ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF A DISCRETE REACTION-DIFFUSION EQUATION

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By means of Bihari type inequalities, we derive sufficient conditions for solutions of a discrete reaction-diffusion equation to be bounded or to converge to zero. Asymptotic representation of solutions are also derived. Our results yield estimates and explicit attractive regions for the solutions.

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1. Introduction. Discrete reaction-diffusion type partial difference equations have recently been introduced by a number of authors as models for the study of spatio-temporal chaos (cf. [4]). Stability criteria have also been derived for such equations, which involves two-level (see [2]) as well as three-level processes (cf. [3]). Besides, the question of stability and asymptotic representation of solutions of such difference schemes are also of fundamental importance. In this paper, we study nonlinear two-level partial difference equations and, by means of comparison theorems, we derive sufficient conditions for the solutions to be bounded or to converge to zero. Further, asymptotic formulae for the solutions of the above equation are obtained. We want to point out that our results compare favorably with results in [1, 2], basically because their main results are valid only for linear or sublinear perturbations. Besides, our results are new and do not overlap with those in [1, 2].

2. Preliminary facts. Let $\mathbb{R}$ be the set of reals and $\mathbb{N}$ the set of nonnegative integers. Consider a discrete reaction-diffusion equation of the form

$$u_i^{(j+1)} = au_{i-1}^{(j)} + bu_i^{(j)} + cu_{i+1}^{(j)} + g_i^{(j)} + \tilde{F}(i, j, u_i^{(j)}), \quad (2.1)$$

where $i = 1, 2, \dots, n$; $j \in \mathbb{N}$; $a, b, c \in \mathbb{R}$; $g = \{g_i^{(j)}\}$ is a real function defined for $i = 1, 2, \dots, n$ and $j \in \mathbb{N}$, and $\tilde{F}$ is a real function. We will also assume that side conditions

$$\begin{aligned} u_0^{(j)} &= h_j \in \mathbb{R}, \quad j \in \mathbb{N}, \\ u_{n+1}^{(j)} &= q_j \in \mathbb{R}, \quad j \in \mathbb{N}, \\ u_i^{(0)} &= \tau_i \in \mathbb{R}, \quad i = 1, 2, \dots, n, \end{aligned} \quad (2.2)$$

are imposed. Let

$$\Omega = \{(i, j) \mid i = 0, 1, \dots, n+1; j \in \mathbb{N}\}. \quad (2.3)$$

A solution of (2.1) and (2.2) is a discrete function $u = \{u_i^{(j)}\}_{(i,j) \in \Omega}$ which satisfies the functional relation (2.1) and also the side conditions (2.2). If we put $u^{(j)} = \text{col}(u_1^{(j)}, u_2^{(j)}, \dots, u_n^{(j)})$ and $\tau = \text{col}(\tau_1, \dots, \tau_n)$, then the sequence $\{u^{(j)}\}_{j=0}^\infty$ will satisfy the two-term vector equation

$$u^{(j+1)} = Au^{(j)} + f_j + F(j, u^{(j)}), \quad j \in \mathbb{N}, \quad (2.4)$$

subject to the initial condition

$$u^{(0)} = \tau, \quad (2.5)$$

where

$$A = \begin{bmatrix} b & c & 0 & \cdots & \cdots & 0 \\ a & b & c & 0 & \cdots & 0 \\ 0 & a & b & c & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & a & b \end{bmatrix}, \quad (2.6)$$

$$f_j = (g_1^{(j)}, \dots, g_n^{(j)}) + \text{col}(ah_j, 0, \dots, 0, cq_j),$$

$$F(j, x) = \text{col}(\tilde{F}(1, j, x_1), \dots, \tilde{F}(n, j, x_n)).$$

Conversely, if $\{u^{(j)}\}_{j=0}^\infty$ is a solution of (2.4) and (2.5), then by augmenting each $u^{(j)} = \text{col}(u_1^{(j)}, \dots, u_n^{(j)})$ with the terms $u_0^{(j)} = h_j$ and $u_{n+1}^{(j)} = q_j$ to form $\{u_0^{(j)}, u_1^{(j)}, \dots, u_n^{(j)}, u_{n+1}^{(j)}\}$, we see that the resulting family forms a solution of (2.1) and (2.2).

