYMPTOTIC HÖLDER ABSOLUTE VALUES

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We prove that asymptotic Hölder absolute values are Hölder equivalent to classical absolute values. As a corollary we obtain a generalization of Ostrowski's theorem and a classical theorem by E. Artin. The theorem presented implies a new, more flexible, definition of classical absolute value.

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1. Introduction. Asymptotic Hölder absolute values generalize the notions of classical absolute value and of Hölder absolute value. A Hölder absolute value (HAV) satisfies an approximate triangle inequality and multiplicative property. More precisely, let $C_1 \geq 1$ and $C_2 \geq 1$. A (C_1, C_2) -Hölder absolute value on a ring R is a mapping $\|\cdot\| : R \rightarrow \mathbb{R}_+$ satisfying:

(HAV1) for $x \in R$, $\|x\| = 0 \Leftrightarrow x = 0$;

(HAV2) for $x, y \in R$, $\|x + y\| \leq C_2(\|x\| + \|y\|)$;

(HAV3) for $x, y \in R$, $C_1^{-1}\|x\|\|y\| \leq \|xy\| \leq C_1\|x\|\|y\|$.

It is known that HAV on a ring are Hölder equivalent to a classical ones. More precisely, we have the following theorem (see [2]).

THEOREM 1.1 (Hölder rigidity). *Let $\|\cdot\| : R \rightarrow \mathbb{R}_+$ be a (C_1, C_2) -Hölder absolute value on a commutative ring R with unit element. There exists an absolute value on R , $|\cdot| : R \rightarrow \mathbb{R}_+$, which is (C_1^α, α) -Hölder equivalent to $\|\cdot\|$ with $\alpha = \log_2(2C_2)$, that is, for $x \in R$,*

$$C_1^{-\alpha}|x|^\alpha \leq \|x\| \leq C_1^\alpha|x|^\alpha. \quad (1.1)$$

Moreover, $|\cdot|$ can be defined by

$$|x| = \lim_{n \rightarrow +\infty} \|\|x^n\|\|^{1/na}. \quad (1.2)$$

For a ring R with unity, a real constant $C_2 \geq 1$, and a function $C_1(\cdot, \cdot)$ defined on $]1, +\infty[\times \mathbb{N}$ taking values in $]1, +\infty[$, we define a (C_1, C_2) -asymptotic Hölder absolute value (AHAV) on R ,

$$|\cdot| : R \longrightarrow \mathbb{R}_+, \quad (1.3)$$

satisfying the three following axioms:

(AHAV1) $|x| = 0$ if and only if $x = 0$;

(AHAV2) for $x, y \in R$, $|x + y| \leq C_2(|x| + |y|)$;

(AHAV3) for $\gamma > 1$ and $n \geq 2$ there is a constant $C_1(\gamma, n) > 1$ such that for $x_1, \dots, x_n \in R$,

$$C_1(\gamma, n)^{-1} |x_1|^{\gamma^{-1}} \cdots |x_n|^{\gamma^{-1}} \leq |x_1 \cdots x_n| \leq C_1(\gamma, n) |x_1|^\gamma \cdots |x_n|^\gamma, \quad (1.4)$$

$$\text{and } L = \overline{\lim}_{n \rightarrow \infty} (1/n) \log C_1(\gamma, n) < +\infty.$$

We prove the following theorem.

THEOREM 1.2. *Let R be a commutative ring with unity. Let $C_2 \geq 1$ be a real constant, $\alpha = 1/\log_2(2C_2)$, and $\|\cdot\|$ a (C_1, C_2) -AHAV on R . We have the following dichotomy:*

(i) *if*

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log C_1(\gamma, n) = 0, \quad (1.5)$$

then $\|\cdot\|^\alpha$ is a classical absolute value on R ;

(ii) *if*

$$0 < L = \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log C_1(\gamma, n) < +\infty, \quad (1.6)$$

then $\|\cdot\|^\alpha$ is a Hölder absolute value on R , more precisely, it is $(e^{L\alpha}, \alpha)$ -Hölder equivalent to an absolute value on R .

