AN APPLICATION OF A SUBORDINATION CHAIN

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Let K denote the class of functions $g(z) = z + a_2z^2 + \dots$ which are regular and univalently convex in the unit disc E . In the present note, we prove that if f is regular in E , $f(0) = 0$, then for $g \in K$, $f(z) + \alpha zf'(z) \prec g(z) + \alpha zg'(z)$ in E implies that $f(z) \prec g(z)$ in E , where $\alpha > 0$ is a real number and the symbol " $\prec$ " stands for subordination.

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1. Introduction. Let S denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are regular and univalent in the unit disc $E = \{z : |z| < 1\}$. A function $f \in S$ is said to be convex of order β , $0 \leq \beta < 1$, if and only if

$$\operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > \beta, \quad z \in E. \quad (1.2)$$

For a given β , $0 \leq \beta < 1$, let $K(\beta)$ denote the subclass of S consisting of convex functions of order β and let $K = K(0)$ be the usual class of convex functions.

A function f given by (1.1) is said to be close-to-convex in E if f is regular in E and if there exists a function $g \in K$ such that

$$\operatorname{Re} \left[\frac{f'(z)}{g'(z)} \right] > 0, \quad z \in E. \quad (1.3)$$

It is well known that if a function is close-to-convex in E , then it is univalent in E .

Suppose that f and g are regular in $|z| < \rho$ and $f(0) = g(0)$. In addition, suppose that g is also univalent in $|z| < \rho$. We say that f is subordinate to g in $|z| < \rho$ (in symbols, $f(z)g \prec (z)$ in $|z| < \rho$) if $f(|z| < \rho) \subset g(|z| < \rho)$.

In 1947, Robinson [4] proved that if $g(z) + zg'(z)$ is in S and $f(z) + zf'(z) \prec g(z) + zg'(z)$ in $|z| < 1$, then $f(z) \prec g(z)$ at least in $|z| < r_0 = 1/5$. S. Singh and R. Singh [6], in 1981, increased the constant r_0 to $2 - \sqrt{3} = 0.268\dots$. Subsequently, in 1984, Miller et al. [2] further increased this constant to $4 - \sqrt{13} = 0.3944\dots$

Recently, R. Singh and S. Singh [5] pursued the problem initiated by Robinson when $g \in K(\beta)$. In fact, they considered the cases when $\beta = 0$ and $\beta = 1/2$ and proved the following results.

THEOREM 1.1. *Let f be regular in E with $f(0) = 0$ and let $g \in K$. Suppose that*

$$f(z) + zf'(z) \prec g(z) + zg'(z) \quad (1.4)$$

in E . Then,

$$f(z) \prec g(z) \quad (1.5)$$

at least in $|z| < r_0$, where $r_0 = \sqrt{5}/3 = 0.745\dots$

THEOREM 1.2. *Let f be regular in E , $f(0) = 0$, and let $g \in K(1/2)$. Then*

$$f(z) + zf'(z) \prec g(z) + zg'(z) \quad (1.6)$$

in E implies that

$$f(z) \prec g(z) \quad (1.7)$$

at least in $|z| < r_1$, where $r_1 = ((51 - 24\sqrt{2})/23)^{1/2} = 0.8612\dots$

In the present note, we consider the subordination $f(z) + \alpha zf'(z) \prec g(z) + \alpha zg'(z)$ in $|z| < 1$, $g \in K$ and $\alpha > 0$, and show that the subordination $f(z) \prec g(z)$ holds in the entire disc $|z| < 1$ and does not depend upon the order of convexity of g as claimed by R. Singh and S. Singh in [5].

2. Preliminaries. We will need the following definition and results to prove our theorem.

DEFINITION 2.1. A function $L(z, t)$, $z \in E$ and $t \geq 0$, is said to be a subordination chain if $L(\cdot, t)$ is analytic and univalent in E for all $t \geq 0$, $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all z in E , and $L(z, t_1) \prec L(z, t_2)$ for $0 \leq t_1 \leq t_2$.

LEMMA 2.2 [3, page 159]. *The function $L(z, t) = a_1(t)z + \dots$, with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$, is a subordination chain if and only if*

$$\operatorname{Re} \left[z \frac{\partial L / \partial z}{\partial L / \partial t} \right] > 0, \quad z \in E, \quad t \geq 0. \quad (2.1)$$

LEMMA 2.3. *Let p be analytic in E and q analytic and univalent in $\bar{E}$ except for points where $\lim_{z \rightarrow \zeta} p(z) = \infty$ with $p(0) = q(0)$. If p is not subordinate to q , then there is a point $z_0 \in E$ and $\zeta_0 \in \partial E$ (boundary of E) such that $p(|z| < |z_0|) \subset q(E)$, $p(z_0) = q(\zeta_0)$, and $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$ for $m \geq 1$.*

Lemma 2.3 is due to Miller and Mocanu [1].