THEOREM 2.1 (see [5]). *Let $B = (b_{ij})_{n \times n}$ be a real matrix and $\rho(B)$ its spectral radius. Then, there exists a constant $\Gamma \geq 1$, such that $\|B^i\| \leq \Gamma(\rho(B))^i$ for $i \in \mathbb{N}$.*

We will use the following theorem which gives an explicit estimate for a function $u = u(m)$ which satisfies the functional inequality

$$u(m) \leq c + \sum_{i=1}^p \sum_{j=0}^{m-1} \lambda_i(j) w_i(u(j)), \quad m \in \mathbb{N}, \quad (2.7)$$

where (G1) $c \geq 0$, p is a positive integer, (G2) $\lambda_1, \lambda_2, \dots, \lambda_p$ are nonnegative sequences in $\ell_1(\mathbb{N})$, where $\ell_1(\mathbb{N})$ is the set of all absolutely summable real sequences defined on $\mathbb{N}$, and (G3) the functions w_i , $1 \leq i \leq p$, are continuous on $[0, \infty)$ and positive on $(0, \infty)$, such that w_{i+1}/w_i , $1 \leq i \leq p-1$, are nondecreasing on $(0, \infty)$.

To this end, we need to define

$$W_i(u) = \int_{u_i}^u \frac{1}{w_i(s)} ds, \quad u > 0, \quad u_i > 0, \quad 1 \leq i \leq p, \quad (2.8)$$

$$\Psi_i(u) = W_i^{-1}(W_i(u) + \alpha_i),$$

with $\alpha_1, \dots, \alpha_p \in \mathbb{R}$, and $\varphi_0(u) = u$, $\varphi_i = \Psi_i \circ \Psi_{i-1} \circ \dots \circ \Psi_1$ for $1 \leq i \leq p$.

THEOREM 2.2 (see [6, 7]). *Under the conditions (G1), (G2), and (G3), if $u = \{u(m)\}_{m \in \mathbb{N}}$ is a nonnegative sequence which satisfies (2.7), $\alpha_i = \sum_{j=0}^{\infty} \lambda_i(j)$, and $c < \varphi_p^{-1}(\infty)$, then*

$$u(m) \leq W_p^{-1} \left(W_p(\varphi_{p-1}(c)) + \sum_{k=0}^{m-1} \lambda_p(k) \right), \quad m \in \mathbb{N}, \quad (2.9)$$

and $u(m) \leq \varphi_p(c)$ for $m \in \mathbb{N}$.

We remark that if

$$\int_1^{\infty} \frac{ds}{w_i(s)} = \infty, \quad 1 \leq i \leq p, \quad (2.10)$$

holds, then the same conclusion $u(m) \leq \varphi_p(c)$ is valid for all $c > 0$. If the dual condition

$$\int_{0+}^1 \frac{ds}{w_i(s)} = \infty, \quad 1 \leq i \leq p, \quad (2.11)$$

holds, then the same conclusion $u(m) \leq \varphi_p(c)$ is valid if c is small enough.

EXAMPLE 2.3. Let $w_i(u) = u^{n_i}$ for $i = 1, 2$. Then

$$\Psi_i(u) = \begin{cases} [u^{1-n_i} + \alpha_i(1-n_i)]^{-1/(n_i-1)}, & n_i \in (0, 1) \cup (1, \infty), \\ u \exp \alpha_i, & n_i = 1, \end{cases} \quad (2.12)$$

for $i = 1, 2$. Thus, for $n_i \neq 1$, $\varphi = \Psi_2 \circ \Psi_1$ takes the form

$$\varphi(u) = \{[u^{1-n_1} + \alpha_1(1-n_1)]^{(n_2-1)/(n_1-1)} + \alpha_2(1-n_2)\}^{-1/(n_2-1)}. \quad (2.13)$$