As a result of [Theorem 1.2\(i\)](#), we can define classical absolute values as AHAV with $C_2 = 1$ having a sequence of constants $(C_1(\gamma, n))_n$ growing sub-exponentially, that is,

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log C_1(\gamma, n) = 0. \quad (1.7)$$

This is far more flexible than the classical definition.

Note that, in general, Hölder equivalence is a metric property which is stronger than the usual topological equivalence, for example, $\{0\} \cup \{1/n; n \geq 1\}$ and $\{0\} \cup \{1/2^n; n \geq 1\}$ are homeomorphic, but not Hölder equivalent.

COROLLARY 1.3. *Consider $|\cdot| : R \rightarrow \mathbb{R}^+$ satisfying*

(AV1) *$|x| = 0$ if and only if $x = 0$,*

(AV2) *for $x, y \in R$, $|x+y| \leq |x| + |y|$ then,*

(AV3) *for $x, y \in R$, $|xy| = |x||y|$ is equivalent to:*

(AV3') *for $\gamma > 1$ and $n \geq 2$ there is a constant $C_1(\gamma, n) > 1$ such that for $x_1, \dots, x_n \in R$,*

$$C_1(\gamma, n)^{-1} |x_1|^{\gamma^{-1}} \cdots |x_n|^{\gamma^{-1}} \leq |x_1 \cdots x_n| \leq C_1(\gamma, n) |x_1|^\gamma \cdots |x_n|^\gamma \quad (1.8)$$

$$\text{with } \overline{\lim}_{n \rightarrow \infty} (1/n) \log C_1(\gamma, n) = 0.$$

Our theorem gives a generalization for discrete rings of Artin's theorem [1].

COROLLARY 1.4. *If $\|\cdot\|$ is a $(1, C_2)$ -AHAV over a discrete field F , there exists an absolute value $|\cdot|$ and an exponent α , such that for all x in F , $\|x\|^\alpha = |x|$.*

Also, our theorem implies a generalization of Ostrowski's theorem [3] for classical absolute values ($C_1 = C_2 = \gamma = 1$) over $\mathbb{Z}$.

COROLLARY 1.5. *If $\|\cdot\|$ is a (C_1, C_2) -AHAV over $\mathbb{Z}$ normalized, so that $\|1\| = 1$, then $\|\cdot\|$ is $(e^{L\alpha}, \alpha)$ -Hölder equivalent to a p -adic absolute value $|\cdot|_p$ or to $|\cdot|_\infty$ or to the trivial absolute value, with $\alpha = 1/\log_2(2C_2)$.*

REMARKS. (1) The constant $C_1(\gamma, n)$ in the definition of AHAV can be chosen to satisfy the inequality

$$C_1(\gamma, n) \leq C_1(\gamma^{1/(\lfloor \log_2 n \rfloor + 1)}, 2)^n, \quad (1.9)$$

where $[a]$ denotes the integer part of a .

(2) Let $C_2 \geq 1$ and let $|\cdot| : R \rightarrow \mathbb{R}_+$ be a (C_1, C_2) -AHAV on R . If $\overline{\lim}_{\gamma \rightarrow 1} C_1(\gamma, 2) = C_1 < +\infty$, then $|\cdot|$ is a (C_1, C_2) -Hölder absolute value.

(3) If R is a ring on which a (C_1, C_2) -AHAV $|\cdot|$ is defined, then R is a discrete ring for the topology defined by $|\cdot|$.

1.1. Weak subadditive lemma. We prove a generalization of a classical lemma on subadditive sequences (which might be of independent interest).

