ON AN \( n \)-TH-ORDER INFINITESIMAL GENERATOR AND TIME-DEPENDENT OPERATOR DIFFERENTIAL EQUATION WITH A STRONGLY ALMOST PERIODIC SOLUTION

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In a Banach space, if \( u \) is a Stepanov almost periodic solution of a certain \( n \)-th-order infinitesimal generator and time-dependent operator differential equation with a Stepanov almost periodic forcing function, then \( u, u', \ldots, u^{(n-2)} \) are all strongly almost periodic and \( u^{(n-1)} \) is weakly almost periodic.

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1. Introduction. Suppose that \( X \) is a Banach space, \( X^* \) is the dual space of \( X \), and \( \mathbb{R} \) is the real line. A continuous function \( f : \mathbb{R} \rightarrow X \) is said to be strongly (or Bochner) almost periodic if, given \( \varepsilon > 0 \), there is a positive real number \( r = r(\varepsilon) \) such that any interval of the real line of length \( r \) contains at least one point \( \tau \) for which

\[
\sup_{t \in \mathbb{R}} \| f(t+\tau) - f(t) \| \leq \varepsilon. \tag{1.1}
\]

A function \( f : \mathbb{R} \rightarrow X \) is weakly almost periodic if the scalar-valued function \( \langle x^*, f(t) \rangle = x^* f(t) \) is almost periodic for each \( x^* \in X^* \).

A function \( f \in L^p_{\text{loc}}(\mathbb{R};X) \) with \( 1 \leq p < \infty \) is said to be Stepanov-bounded or \( S^p \)-bounded on \( \mathbb{R} \) if

\[
\| f \|_{S^p} = \sup_{t \in \mathbb{R}} \left[ \int_t^{t+1} \| f(s) \|^p ds \right]^{1/p} < \infty. \tag{1.2}
\]

A function \( f \in L^p_{\text{loc}}(\mathbb{R};X) \) with \( 1 \leq p < \infty \) is said to be Stepanov almost periodic or \( S^p \)-almost periodic if, given \( \varepsilon > 0 \), there is a positive real number \( r = r(\varepsilon) \) such that any interval of the real line of length \( r \) contains at least one point \( \tau \) for which

\[
\sup_{t \in \mathbb{R}} \left[ \int_t^{t+1} \| f(s+\tau) - f(s) \|^p ds \right]^{1/p} \leq \varepsilon. \tag{1.3}
\]

We designate by \( L(X;X) \) the set of all bounded linear operators on \( X \) into itself. An operator-valued function \( T : \mathbb{R} \rightarrow L(X;X) \) is called a strongly continuous group if

\[
T(t_1 + t_2) = T(t_1) T(t_2) \quad \forall t_1, t_2 \in \mathbb{R}, \tag{1.4}
\]

\[
T(0) = I = \text{the identity operator on } X, \tag{1.5}
\]

\[
T(t)x, \quad t \in \mathbb{R} \rightarrow X, \text{ is continuous for each } x \in X. \tag{1.6}
\]
The infinitesimal generator $A$ of a strongly continuous group $T : \mathbb{R} \to L(X;X)$ is a closed linear operator, with its domain $D(A)$ dense in $X$, defined by
\[
Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} \quad \text{for } x \in D(A) \tag{1.7}
\]
(see Dunford and Schwartz [4]).

An operator-valued function $T : \mathbb{R} \to L(X;X)$ is said to be strongly (weakly) almost periodic if $T(t)x, t \in \mathbb{R} \to X$ is strongly (weakly) almost periodic for each $x \in X$.

Assume that $A$ and $B$ are two densely defined closed linear operators, having their domains and ranges in a Banach space $X$, and $f : \mathbb{R} \to X$ is a continuous function. Then, a strong solution of the differential equation
\[
u^{(n)}(t) = Au^{(n-1)}(t) + Bu(t) + f(t) \quad \text{a.e. on } \mathbb{R} \tag{1.8}
\]
is an $n$ times strongly differentiable function $u : \mathbb{R} \to D(B)$ with $u^{(n-1)}(t) \in D(A)$ for all $t \in \mathbb{R}$, and satisfying equation (1.8) a.e. (almost everywhere) on $\mathbb{R}$.