Moreover, since $\Psi_i^{-1}(u) = W_i^{-1}[W_i(u) - \alpha_i]$, $\varphi^{-1}(u)$ is obtained from (2.13) by replacing α_i with $-\alpha_i$. Thus, for $n_i > 1$, $\varphi(u)$ is defined for all $u < \varphi^{-1}(\infty)$, where

$$\varphi^{-1}(\infty) = \{[\alpha_1(n_1-1)]^{(n_2-1)/(n_1-1)} + \alpha_2(n_2-1)\}^{-1/(n_2-1)}. \quad (2.14)$$

For the case $n_1 = 1 < n_2$, we get

$$\begin{aligned} \varphi(u) &= [(ue^{\alpha_1})^{1-n_2} + \alpha_2(1-n_2)]^{-1/(n_2-1)} \\ &= ue^{\alpha_1} [1 + \alpha_2(1-n_2)(ue^{\alpha_1})^{n_2-1}]^{-1/(n_2-1)}, \end{aligned} \quad (2.15)$$

which is defined for

$$u < \varphi^{-1}(\infty) = [\alpha_2(n_2-1)]^{-1/(n_2-1)}, \quad (2.16)$$

and if $n_1 < 1 = n_2$, then

$$\varphi(u) = e^{\alpha_2} [u^{1-n_1} + \alpha_1(1-n_1)]^{-1/(n_1-1)} \quad (2.17)$$

is defined for $u < \varphi^{-1}(\infty) = (\infty)$.

3. Boundedness. Now, we can establish our boundedness criteria.

THEOREM 3.1. *Assume that*

$$\|F(j, x)\| \leq \sum_{l=1}^p \lambda_l(j) w_l(\|x\|), \quad j \in \mathbb{N}, \quad (3.1)$$

for all $x \in \mathbb{R}^n$, where $w_1, \dots, w_p$ satisfy condition (G3), and $\lambda_1, \dots, \lambda_p$ are nonnegative sequences in $\ell_1(\mathbb{N})$. In addition, suppose $\sum_{k=0}^{\infty} \|f_k\| < \infty$, $\rho(A) = \rho \leq 1$, and $\|A^i\| \leq \Gamma \rho^i$ for $i \in \mathbb{N}$. If there exists a constant $d > 0$ such that

$$\sum_{k=0}^{\infty} \lambda_l(k) < \frac{1}{\Gamma} \int_{\varphi_{l-1}(d)}^{\infty} \frac{1}{w_l(s)} ds, \quad 1 \leq l \leq p, \quad (3.2)$$

and $\Gamma(\|\tau\| + \sum_{k=0}^{\infty} \|f_k\|) < d$, then all solutions of (2.4) are bounded.

PROOF. By inductive arguments, it is easily seen that the unique solution $\{u^{(j)}\}_{j=0}^{\infty}$ of (2.4), subject to $u^{(0)} = \tau$, satisfies

$$u^{(j)} = A^j \tau + \sum_{k=0}^{j-1} A^{j-k-1} f_k + \sum_{k=0}^{j-1} A^{j-k-1} F(k, u^{(k)}), \quad j \in \mathbb{N}. \quad (3.3)$$

Thus, in view of Theorem 2.1 and our assumptions,

$$\|u^{(j)}\| \leq \Gamma \rho^j \|\tau\| + \Gamma \sum_{k=0}^{j-1} \rho^{j-k-1} \|f_k\| + \Gamma \sum_{l=1}^p \sum_{k=0}^{j-1} \rho^{j-k-1} \lambda_l(k) w_l(\|u^{(k)}\|), \quad (3.4)$$

where Γ is some positive number greater than or equal to 1. Put $v(j) = \|u^{(j)}\|$ for $j \in \mathbb{N}$. Then

$$\begin{aligned} v(j) &\leq \Gamma \|\tau\| + \Gamma \sum_{k=0}^{\infty} \|f_k\| + \Gamma \sum_{l=1}^p \sum_{k=0}^{j-1} \lambda_l(k) w_l(v(k)) \\ &\leq d + \Gamma \sum_{l=1}^p \sum_{k=0}^{j-1} \lambda_l(k) w_l(v(k)). \end{aligned} \quad (3.5)$$