DEFINITION 1.6. The real sequence $(b_m)_{m \in \mathbb{N}}$ is weakly subadditive if

(i) for $\gamma > 1$ and $k \geq 1$, there is a constant $K(\gamma, k) > 0$ such that for $m_1, \dots, m_k \in \mathbb{N}$,

$$b_{m_1 + \dots + m_k} \leq \gamma \sum_{i=1}^k b_{m_i} + K(\gamma, k); \quad (1.10)$$

(ii) for $\gamma > 1$, we have $K^*(\gamma) = \overline{\lim}_{k \rightarrow \infty} (1/k)K(\gamma, k) < +\infty$.

LEMMA 1.7. *If $(b_m)_{m \in \mathbb{N}}$ is weakly subadditive, then*

$$\lim_{m \rightarrow \infty} \frac{b_m}{m} = \overline{\lim}_{m \rightarrow \infty} \frac{b_m}{m}. \quad (1.11)$$

PROOF. Fix $n \geq 1$. For any $m \in \mathbb{Z}$, we consider the Euclidean division

$$m = nq + r, \quad 0 \leq r < n. \quad (1.12)$$

Now,

$$b_m = b_{nq+r} \leq \gamma(qb_n + b_r) + K(\gamma, q+1). \quad (1.13)$$

Dividing by m ,

$$\frac{b_m}{m} = \frac{b_{nq+r}}{nq+r} \leq \gamma \left(\frac{q}{nq+r} b_n + \frac{b_r}{nq+r} \right) + \left(\frac{q+1}{nq+r} \right) \frac{K(\gamma, q+1)}{q+1}. \quad (1.14)$$

Taking the upper limit when $m \rightarrow \infty$,

$$\overline{\lim}_{m \rightarrow \infty} \frac{b_m}{m} \leq \gamma \left(\frac{b_n}{n} + 0 \right) + \frac{1}{n} K^*(\gamma). \quad (1.15)$$

That is, for all $q \geq 1$,

$$\overline{\lim}_{m \rightarrow \infty} \frac{b_m}{m} \leq \gamma \frac{b_n}{n} + \frac{1}{n} K^*(\gamma). \quad (1.16)$$

Now, taking the lower limit on the right side when $n \rightarrow \infty$,

$$\overline{\lim}_{m \rightarrow \infty} \frac{b_m}{m} \leq \gamma \lim_{n \rightarrow \infty} \frac{b_n}{n}. \quad (1.17)$$

This holds for all $\gamma > 1$, thus making $\gamma \rightarrow 1$,

$$\overline{\lim}_{m \rightarrow \infty} \frac{b_m}{m} \leq \lim_{m \rightarrow \infty} \frac{b_m}{m}, \quad \overline{\lim}_{m \rightarrow \infty} \frac{b_m}{m} = \lim_{m \rightarrow \infty} \frac{b_m}{m}. \quad (1.18)$$

□

1.2. Proof of Theorem 1.1

LEMMA 1.8. Define $|| \cdot || : R \rightarrow \mathbb{R}_+$ by $||x|| = \|x\|^\alpha$. Then, $|| \cdot ||$ is a $(C_1^\alpha, 2)$ -AHAV on R .

PROOF. (AHAV1) $||x|| = 0$ if and only if $\|x\| = 0$ if and only if $x = 0$.

(AHAV2) $||x + y|| = \|x + y\|^\alpha \leq (2C_2)^\alpha (\max(\|x\|, \|y\|))^\alpha \leq 2(\|x\|^\alpha + \|y\|^\alpha) = 2(||x|| + ||y||)$.