Our first result is as follows (see Zaidman [7] for a first-order infinitesimal generator differential equation).

**Theorem 1.1.** In a Banach space $X$, suppose that $f : \mathbb{R} \to X$ is an $S^1$-almost periodic continuous function, $A$ is the infinitesimal generator of a weakly almost periodic strongly continuous group $T : \mathbb{R} \to L(X;X)$, $B : \mathbb{R} \to L(X;X)$ is a strongly almost periodic operator-valued function, and $u : \mathbb{R} \to X$ is a strong solution of the differential equation
\[
u^{(n)}(t) = Au^{(n-1)}(t) + Bu(t) + f(t) \quad \text{a.e. on } \mathbb{R}. \tag{1.9}
\]
If $u$ is $S^1$-almost periodic from $\mathbb{R}$ to $X$ and $u^{(n-1)}$ is $S^1$-bounded on $\mathbb{R}$, then $u, u', \ldots, u^{(n-2)}$ are all strongly almost periodic from $\mathbb{R}$ to $X$, $T(-t)u^{(n-1)}(t)$ is weakly almost periodic from $\mathbb{R}$ to $X$, and $u^{(n-1)}$ is bounded on $\mathbb{R}$.


2. Lemmas

**Lemma 2.1.** The $(n-1)$th derivative of any solution of (1.9) admits the representation
\[
u^{(n-1)}(t) = T(t)u^{(n-1)}(0) + \int_0^t T(t-s)[B(s)u(s) + f(s)] ds \quad \text{on } \mathbb{R}. \tag{2.1}
\]

**Proof.** For an arbitrary but fixed $t \in \mathbb{R}$, we have
\[
d {ds} [T(t-s)u^{(n-1)}(s)] = T(t-s)[u^{(n)}(s) - Au^{(n-1)}(s)]
\]
\[= T(t-s)[B(s)u(s) + f(s)] \quad \text{a.e. on } \mathbb{R}, \text{ by (1.9)}. \tag{2.2}
\]
Hence,
\[
\int_0^t \frac {d}{ds} [T(t-s)u^{(n-1)}(s)] ds = \int_0^t T(t-s)[B(s)u(s) + f(s)] ds, \tag{2.3}
\]
which gives the desired representation by (1.5).
**Lemma 2.2.** In a Banach space $X$, if $g : \mathbb{R} \to X$ is a strongly almost periodic function and if $G : \mathbb{R} \to L(X;X)$ is a strongly (weakly) almost periodic operator-valued function, then $G(t)g(t)$, $t \in \mathbb{R} \to X$, is a strongly (weakly) almost periodic function.

**Proof.** See Rao [6, Theorem 1] for weak almost periodicity. 

**Lemma 2.3.** In a Banach space $X$, if $g : \mathbb{R} \to X$ is an $S^1$-almost periodic continuous function and if $G : \mathbb{R} \to L(X;X)$ is a weakly almost periodic operator-valued function, then $x^*G(t)g(t)$, $t \in \mathbb{R} \to$ scalars, is an $S^1$-almost periodic continuous function for each $x^* \in X^*$.

**Proof.** By our assumption, for an arbitrary but fixed $x^* \in X^*$, the scalar-valued function $x^*G(t)x$ is almost periodic, and hence is bounded on $\mathbb{R}$, for each $x \in X$. So, by the uniform-boundedness principle,

$$\sup_{t \in \mathbb{R}}||x^*G(t)|| = M < \infty. \quad (2.4)$$

The function $x^*G(t)g(t)$ is continuous on $\mathbb{R}$ (see the proof of Theorem 1 of Rao [6]).

