Lp-INVERSE THEOREM FOR MODIFIED BETA OPERATORS

VIJAY GUPTA, PRERNA MAHESHWARI, and V. K. JAIN

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We obtain a converse theorem for the linear combinations of modified beta operators whose weight function is the Baskakov operators. To prove our inverse theorem, we use the technique of linear approximating method, namely, Steklov mean.

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1. Introduction. For \( f \in L_p[0, \infty), \) \( p \geq 1, \) modified beta operators with the weight function of Baskakov operators are defined as

\[
B_n(f,x) = \frac{(n-1)}{n} \sum_{\nu=0}^{\infty} b_{n,\nu}(x) \int_0^\infty p_{n,\nu}(t) f(t) dt, \quad x \in [0, \infty),
\]  

(1.1)

where

\[
b_{n,\nu}(x) = \frac{1}{B(v+1,n)} x^\nu (1+x)^{-n-\nu-1},
\]

\[
p_{n,\nu}(x) = \binom{n+v-1}{\nu} x^\nu (1+x)^{-n-\nu},
\]

and \( B(v+1,n) \) being the beta function (see, e.g., [3]).

It is easily verified that the operators \( B_n \) are linear positive operators. Also, \( B_n(1,x) = 1. \) It turns out that the order of approximation for the operators (1.1) is at best \( O(n^{-1}) \) however smooth the function may be. With the aim of improving the order of approximation, we have to slack the positive condition of these operators for which we may take appropriate linear combinations of the operators (1.1). Now we consider the linear combinations \( B_n(f,k,x) \) of the operators \( B_{d_j,n}(f,x) \) as

\[
B_n(f,k,x) = \sum_{j=0}^k C(j,k) B_{d_j,n}(f,x),
\]

(1.3)

where

\[
C(j,k) = \prod_{\substack{i=0 \atop i \neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0, \quad C(0,0) = 1
\]

(1.4)

and \( d_0, d_1, d_2, \ldots, d_k \) are \( (k+1) \) arbitrary but fixed distinct positive integers.
Throughout this note, let $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, $0 < a < b < \infty$, and $I_i = [a_i, b_i]$, $i = 1, 2, 3$.

For $f \in L_p[0, \infty)$, $1 \leq p < \infty$, the Steklov mean $f_{\eta,m}$ of $m$th order corresponding to $f$ is defined as

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[ f(t) + (-1)^{m-1} \Delta_m \sum_{i=1}^{m} f(t_i) dt_1, dt_2, \ldots, dt_m \right],$$

(1.5)

where $t \in I$ and $\Delta_m f(t)$ is the $m$th order forward difference of the function $f$ with step length $h$. It follows from [5, 7] that

(i) $f_{\eta,m}$ has derivatives up to order $m$, $f^{(m-1)}_{\eta,m} \in AC(I_1)$, and $f^{(m-1)}_{\eta,m}(t)$ exists a.e. and belongs to $L_p(I_1)$;

(ii) $\|f^{(r)}_{\eta,m}\|_{L_p(I_1)} \leq K_1 \eta^{-r} \omega_r(f, \eta, p, I_1)$, $r = 1, 2, \ldots, m$;

(iii) $\|f - f_{\eta,m}\|_{L_p(I_1)} \leq K_2 \omega_m(f, \eta, p, I_1)$;

(iv) $\|f_{\eta,m}\|_{L_p(I_1)} \leq K_3 \|f\|_{L_p(I_1)}$;

(v) $\|f^{(m)}_{\eta,m}\|_{L_p(I_1)} \leq K_4 \eta^{-m} \|f\|_{L_p(I_1)}$.

In this note, we obtain an inverse theorem in $L_p$-approximation for the linear combinations of the operators (1.1).