(AHAV3) For all $\gamma > 1$ and for all $n \geq 2$ there is a constant $C_1(\gamma, n)^\alpha > 1$ such that for all $x_1, \dots, x_n$ in R ,

$$\begin{aligned} & (C_1(\gamma, n))^{-\alpha} ||x_1||^{\gamma^{-1}} \cdots ||x_n||^{\gamma^{-1}} \\ & \leq ||x_1 \cdots x_n|| \leq C_1(\gamma, n)^\alpha ||x_1||^\gamma \cdots ||x_n||^\gamma. \end{aligned} \quad (1.19)$$

□

LEMMA 1.9. Let $x \in R$ and define the real sequence $(a_n)_{n \in \mathbb{N}}$ by $a_n = ||x^n||$. The sequence $(a_n^{1/n})$ is converging and

$$e^{-L} ||x|| \leq \lim_{n \rightarrow \infty} a_n^{1/n} \leq e^L ||x||, \quad (1.20)$$

where $L = \overline{\lim}_{n \rightarrow \infty} (1/n) \log C_1(\gamma, n) < +\infty$.

PROOF. Let $b_m = \log a_m$. The sequence $\{b_m\}$ is weakly subadditive, since for all $\gamma > 1$ and for all $k \geq 1$ there is a constant $K(\gamma, k) = (C_1(\gamma, k))^\alpha$, such that

$$b_{m_1 + \dots + m_k} \leq \gamma \sum_{i=1}^k b_{m_i} + \log K(\gamma, k), \quad (1.21)$$

and for all $\gamma > 1$,

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \log K(\gamma, k) < +\infty. \quad (1.22)$$

Therefore, by Lemma 1.7,

$$\lim_{m \rightarrow \infty} \frac{b_m}{m} = \overline{\lim}_{m \rightarrow \infty} \frac{b_m}{m}. \quad (1.23)$$

Thus, to prove the convergence of $\{a_n^{1/n}\}$, we only have to prove that $\{a_n^{1/n}\}$ is bounded.

Let $\gamma > 1$, for $n \in \mathbb{N}$ there is $C_1(\gamma, n)^\alpha$ satisfying

$$C_1(\gamma, n)^{-\alpha} ||x||^{n/\gamma} \leq ||x^n|| \leq C_1(\gamma, n)^\alpha ||x||^{n\gamma}. \quad (1.24)$$

Taking n th roots,

$$C_1(\gamma, n)^{-\alpha/n} \|x\|^{1/\gamma} \leq a_n^{1/n} \leq C_1(\gamma, n)^{\alpha/n} \|x\|^\gamma. \quad (1.25)$$

Since $L = \overline{\lim}_{n \rightarrow \infty} (1/n) \log C_1(\gamma, n) < +\infty$, we obtain

$$e^{-\alpha L} \|x\|^{1/\gamma} \leq \lim_{n \rightarrow \infty} a_n^{1/n} \leq e^{\alpha L} \|x\|^\gamma. \quad (1.26)$$

This inequality holds for any $\gamma > 1$. Taking the limit when $\gamma \rightarrow 1$,

$$e^{-\alpha L} \|x\| \leq a_n^{1/n} \leq e^{\alpha L} \|x\|. \quad (1.27)$$

□

Now we define that $|\cdot| : R \rightarrow \mathbb{R}_+$ by $|0| = 0$ and that $|x| = \lim_{n \rightarrow \infty} \|x^n\|^{1/n}$ for $x \neq 0$.

LEMMA 1.10. *The function $|\cdot| : R \rightarrow \mathbb{R}_+$ defined as above is an absolute value on R . Moreover, if $\overline{\lim}_{n \rightarrow +\infty} (1/n) \log C_1(\gamma, n) = 0$, then $|x| = \|x\|^\alpha$ for all $x \in R$.*

PROOF. From Lemma 1.9, if $\overline{\lim}_{n \rightarrow \infty} (1/n) \log C_1(\gamma, n) = 0$, we obtain

$$\|x\| \leq |x| \leq \|x\|. \quad (1.28)$$

That is, $|x| = \|x\|$. □

It is clear that, $|x| = 0$ if and only if $x = 0$. Next we check the multiplicative property. For $\gamma > 1$ and for $n \geq 2$ there exists $C_1(\gamma, 2)^\alpha > 1$, such that for $n \in \mathbb{N}$ and x, y in R ,

$$\begin{aligned} C_1(\gamma, 2)^{-\alpha} \|x^n\|^{y^{-1}} \|y^n\|^{y^{-1}} \\ \leq \|x^n\| \|y^n\| \leq C_1(\gamma, n)^\alpha \|x^n\|^\gamma \|y^n\|^\gamma. \end{aligned} \quad (1.29)$$