Consider the functions on $\mathbb{R}$

$$g_\delta(t) = \frac{1}{\delta} \int_0^\delta g(t+s)ds \quad \text{for } \delta > 0. \quad (2.5)$$

Since $g$ is $S^1$-almost periodic from $\mathbb{R}$ to $X$, it follows that $g_\delta$ is strongly almost periodic from $\mathbb{R}$ to $X$ for each fixed $\delta > 0$. Further, as shown for scalar-valued functions in Besicovitch [2, pages 80–81], we can prove that $g_\delta \to g$ as $\delta \to 0^+$ in the $S^1$-sense, that is,

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} ||g(s) - g_\delta(s)||ds \to 0 \quad \text{as } \delta \to 0^+. \quad (2.6)$$

Furthermore, we have

$$x^*G(s)g(s) = x^*G(s)[g(s) - g_\delta(s)] + x^*G(s)g_\delta(s) \quad \text{on } \mathbb{R}, \quad (2.7)$$

and, by (2.4) and (2.6),

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} |x^*G(s)[g(s) - g_\delta(s)]|ds$$

$$\leq M \sup_{t \in \mathbb{R}} \int_t^{t+1} ||g(s) - g_\delta(s)||ds \to 0 \quad \text{as } \delta \to 0^+. \quad (2.8)$$

By Lemma 2.2, the functions $x^*G(s)g_\delta(s)$ are almost periodic from $\mathbb{R}$ to the scalars. Therefore, from (2.7) and (2.8), it follows that $x^*G(s)g(s)$ is $S^1$-almost periodic from $\mathbb{R}$ to the scalars. 

**Lemma 2.4.** In a Banach space $X$, if $g : \mathbb{R} \to X$ is an $S^1$-almost periodic continuous function and if $G : \mathbb{R} \to L(X;X)$ is a strongly almost periodic operator-valued function, then $G(t)g(t)$, $t \in \mathbb{R} \to X$, is an $S^1$-almost periodic continuous function.
The proof of this lemma is analogous to that of Lemma 2.3.

**Lemma 2.5.** In a reflexive Banach space $X$, let $h: \mathbb{R} \to X$ be an $S^1$-almost periodic continuous function and

$$H(t) = \int_0^t h(s) \, ds \quad \text{on } \mathbb{R}. \tag{2.9}$$

If $H$ is $S^1$-bounded on $\mathbb{R}$, then it is strongly almost periodic from $\mathbb{R}$ to $X$.

**Proof.** See Rao [5, Notes (ii)]. \hfill \square

**Lemma 2.6.** For an operator-valued function $G: \mathbb{R} \to L(X^*; X^*)$, assume that $G^*(t)$ is the adjoint (conjugate) of the operator $G(t)$. If $G^*: \mathbb{R} \to L(X^*; X^*)$ is strongly almost periodic and if $g: \mathbb{R} \to X$ is weakly almost periodic, then $G(t)g(t)$, $t \in \mathbb{R} \to X$, is weakly almost periodic ($X$ a Banach space).

**Proof.** See Rao [6, Remarks (iii)]. \hfill \square

3. Proof of Theorem 1.1. From (2.1), we obtain

$$T(-t)u^{(n-1)}(t) = u^{(n-1)}(0) + \int_0^t T(-s)[B(s)u(s) + f(s)] \, ds \quad \text{on } \mathbb{R}. \tag{3.1}$$

So, for an arbitrary but fixed $x^* \in X^*$, we have

$$x^*T(-t)u^{(n-1)}(t) = x^*u^{(n-1)}(0) + \int_0^t x^*T(-s)[B(s)u(s) + f(s)] \, ds \quad \text{on } \mathbb{R}. \tag{3.2}$$

By Lemma 2.4, $B(s)u(s)$, $s \in \mathbb{R} \to X$ is an $S^1$-almost periodic continuous function. Hence, $[B(s)u(s) + f(s)]$, $s \in \mathbb{R} \to X$, is an $S^1$-almost periodic continuous function.