3. Main theorem

THEOREM 3.1. *Let f be regular in E with $f(0) = 0$ and let $g \in K$. For any real number α , $\alpha > 0$, suppose that*

$$f(z) + \alpha z f'(z) \prec g(z) + \alpha z g'(z) \quad (3.1)$$

in E . Then,

$$f(z) \prec g(z) \quad (3.2)$$

in E .

PROOF. First, we observe that $g(z) + \alpha z g'(z) = h(z)$, say, is close-to-convex and hence univalent in E whenever $g \in K$. Without any loss of generality, we can assume that g is regular and univalent in the closed disc $\bar{E}$. If possible, suppose that $f(z)$ is not subordinate to $g(z)$ whenever (3.1) holds. Then by Lemma 2.3, there exist points $z_0 \in E$, $\zeta_0 \in \partial E$, and $m \geq 1$ such that $f(|z| < |z_0|) \subset g(E)$, $f(z_0) = g(\zeta_0)$, and $z_0 f'(z_0) = m \zeta_0 g'(\zeta_0)$. This gives

$$f(z_0) + \alpha z_0 f'(z_0) = g(\zeta_0) + m \alpha \zeta_0 g'(\zeta_0). \quad (3.3)$$

Define a function

$$L(z, t) = h(z) + \alpha t z g'(z) = a_1(t)z + \cdots. \quad (3.4)$$

Since $h(z)$ and $z g'(z)$ are analytic in E , $L(z, t)$ is also analytic in E for all $t \geq 0$, and is continuously differentiable on $[0, \infty)$ for all $z \in E$. Now, from (3.4), we get

$$a_1(t) = \frac{\partial L}{\partial z}(0, t) = 1 + \alpha(1 + t) \neq 0 \quad (3.5)$$

for all $t \geq 0$ and $\alpha > 0$. Also

$$\lim_{t \rightarrow \infty} |a_1(t)| = \infty. \quad (3.6)$$

As $g \in K$, a simple calculation yields

$$\operatorname{Re} \left[z \frac{\partial L / \partial z}{\partial L / \partial t} \right] = \operatorname{Re} \left[\frac{1}{\alpha} + (1 + t) \left(1 + \frac{z g''(z)}{g'(z)} \right) \right] > 0 \quad (3.7)$$

for $z \in E$, $t \geq 0$, and $\alpha > 0$. Hence, by Lemma 2.2, $L(z, t)$ is a subordination chain. Therefore, in view of Definition 2.1, we have $L(z, t_1) \prec L(z, t_2)$ for $0 \leq t_1 \leq t_2$. Since, from (3.4), $L(z, 0) = h(z)$, we deduce that

$$L(\zeta_0, t) \notin h(E) \quad (3.8)$$

for $|\zeta_0| = 1$ and $t \geq 0$.

Now, in view of (3.4) and (3.3), we can write

$$f(z_0) + \alpha z_0 f'(z_0) = L(\zeta_0, m-1), \quad (3.9)$$

where $z_0 \in E$, $|\zeta_0| = 1$, and $m \geq 1$. Formula (3.9), when combined with (3.8), contradicts (3.1). Hence, we must have $f(z) < g(z)$ in E . This completes the proof of our theorem. $\square$

Letting α approach infinity, we arrive at the following well-known result of Suffridge [7].

COROLLARY 3.2. *Let f be regular in E with $f(0) = 0$ and let $g \in K$. If $zf'(z) < zg'(z)$ in E , then $f(z) < g(z)$ in E .*

We now present some interesting examples choosing g as some distinguished member of the class K .

Let f be regular in E , $f(0) = 0$, and let $\alpha > 0$. Then

- (a) $f(z) + \alpha z f'(z) < z/(1-z) + \alpha z/(1-z)^2$ in $E \Rightarrow f(z) < z/(1-z)$ in E ;
- (b) $f(z) + \alpha z f'(z) < e^z(1 + \alpha z) - 1$ in $E \Rightarrow f(z) < e^z - 1$ in E ;
- (c) $f(z) + \alpha z f'(z) < -\log(1-z) + \alpha z/(1-z)$ in $E \Rightarrow f(z) < -\log(1-z)$ in E .

Note that the function $-\log(1-z)$ is convex of order $1/2$ in E .

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