Taking n th roots and passing to the limit when $n \rightarrow +\infty$, we obtain

$$|x|^{y^{-1}} |y|^{y^{-1}} \leq |xy| \leq |x|^\gamma |y|^\gamma. \quad (1.30)$$

Taking the limit when $\gamma \rightarrow 1$, we have the desired multiplicative property.

Finally, we have to check the triangle inequality. This is a corollary of the following general proposition that gives an equivalent, apparently weaker, definition of absolute value.

PROPOSITION 1.11. *Let R be a commutative ring with unity. Let $|\cdot| : R \rightarrow \mathbb{R}_+$ be a function satisfying the following three properties:*

- (A1) $|x| = 0$ if and only if $x = 0$;
- (A2) (approximate triangle inequality) there exists a real constant $B > 0$, such that for all x, y in R , $|x + y| \leq B(|x| + |y|)$;
- (A3) for x, y in R , $|xy| = |x||y|$.

Then, $|\cdot|$ is an absolute value on R , that is, $|\cdot|$ satisfies the triangle inequality.

LEMMA 1.12. *For $x, y \in R$,*

$$|x + y| \leq B(|x| + |y|) \leq 2B \max(|x|, |y|). \quad (1.31)$$

LEMMA 1.13. Let $|\cdot|' : R \rightarrow \mathbb{R}_+$, such that for $x, y \in R$,

$$|x + y|' \leq M \max(|x|', |y|'), \quad (1.32)$$

for some positive constant M . Then for $x_1, x_2, \dots, x_n \in R$,

$$\left| \sum_{i=1}^n x_i \right|' \leq M^{\lceil \log_2 n \rceil + 1} \max_{1 \leq i \leq n} (|x_i|'), \quad (1.33)$$

where $[a]$ denotes the integer part of a .

PROOF. Let $m = \lceil \log_2 n \rceil + 1$ and complete the sequence $(x_i)_{1 \leq i \leq n}$ into $(x_i)_{1 \leq i \leq 2^m}$ adjoining 0 elements.

$$\begin{aligned} \left| \sum_{i=1}^{2^m} x_i \right|' &\leq M \max \left(\left| \sum_{i=1}^{2^{m-1}} x_i \right|', \left| \sum_{i=2^{m-1}+1}^{2^m} x_i \right|' \right) \\ &\leq M^2 \max \left(\left| \sum_{i=1}^{2^{m-2}} x_i \right|', \left| \sum_{i=2^{m-2}+1}^{2^{m-1}} x_i \right|', \left| \sum_{i=2^{m-1}+1}^{3 \cdot 2^{m-2}} x_i \right|', \left| \sum_{i=3 \cdot 2^{m-2}+1}^{2^m} x_i \right|' \right) \\ &\leq \dots \leq M^m \max_{1 \leq i \leq 2^m} |x_i|. \quad \square \end{aligned} \quad (1.34)$$

LEMMA 1.14. Let $\bar{\mathbb{Z}}$ be the image of $\mathbb{Z}$ in R . For $n \in \mathbb{N}$,

$$|\bar{n}| \leq 2n|1|. \quad (1.35)$$

PROOF. We use Lemma 1.13 with $M = 2$ and $|\cdot|' = |\cdot|$. Take $m = \lceil \log_2 n \rceil + 1$, $n \leq 2^m \leq 2n$, and $x_i = 1$ for $1 \leq i \leq n$. We have

$$|n| = \left| \sum_{i=1}^n x_i \right| \leq 2^m |1| \leq 2n|1|. \quad \square \quad (1.36)$$