So, by Theorem 2.2, we obtain

$$v(j) \leq W_p^{-1} \left(W_p(\varphi_{p-1}(d)) + \Gamma \sum_{k=0}^{j-1} \lambda_p(k) \right), \quad j \in \mathbb{N}. \quad (3.6)$$

Inequalities (3.2) show that this estimation is valid for all $j \in \mathbb{N}$, and that the function in the right-hand side is bounded, in fact,

$$W_p^{-1} \left(W_p(\varphi_{p-1}(d)) + \Gamma \sum_{k=0}^{\infty} \lambda_p(k) \right) \leq \varphi_p(d). \quad (3.7)$$

Hence,

$$v(j) \leq \varphi_p(d); \quad j \in \mathbb{N}, \quad (3.8)$$

concluding the proof. $\square$

REMARKS. (a) If $\int_1^\infty (1/w_l(s))ds = \infty$, $1 \leq l \leq p$, then conditions (3.2) of [Theorem 3.1](#) are satisfied for all $d > 0$.

(b) If $\int_{0+}^1 (1/w_l(s))ds = \infty$, $1 \leq l \leq p$, then there always exists a small d satisfying conditions (3.2) of [Theorem 3.1](#).

(c) If $\int_{0+}^1 (1/w_l(s))ds < \infty$, $1 \leq l \leq p$, then the inequality $\sum_{k=0}^\infty \lambda_l(k) \geq \int_0^\infty (1/w_l(s))ds$ for some l , $1 \leq l \leq p$, implies that there is no $d > 0$ satisfying (3.2). In every case, the biggest d satisfying conditions (3.2) is $d = \varphi_p^{-1}(\infty)$.

Thus, we can establish the following corollary of [Theorem 3.1](#).

COROLLARY 3.2. (A) If $\int_1^\infty (1/w_l(s))ds = \infty$, $1 \leq l \leq p$, holds, then the result of [Theorem 3.1](#) is valid for all solutions.

(B) If $\int_{0+}^1 (1/w_l(s))ds = \infty$, $1 \leq l \leq p$, holds, then the statement of [Theorem 3.1](#) is valid for all solutions of (2.4) such that $\Gamma(\|u^{(0)}\| + \sum_{k=0}^\infty \|f_k\|)$ is small enough, namely, $\Gamma(\|u^{(0)}\| + \sum_{k=0}^\infty \|f_k\|) < \varphi_p^{-1}(\infty)$.

COROLLARY 3.3. Suppose that

$$\|F(j, x)\| \leq \sum_{l=1}^p \lambda_l(j) \|x\|^{\eta_l}, \quad j \in \mathbb{N}, \quad (3.9)$$

where $\eta_i \geq 1$, $1 \leq i \leq p$ and $\lambda_1, \dots, \lambda_p$ are positive sequences in $\ell_1(\mathbb{N})$. Then, the statement of [Theorem 3.1](#) is valid for all sufficiently small positive d .

REMARK 3.4. If $0 < \eta_i < 1$, $1 \leq i \leq p$, in (3.9), then the statement of [Theorem 3.1](#) remains valid for all $d > 0$, and consequently for all solutions of (2.4).

THEOREM 3.5. Suppose that the function $F = F(j, x)$ satisfies (3.9), where

- (I) the functions w_l , $1 \leq l \leq p$, satisfy conditions (G3), and for any l , $1 \leq l \leq p$, there is a function r_l defined on $(0, \infty)$ such that $w_l(\alpha u) \leq r_l(\alpha) w_l(u)$ for $\alpha \geq 0$, $u \geq 0$;
- (II) the functions λ_l , $1 \leq l \leq p$, satisfy $\sum_{k=0}^\infty \rho^{-k} \cdot \lambda_l(k) r_l(\rho^k) < \infty$;
- (III) there is a constant $d > 0$ such that

$$\sum_{k=0}^\infty \rho^{-k} \cdot \lambda_l(k) r_l(\rho^k) < \frac{\rho}{\Gamma} \int_{\varphi_{l-1}(d)}^\infty \frac{ds}{w_l(s)}, \quad 1 \leq l \leq p; \quad (3.10)$$

- (IV) $\sum_{k=0}^\infty \rho^{-k} \|f_k\| < \infty$. If $\rho(A) = \rho < 1$ and $\|A^i\| \leq \Gamma \rho^i$ for $i \in \mathbb{N}$, then any solution $u^{(j)}$ of (2.4), such that $\Gamma(\|\tau\| + \sum_{k=0}^\infty \rho^{-(k+1)} \|f_k\|) < d$, converges to zero, as $j \rightarrow \infty$.