Obviously, $T(-s)$, $s \in \mathbb{R} \to L(X; X)$, is a weakly almost periodic strongly continuous group. Therefore, by Lemma 2.3, $x^*T(-s)[B(s)u(s) + f(s)]$, $s \in \mathbb{R} \to$ scalars, is an $S^1$-almost periodic continuous function. By (2.4) and our assumption on $u^{(n-1)}$, $x^*T(-t)u^{(n-1)}(t)$ is $S^1$-bounded on $\mathbb{R}$. Consequently, by Lemma 2.5, $x^*T(-t)u^{(n-1)}(t)$ is almost periodic from $\mathbb{R}$ to the scalars. That is, $T(-t)u^{(n-1)}(t)$ is weakly almost periodic from $\mathbb{R}$ to $X$ and so is bounded on $\mathbb{R}$.

From (2.4), again by the uniform-boundedness principle,

$$\sup_{t \in \mathbb{R}} \| T(t) \| < \infty. \tag{3.3}$$

Therefore, $u^{(n-1)}(t) = T(t)[T(-t)u^{(n-1)}(t)]$ is bounded on $\mathbb{R}$.

Consider a sequence $\{ \varphi_k(t) \}_{k=1}^\infty$ of infinitely differentiable nonnegative functions on $\mathbb{R}$ such that

$$\varphi_k(t) = 0 \quad \text{for } |t| \geq \frac{1}{k}, \quad \int_{-1/k}^{1/k} \varphi_k(t) \, dt = 1. \tag{3.4}$$

The convolution of $u$ and $\varphi_k$ is defined by

$$(u * \varphi_k)(t) = \int_{\mathbb{R}} u(t-s) \varphi_k(s) \, ds = \int_{\mathbb{R}} u(s) \varphi_k(t-s) \, ds \quad \text{on } \mathbb{R}. \tag{3.5}$$
Since $u$ is $S^1$-almost periodic from $\mathbb{R}$ to $X$, $u \ast \varphi_\kappa$ is strongly almost periodic from $\mathbb{R}$ to $X$ and hence is bounded on $\mathbb{R}$.

We note that

$$\sup_{t \in \mathbb{R}} \| (u^{(n-1)} \ast \varphi_\kappa)(t) \| \leq \sup_{t \in \mathbb{R}} \| u^{(n-1)}(t) \|, \tag{3.6}$$

and, for $m = 1, 2, \ldots, n-1$ and $\kappa = 1, 2, \ldots,$

$$(u \ast \varphi_\kappa)^{(m)}(t) = (u^{(m)} \ast \varphi_\kappa)(t) \quad \text{on } \mathbb{R}. \tag{3.7}$$

Therefore, $y = u \ast \varphi_\kappa$ is a bounded solution of the differential equation

$$y^{(n-1)}(t) = (u \ast \varphi_\kappa)^{(n-1)}(t) \quad \text{on } \mathbb{R}. \tag{3.8}$$

Hence, by Cooke [3, Lemma 2], $u' \ast \varphi_\kappa, u'' \ast \varphi_\kappa, \ldots, u^{(n-1)} \ast \varphi_\kappa$ are all bounded on $\mathbb{R}$. Consequently, $u' \ast \varphi_\kappa, u'' \ast \varphi_\kappa, \ldots, u^{(n-2)} \ast \varphi_\kappa$ are all uniformly continuous on $\mathbb{R}$.

So, by Amerio and Prouse [1, Theorem 6, page 6], we conclude successively that $u' \ast \varphi_\kappa, u'' \ast \varphi_\kappa, \ldots, u^{(n-2)} \ast \varphi_\kappa$ are all strongly almost periodic from $\mathbb{R}$ to $X$.

Since $u^{(n-1)}$ is bounded on $\mathbb{R}$, $u^{(n-2)}$ is uniformly continuous on $\mathbb{R}$. Hence, $(u^{(n-2)} \ast \varphi_\kappa)(t) \to u^{(n-2)}(t)$ as $\kappa \to \infty$, uniformly on $\mathbb{R}$. Therefore, $u^{(n-2)}$ is strongly almost periodic from $\mathbb{R}$ to $X$ and so is bounded on $\mathbb{R}$. We thus conclude successively that $u^{(n-2)}, \ldots, u', u$ are all strongly almost periodic from $\mathbb{R}$ to $X$, which completes the proof of the theorem.