LEMMA 1.15. Let $\bar{\mathbb{Z}}$ be the image of $\mathbb{Z}$ in R . For $n \in \mathbb{N}$,

$$|\bar{n}| \leq n. \quad (1.37)$$

PROOF. Using Lemma 1.14,

$$|\bar{n^k}| = |\bar{n}^k| \leq 2n^k|1|, \quad (1.38)$$

and $|\bar{n}^k|^{1/k} \leq 2^{1/k} n|1|^{1/k}$. Taking $k \rightarrow +\infty$, we have $|\bar{n}| \leq n$. $\square$

PROOF OF PROPOSITION 1.11. Let $x, y \in R$ and $n \geq 1$. Let $m = \lceil \log_2 n \rceil + 1$. Using Lemmas 1.12 and 1.14, we have

$$\begin{aligned} |(x + y)^n| &= \left| \sum_{i=0}^n \binom{n}{i} x^i y^{n-i} \right| \\ &\leq (B)^m \max_{0 \leq i \leq n} \left| \binom{n}{i} x^i y^{n-i} \right|. \end{aligned} \quad (1.39)$$

Now using [Lemma 1.14](#),

$$\begin{aligned}
 |(x+y)^n| &\leq (2B)^m \max_{0 \leq i \leq n} \binom{n}{i} |x|^i |y|^{n-i} \\
 &\leq (2B)^m \max_{0 \leq i \leq n} \binom{n}{i} |x|^i |y|^{n-i} \\
 &\leq (2B)^m \sum_{i=0}^n \binom{n}{i} |x|^i |y|^{n-i} \\
 &\leq (2B)^m (|x| + |y|)^n.
 \end{aligned} \tag{1.40}$$

Finally,

$$|x+y| = |(x+y)^n|^{1/n} \leq (2B)^{(1/n)(\log_2 n + 1)} (|x| + |y|), \tag{1.41}$$

and passing to the limit $n \rightarrow +\infty$ we get the sharp triangle inequality $|x+y| \leq |x| + |y|$. $\square$

PROOF OF [THEOREM 1.2](#).

CASE 1. Assume $\overline{\lim}_{n \rightarrow \infty} (1/n) \log C_1(\gamma, n) = 0$. By [Lemma 1.8](#), for all x, y in R we have

$$|x+y| = ||x+y|| \leq 2(||x|| + ||y||) \leq 4 \max(||x||, ||y||) = 4 \max(|x|, |y|). \tag{1.42}$$

Therefore, by [Proposition 1.11](#), the function $|\cdot|$ satisfies the triangle inequality.

CASE 2. Assume $0 < L = \overline{\lim}_{n \rightarrow \infty} (1/n) \log C_1(\gamma, n) < +\infty$. From [Lemma 1.9](#), for any x in R ,

$$e^{-\alpha L} ||x|| \leq |x| \leq e^{\alpha L} ||x||. \tag{1.43}$$

Therefore,

$$|x+y| \leq e^{\alpha L} ||x+y|| \leq 2e^{\alpha L} (||x|| + ||y||) \leq 2e^{2\alpha L} (|x| + |y|). \tag{1.44}$$

Thus by [Proposition 1.11](#), the function $|\cdot|$ satisfies the triangle inequality, it is an absolute value, and $\|\cdot\|^\alpha$ is $(e^{L\alpha}, \alpha)$ -equivalent to $|\cdot|$. $\square$

REFERENCES

- [1] E. Artin, *Theory of Algebraic Numbers*, George Striker, Göttingen, 1959.
- [2] E. Muñoz Garcia, *Hölder absolute values are equivalent to classical ones*, Proc. Amer. Math. Soc. **127** (1999), no. 7, 1967-1971.
- [3] A. Ostrowski, *Über einige Lösungen der Funktionalgleichung $\phi(x) \cdot \phi(y) = \phi(x \cdot y)$* , Acta Math. **41** (1914), 271-284 (German).

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