2. Auxiliary results. In this section, we give certain results which are necessary to prove the inverse result.

**Lemma 2.1** [3]. Let the $m$th order moment be defined as

$$T_{n,m}(x) = \frac{n-1}{n} \sum_{v=0}^{\infty} b_{n,v}(x) \int_{0}^{\infty} p_{n,v}(t)(t-x)^m dt,$$

(2.1)

then $T_{n,0}(x) = 1$ and $T_{n,1}(x) = (1 + 3x)/(n - 2)$ and there holds the recurrence relation

$$(n - m - 2)T_{n,m+1}(x) = x(1 + x)[T_{n,m}(x) + 2mT_{n,m-1}(x)] + [(1 + 2x)(m + 1) + x]T_{n,m}(x), \quad n > m + 2.$$

(2.2)

Consequently for each $x \in [0, \infty)$,

$$\mu_{n,m}(x) = O(n^{-(m+1)/2}).$$

(2.3)

**Lemma 2.2.** Let $h \in L_1[0, \infty)$ have a compact support, then, for $n \in \mathbb{N}$,

$$\left\| \frac{n-1}{n} \int_{0}^{\infty} b_{n,v}(x)p_{n,v}(t)(\frac{v}{n} - t)^m h(t) dt \right\|_{L_1[0, \infty)} \leq K_5 n^{-m/2} \|h\|_{L_1[0, \infty)},$$

(2.4)

where the constant $K_5$ is independent of $n$ and $h$. 


**Proof.** Applying Fubini’s theorem and Holder’s inequality, we obtain

\[
\frac{n-1}{n} \int_0^\infty \int_0^\infty \sum_{v=0}^\infty b_{n,v}(x) p_{n,v}(t) \left| \frac{v}{n} - x \right|^m |h(t)| \, dt \, dx
\]

\[
= \int_0^\infty \sum_{v=0}^\infty \left\{ \int_0^\infty \frac{n-1}{n} b_{n,v}(x) \left| \frac{v}{n} - x \right|^m \, dx \right\} p_{n,v}(t) |h(t)| \, dt
\]

\[
\leq \int_0^\infty \sum_{v=0}^\infty \left\{ \int_0^\infty \frac{n-1}{n} b_{n,v}(x) \left( \frac{v}{n} - x \right)^{2m} \, dx \right\}^{1/2} p_{n,v}(t) |h(t)| \, dt
\]

\[
= \int_0^\infty \sum_{v=0}^\infty \frac{2m}{j} \binom{2m}{j} \left( \frac{v}{n} \right)^{2m-j} (-1)^j \int_0^\infty \frac{n-1}{n} b_{n,v}(x) x^j \, dx \right\}^{1/2}
\]

\[
\times p_{n,v}(t) |h(t)| \, dt
\]

\[
= \int_0^\infty \sum_{v=0}^\infty \frac{2m}{j} \binom{2m}{j} \left( \frac{v}{n} \right)^{2m-j} (-1)^j \frac{(v+n)!}{n \cdot v!(n-2)!} B(v+j+1, n-j)
\]

\[
\times p_{n,v}(t) |h(t)| \, dt
\]

\[
= \int_0^\infty \left\{ \sum_{j=0}^{2m} \binom{2m}{j} \left( \frac{v}{n} \right)^{2m-j} (-1)^j \frac{(v/n + j/n)(v/n + (j-1)/n) \cdots (v/n + 1/n)}{(1-2/n)(1-3/n) \cdots (1-j/n)} \right\}
\]

\[
\times p_{n,v}(t) |h(t)| \, dt.
\]  

(2.5)

Now, we use the identities

\[
\prod_{i=1}^j \left( \frac{v}{n} + \frac{i}{n} \right) = \left( \frac{v}{n} \right)^j + \binom{j}{1} \frac{v}{n} P_1(j) + \binom{j}{2} \frac{1}{n^2} P_2(j) + \cdots,
\]

\[
\prod_{i=1}^j \left( 1 - \frac{i}{n} \right) = 1 + \frac{1}{n} Q_1(j) + \frac{1}{n^2} Q_2(j) + \cdots,
\]  

(2.6)

where \( P_k(j) \) and \( Q_k(j) \) are polynomials in \( j \) of degree \( 2k \).