4. A consequence of Theorem 1.1. We demonstrate the following result.

**Theorem 4.1.** In a Banach space $X$, assume that $A$ is the infinitesimal generator of a strongly continuous group $T : \mathbb{R} \to L(X; X)$, with the group of adjoint operators $T^* : \mathbb{R} \to L(X^*; X^*)$ being strongly almost periodic, and $f, B$, and $u$ are defined as in Theorem 1.1. If $u$ is $S^1$-almost periodic from $\mathbb{R}$ to $X$ and $u^{(n-1)}$ is $S^1$-bounded on $\mathbb{R}$, then $u, u', \ldots, u^{(n-2)}$ are all strongly almost periodic and $u^{(n-1)}$ is weakly almost periodic from $\mathbb{R}$ to $X$.

**Proof.** By our assumption, for an arbitrary but fixed $x^* \in X^*$, $T^*(t)x^*, t \in \mathbb{R} \to X^*$, is strongly almost periodic, and so, $x^*T(t)x$, $t \in \mathbb{R} \to$ scalars, is almost periodic for each $x \in X(x^*T(t) = T^*(t)x^*)$. Consequently, it follows that $T : \mathbb{R} \to L(X; X)$ is a weakly almost periodic group. Hence, by Theorem 1.1, $u, u', \ldots, u^{(n-2)}$ are all strongly almost periodic, and $T(-t)u^{(n-1)}(t)$ is weakly almost periodic from $\mathbb{R}$ to $X$. So, by Lemma 2.6, $u^{(n-1)}(t) = T(t)[T(-t)u^{(n-1)}(t)]$ is weakly almost periodic from $\mathbb{R}$ to $X$, which completes the proof of the theorem.

**Remark 4.2.** Theorem 4.1 remains valid if $f : \mathbb{R} \to X$ is a weakly almost periodic continuous function.

**Proof.** By Lemma 2.6, $T(-s)f(s), s \in \mathbb{R} \to X$, is weakly almost periodic.
5. Note. Now, the proof of the following result is obvious.

**Theorem 5.1.** In a reflexive Banach space \( X \), suppose that \( A \) is the infinitesimal generator of a strongly almost periodic group \( T : \mathbb{R} \to L(X;X) \) and \( f, B, \) and \( u \) are defined as in Theorem 1.1. If \( u \) is \( S^1 \)-almost periodic from \( \mathbb{R} \) to \( X \) and \( u^{(n-1)} \) is \( S^1 \)-bounded on \( \mathbb{R} \), then \( u, u', \ldots, u^{(n-1)} \) are all strongly almost periodic from \( \mathbb{R} \) to \( X \).

**Remark 5.2.** For \( n = 1 \), Theorem 5.1 holds in a Banach space \( X \).

**Proof.** For \( n = 1 \), (3.1) becomes

\[
T(-t)u(t) = u(0) + \int_0^t T(-s)[B(s)u(s) + f(s)]ds \quad \text{on } \mathbb{R}.
\]

(5.1)

Using Lemma 2.4 twice, we can show that \( T(-s)[B(s)u(s) + f(s)] \) is \( S^1 \)-almost periodic from \( \mathbb{R} \) to \( X \). So, by Amerio and Prouse [1, Theorem 8, page 79], \( T(-t)u(t) \) is uniformly continuous on \( \mathbb{R} \). Further, By Lemma 2.4, \( T(-t)u(t) \) is \( S^1 \)-almost periodic from \( \mathbb{R} \) to \( X \). Consequently, by Amerio and Prouse [1, Theorem 7, page 78], \( T(-t)u(t) \) is strongly almost periodic from \( \mathbb{R} \) to \( X \). So, by Lemma 2.2, \( u(t) = T(t)[T(-t)u(t)] \) is strongly almost periodic from \( \mathbb{R} \) to \( X \).

\[
\square
\]

**References**