Indeed, it suffices to proceed in a way similar to the proof of [Theorem 3.1](#), thus we omit it.

COROLLARY 3.6. Suppose that

$$\|F(j, x)\| \leq \sum_{l=1}^p \lambda_l(j) \|x\|^{\eta_l}, \quad j \in \mathbb{N}, \quad (3.11)$$

where $\eta_i \geq 1$, $1 \leq i \leq p$ and the sequences $\lambda_1, \dots, \lambda_p$ satisfy

$$\sum_{k=0}^\infty \rho^{-k(1-\eta_l)} \cdot \lambda_l(k) < \infty, \quad 1 \leq l \leq p. \quad (3.12)$$

Further, suppose that there exists a constant $d > 0$ such that

$$\sum_{k=0}^{\infty} \rho^{-k(1-\eta_l)} \cdot \lambda_l(k) < \frac{\rho}{\Gamma} \frac{(\varphi_{l-1}(d))^{1-\eta_l}}{1-\eta_l}, \quad 1 \leq l \leq p. \quad (3.13)$$

Then the statement of [Theorem 3.5](#) is valid for all sufficiently small positive d .

REMARK 3.7. We want to point out that $\varphi_p(d)$ and $\varphi_p^{-1}(\infty)$ can be explicitly calculated (see [Section 2](#), [Example 2.3](#)), and consequently, we can show the radius of attraction for bounded and convergent to zero solutions (see [\[6\]](#)).

4. Asymptotic representation. In this section, our objective is to obtain asymptotic formulae for the solution of discrete reaction-diffusion equations of the form [\(2.1\)](#), but under the additional conditions $u_0^{(j)} = 0 = u_{n+1}^{(j)}$; $j \in \mathbb{N}$ and $g_i^{(j)} = 0$ for all $i = 1, 2, \dots, n$; $j \in \mathbb{N}$.

By iteration and induction, it can be proved that the unique vector solution $\{u^{(j)}\}_{j=0}^{\infty}$ of the reduced equation

$$u^{(j+1)} = Au^{(j)} + F(j, u^{(j)}); \quad j \in \mathbb{N}, \quad (4.1)$$

subject to $u^{(0)} = \tau$, satisfies

$$u^{(j)} = A^j \tau + \sum_{k=0}^{j-1} A^{j-k-1} F(k, u^{(k)}), \quad j \in \mathbb{N}. \quad (4.2)$$

THEOREM 4.1. Assume that $\|F(j, x)\| \leq \lambda(j)w(\|x\|)$, for all $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$, where $w : [0, \infty) \rightarrow [0, \infty)$ is a continuous, positive, and nondecreasing function on $[0, \infty)$. Further, suppose that $\lambda \in \ell_1(\mathbb{N})$, and there exists a constant $d > 0$ such that

$$\sum_{k=0}^{\infty} \lambda(k) < \frac{1}{\Gamma} \int_{d\|\tau\|}^{\infty} \frac{ds}{w(s)}. \quad (4.3)$$

If $\rho(A) = \rho \leq 1$, and $\|A^i\| \leq \Gamma\rho^i$ for all $i \in \mathbb{N}$, then corresponding to each bounded solution of [\(4.1\)](#) there is $z_0 \in \mathbb{R}^n$ such that $u^{(j)} = A^j z_0 + \tilde{o}(1)$, as $j \rightarrow \infty$, where $\tilde{o}(1)$ represents a vector function of j which is bounded at infinity.