Using the fact \( \sum_{j=0}^{2m} \binom{2m}{j} (-1)^j j^k = 0, k < 2n, \) we have

\[
\left\| \frac{n-1}{n} \int_0^\infty \sum_{v=0}^\infty b_{n,v}(x) p_{n,v}(t) \left( \frac{v}{n} - x \right)^m \, dt \right\|_{L_1[0,\infty)}
\]

\[
\leq K_6 \int_0^\infty \sum_{v=0}^\infty \left\{ \left( \frac{v}{n} \right)^{2m} + \frac{1}{n^m} \left( \frac{v}{n} \right)^{2m-1} + \cdots + \frac{1}{n^{2m}} \right\}^{1/2} p_{n,v}(t) |h(t)| \, dt.
\]  

(2.7)

Now applying Holder’s inequality for summation and by the compactness of \( h \), we obtain the required result. \( \square \)
**Lemma 2.3.** Let $h \in L_p[0, \infty)$, $p > 1$, have a compact support, $i, j \in \mathbb{N} \cup \{0\}$. Then for $m > 0$, there holds

$$\left\| \frac{n-1}{n} \int_0^\infty \sum_{v=0}^\infty b_{n,v}(x) p_{n,v}(t) \left( \frac{v}{n} - x \right)^i \int_x^t (t-w)^j h(w) \, dw \, dt \right\|_{L^p(I_2)} \leq K_7 \left\{ n^{-(i+j+1)/2} \| h \|_{L^p(I_1)} + n^{-m} \| h \|_{L^p[0,\infty)} \right\}. \quad (2.8)$$

**Proof.** Applying Jensen’s inequality repeatedly,

$$\left\| \sum_{v=0}^\infty b_{n,v}(x) \left( \frac{v}{n} - x \right)^i \int_0^\infty \frac{n-1}{n} p_{n,v}(t) \int_x^t (t-w)^j h(w) \, dw \, dt \right\|^p \leq \sum_{v=0}^\infty b_{n,v}(x) \left| \frac{v}{n} - x \right|^i \int_0^\infty \frac{n-1}{n} p_{n,v}(t)|t-x|^s \int_t^u |h(w)|^p \, dw \, dt, \quad s = jp + p - 1 \quad (2.9)$$

$$= \sum_{v=0}^\infty b_{n,v}(x) \left| \frac{v}{n} - x \right|^i \int_0^\infty \frac{n-1}{n} \varphi(t) p_{n,v}(t)|t-x|^s \int_t^u |h(w)|^p \, dw \, dt + \sum_{v=0}^\infty b_{n,v}(x) \left| \frac{v}{n} - x \right|^i \int_0^\infty \frac{n-1}{n} (1-\varphi(t)) p_{n,v}(t)|t-x|^s \int_t^u |h(w)|^p \, dw \, dt.$$

We break the interval $[x,t]$ in the first term as

$$\bigcup_{l=0}^m ([x,x+(l+1)n^{-1/2}] \cup [x-(l+1)n^{-1/2},x]), \quad (2.10)$$

where $mn^{-1/2} \leq \max\{b_1-a_2,b_2-a_1\} < (m+1)n^{-1/2}$, which is similar to [4, Theorem 2] and [2, Theorem 3.2]. A typical element of the first term is now $L^p$-bounded by

$$\frac{n^2}{l^4} \int_{a_2}^{b_2} \sum_{v=0}^\infty b_{n,v}(x) \left| \frac{v}{n} - x \right|^i \int_x^{x+(l+1)n^{-1/2}} \frac{n-1}{n} p_{n,v}(t)|t-x|^{s+4} \, dt \int_{x+(l+1)n^{-1/2}}^{x+(l+1)n^{-1/2}} \varphi(w) |h(w)|^p \, dw \, dx.$$
We apply Holder’s inequality for infinite sum, Lemma 2.1, and Fubini’s theorem to obtain the required estimate. The presence of the factor \((1 - \phi(t))\) in the second term of (2.9) implies that \(|t - x|/\delta > 1\), which gives arbitrary order \(O(n^{-m})\). This completes the proof of the lemma.

**Lemma 2.4.** There exist polynomials \(q_{i,j,r}(x)\) independent of \(n\) and \(v\) such that

\[
\{x(1 + x)^{r} \frac{d^{r}}{dx^{r}} (b_{n,v}(x))
= \sum_{2i+j \leq r \atop i,j \geq 0} (n+1)^{i} [v - (n+1)x]^{j} q_{i,j,r}(x) b_{n,v}(x). \tag{2.12}\]

The proof of the above lemma is similar to [3, Lemma 2.2].

**Lemma 2.5.** Let \(h \in L_{p}[0, \infty), p \geq 1\) and \(\text{supp} \ h \subset I_{2}\), then

\[
\|B_{n}^{(2k+2)}(h, \cdot)\|_{L_{p}(I_{2})} \leq K_{8} n^{k+1} \|h\|_{L_{p}(I_{2})}. \tag{2.13}\]

Moreover, if \(h^{(2k+1)} \in AC(I_{2})\) and \(h^{(2k+2)} \in L_{p}(I_{2})\), then

\[
\|B_{n}^{(2k+2)}(h, \cdot)\|_{L_{p}(I_{2})} \leq K_{9} \|h^{(2k+2)}\|_{L_{p}(I_{2})}, \tag{2.14}\]

the constants \(K_{8}\) and \(K_{9}\) are independent of \(n\) and \(h\).

**Proof.** Since functions \(q_{i,2k+2}(x)\) and \(\{x(1 + x)^{-(2k+2)}\) are bounded on \(I_{2}\), it follows from Lemmas 2.2 and 2.4 that, for \(h \in L_{1}[0, \infty),\)

\[
\|B_{n}^{(2k+2)}(h, \cdot)\|_{L_{p}(I_{2})} \leq K_{10} n^{k+1} \|h\|_{L_{p}(I_{2})}. \tag{2.15}\]

If \(h \in L_{\infty}[0, \infty),\) then by Lemmas 2.1 and 2.4, we get

\[
\|B_{n}^{(2k+2)}(h, \cdot)\|_{L_{\infty}(I_{2})} \leq K_{11} n^{k+1} \|h\|_{L_{\infty}(I_{2})}. \tag{2.16}\]

Now applying Riesz-Thorin interpolation theorem [6], we get (2.13). To prove (2.14), we have

\[
h(t) = \sum_{r=0}^{k+1} \frac{(t-x)^{r}}{r!} h^{(r)}(x) + \frac{1}{(2k+1)!} \int_{x}^{t} (t-w)^{2k+1} h^{(2k+2)}(w) dw. \tag{2.17}\]
Using Lemmas 2.1 and 2.3, it is easily verified that
\[
B_n^{(2k+2)}(h,x) = \frac{(n-1)}{n(2k+1)} \{x(1+x)\}^{2k+2} \sum_{v=0}^{\infty} b_{n,v}(x)
\cdot \left\{ \sum_{2i+j \leq 2k+2} (n+1)^{j} \delta_{i,j,2k+2}(x) \right\}
\times \int_{0}^{\infty} p_{n,v}(t) \int_{x}^{t} (t-w)^{2k+1} h^{(2k+2)}(w) dw \ dt \right\}.
\] (2.18)

Now applying Lemma 2.3, we obtain the estimate (2.14).