PROOF. From [Theorem 3.1](#), we infer that all solutions $u^{(j)}$ of [\(4.1\)](#), such that $\Gamma\|u^{(0)}\| < d$, are bounded. On the other hand, we have

$$\begin{aligned} \left\| \sum_{k=0}^{j-1} A^{j-k-1} F(k, u^{(k)}) \right\| &\leq \sum_{k=0}^{j-1} \Gamma\rho^{j-k-1} \lambda(k) w(\|u^{(k)}\|) \\ &\leq \Gamma w(K) \sum_{k=0}^{j-1} \lambda(k) \leq \Gamma w(K) \sum_{k=0}^{\infty} \lambda(k), \end{aligned} \quad (4.4)$$

where $K > 0$ is a constant such that $\|u^{(j)}\| \leq K$, for $j \in \mathbb{N}$. Then, for every solution u of [\(4.1\)](#), we see that the solution v of equation

$$v^{(j+1)} = Av^{(j)}; \quad j \in \mathbb{N}, \quad (4.5)$$

given by

$$v^{(j)} = u^{(j)} - \sum_{k=0}^{j-1} A^{j-k-1} F(k, u^{(k)}), \quad (4.6)$$

has the property

$$u^{(j)} = v^{(j)} + \tilde{o}(1), \quad \text{as } j \rightarrow \infty. \quad (4.7)$$

Moreover, if $v^{(j)}$ is a solution of (4.5), then there is $z_0 \in \mathbb{R}^n$ such that

$$v^{(j)} = A^j z_0. \quad (4.8)$$

Therefore,

$$u^{(j)} = A^j z_0 + \tilde{o}(1), \quad \text{as } j \rightarrow \infty. \quad (4.9)$$

□

REMARK 4.2. In [Theorem 4.1](#), we assume that $\|\tau\|$ is small enough so that $W(0^+) = -\infty$, where

$$W(u) = \int_{u_0}^u \frac{ds}{w(s)}, \quad u > 0, \quad u_0 > 0, \quad (4.10)$$

in order that W^{-1} has meaning, that is, the inverse function $W^{-1}(v)$ is defined for $v \in (0, \delta_0)$, for δ_0 small enough.

COROLLARY 4.3. (I) If (2.10) holds, then the result of [Theorem 4.1](#) is valid for every solution of (4.1).

(II) If (2.11) holds, then the result of [Theorem 4.1](#) is valid for every solution $u^{(j)}$ of (4.1), such that $\Gamma \|u^{(0)}\| < \varphi_p^{-1}(\infty)$.

A more precise asymptotic formula is given in the following theorem.

THEOREM 4.4. Under the hypotheses of [Theorem 4.1](#), if in addition,

$$\sum_{k=0}^{j-1} A^{j-k-1} F(k, u^{(k)}) \rightarrow 0, \quad \text{as } j \rightarrow \infty, \quad (4.11)$$

then

$$u^{(j)} = A^j \cdot z_0 + o(1), \quad \text{as } j \rightarrow \infty. \quad (4.12)$$

THEOREM 4.5. Under the hypotheses of [Corollary 4.3\(II\)](#). Suppose that $\Phi(j, j_0) = A^{j-j_0}$, $j \geq j_0$, satisfies

$$\|\Phi^{-1}(j+1, 0)F(j, \Phi(j, 0)z)\| \leq \lambda(j)w(\|z\|), \quad (4.13)$$

for $j \in \mathbb{N}$ and $z \in \mathbb{R}^n$; and $\lambda \in \ell_1(\mathbb{N})$. Then, for every solution $u^{(j)}$ of (4.1), with $\|u^{(0)}\|$ small enough, there is $z_0 \in \mathbb{R}^n$, such that

$$u^{(j)} = A^j \left[z_0 + O \left(\sum_{\ell=j}^{\infty} \lambda(\ell) \right) \right], \quad \text{as } j \rightarrow \infty. \quad (4.14)$$

PROOF. Making $u^{(j)} = \Phi(j, 0)z(j)$ in (4.1), we get

$$z(j) = \tau + \sum_{k=0}^{j-1} A^{-k-1} F(k, A^k z(k)). \quad (4.15)$$

The rest of the proof follows by arguments similar to those in the proof of [Theorem 4.1](#), so we omit it. $\square$

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