3. Inverse theorem. In this section, we prove the following inverse theorem.

**Theorem 3.1.** Let \(0 < \alpha < 2k+2\), \(f \in L_p[0, \infty)\), \(p \geq 1\), and
\[
\|B_n(f,k,x) - f(x)\|_{L_p(I_1)} = O(n^{-\alpha/2}), \quad n \to \infty,
\] (3.1)
then
\[\omega_{2k+2}(f,\tau,p,I_2) = O(\tau^\alpha), \quad \tau \to 0.\] (3.2)

**Proof.** We choose a function \(g \in C_0^{2k+2}\) such that \(supp\ h \subset (x_2,y_2)\), \(h(t) = 1\) on \([x_3,y_3]\), and \(a_1 < x_1 < x_2 < x_3 < a_2 < b_2 < y_3 < y_2 < y_1 < b_1\). Writing \(f h = \tilde{f}\) for all values of \(\gamma \leq \tau\), we have
\[
\Delta_y^{2k+2} \tilde{f}(x) = \Delta_y^{2k+2} (\tilde{f}(t) - B_n(\tilde{f},k,x)) + \Delta_y^{2k+2} B_n(\tilde{f},k,x),
\] (3.3)
where \(\Delta_y^{2k+2}\) denotes the \((2k+2)\)th order forward difference. Applying Jensen’s inequality repeatedly and Fubini’s theorem for the second term, we have
\[
\|\Delta_y^{2k+2} \tilde{f}\|_{L_p[x_2,y_2]} \leq \|\Delta_y^{2k+2} (\tilde{f} - B_n(\tilde{f},k,\cdot))\|_{L_p[x_2,y_2]}
+ \nu^{2k+2} \|B_n^{(2k+2)}(\tilde{f},k,\cdot)\|_{L_p[x_2,y_2+(2k+2)\nu]} \leq \|\Delta_y^{2k+2} (\tilde{f} - \tilde{f}_{\eta,2k+2},k,\cdot)\|_{L_p[x_2,y_2+(2k+2)\nu]}
+ \nu^{2k+2} \| B_n^{(2k+2)}(\tilde{f},k,\cdot)\|_{L_p[x_2,y_2+(2k+2)\nu]}.
\] (3.4)

Applying Lemma 2.5 and using the properties of Steklov mean, we get
\[
\|\Delta_y^{2k+2} \tilde{f}\|_{L_p[x_2,y_2]} \leq \|\Delta_y^{2k+2} (\tilde{f} - B_n(\tilde{f},k,\cdot))\|_{L_p[x_2,y_2]}
+ K_{12} \nu^{2k+2} (n^{k+1} + \eta^{-(2k+2)}) \omega_{2k+2}(\tilde{f},\eta,p,[x_2,y_2]).
\] (3.5)
Following [1], we can complete the proof of the theorem if we show that
\[
\| \Delta_x^{2k+2} (\tilde{f} - B_n(f, k, \cdot)) \|_{L_p[x_2, y_2]} = O(n^{-\alpha/2}), \quad n \to \infty,
\] (3.6)

therefore,
\[
\omega_{2k+2}(\tilde{f}, \tau, p, [x_2, y_2]) = O(\tau^\alpha), \quad \tau \to 0. \tag{3.7}
\]

For \( t \in [x_3, y_3] \), \( \tilde{f}(t) = f(t) \), thus
\[
\omega_{2k+2}(f, \tau, p, [c, d]) = O(\tau^{m-1+\beta}), \quad \tau \to 0 \quad \text{as required.} \tag{3.8}
\]

We will prove (3.6) by the principle of mathematical induction on \( \alpha \). First, we consider the case \( \alpha \leq 1 \), thus
\[
\| B_n(fh, k, \cdot) - fh \|_{L_p[x_2, y_2]} \leq \| B_n(h(x)(f(t) - f(x)), k, \cdot) \|_{L_p[x_2, y_2]}
+ \| B_n(f(t)(h(t) - h(x)), k, \cdot) \|_{L_p[x_2, y_2]} . \tag{3.9}
\]

Now \( |h(t) - h(x)| = |t - x||h'(\xi)| \) for some \( \xi \) lying between \( t \) and \( x \). Using Lemma 2.1 and the compactness of \( f \) to estimate the second term, and the assumption of the theorem for the first term, we get
\[
\| B_n(fh, k, \cdot) - fh \|_{L_p[x_2, y_2]} = O(n^{-1/2}) + O(n^{-\alpha/2}). \tag{3.10}
\]

This completes the proof of (3.6) for the case \( \alpha \leq 1 \).

Now, assume that (3.6) holds for all values of \( \alpha \) satisfying \( m - 1 < \alpha < m \) and prove that the same holds true for \( m < \alpha < m + 1 \). Thus, we have
\[
\omega_{2k+2}(f, \tau, p, [c, d]) = O(\tau^{m-1+\beta}), \quad \tau \to 0, \quad 0 < \beta < 1 \tag{3.11}
\]

for any \( [c, d] \subset (a_1, b_1) \). Let \( \phi(t) \) denote the characteristic function of \( [x_1, y_1] \). The assumed smoothness of \( f \) implies that
\[
\| B_n(fh, k, \cdot) - fh \|_{L_p[x_2, y_2]} \\
\leq \sum_{i=0}^{r-2} \frac{1}{i!} \| f^{(i)}(x)B_n((t-x)^i(h(t) - h(x)), k, \cdot) \|_{L_p[x_2, y_2]} \\
+ \frac{1}{(r-2)!} \| B_n(\phi(t)(h(t) - h(x)) \\
\times \left( \int_x^t (t-w)^{r-2}(f^{(r-1)}(w) - f^{(r-1)}(x)) dw \right), k, \cdot) \|_{L_p[x_2, y_2]} \\
+ \| B_n(F(t,x)(1-\phi(t))(h(t) - h(x)), k, \cdot) \|_{L_p[x_2, y_2]} \\
= J_1 + J_2 + J_3, \tag{3.12}
\]

where \( F(t,x) = f(x) - \sum_{i=0}^{r-2} ((t-x)^i/i!)f^{(i)}(x), \quad t \in [0, \infty), \quad \text{and} \quad x \in [x_2, y_2] \).
The direct theorem in [4] and Lemma 2.1 imply that \( J_1, J_3 = O(n^{-(k+1)}) \), \( n \to \infty \), using Jensen's inequality, mean value theorem on \( h \), and breaking \( [x, t] \) as in Lemma 2.3, we have

\[
\int_{y_2}^{y_1} \left| B_n \left( \phi(t) \left( h(t) - h(x) \left( \int_x^t (t-w)^{r-2} (f^{(r-1)}(w) - f^{(r-1)}(x)) \, dw \right), x \right) \right) \right|^p \, dx
\]

\[
\leq K_1 \int_{y_2}^{y_1} \int_{x_1}^{y_1} W_n(x, t) |t-x|^{r-1} \phi(t) \left| f^{(r-1)}(w) - f^{(r-1)}(x) \right|^p \, dw \, dt \, dx
\]

\[
\leq K_1 \sum_{l=1}^{p-1} \int_{x_2}^{x_2+\frac{n-1}{2}} W_n(x, t) \left( n^2 l^{-4} \right)^p |t-x|^{r+4p-1} \phi(t) \left| f^{(r-1)}(w) - f^{(r-1)}(x) \right|^p \, dw \, dt \, dx
\]

\[
- f^{(r-1)}(x) \right|^p \, dw \, dt \left\{ \int_{x_2}^{x_2+n^{-1/2}} W_n(x, t) |t-x|^{r-1} - f^{(r-1)}(x) \right|^p \, dw \right\} \, dx
\]

\[
\leq K_1 \sum_{l=1}^{p-1} \left( n^2 l^{-4} \right)^p n^{-((r+4)p-1)/2} \phi(t) \left| f^{(r-1)}(w) - f^{(r-1)}(t) \right|^p \, dw \, dt \, dx
\]

\[
\int_{0}^{(l+1)n^{-1/2}} \omega(f^{(r-1)}(w, p, [x_1, y_1]))^p \, dw + n^{-(r-1)/2} \int_{0}^{n^{-1/2}} \omega(f^{(r-1)}(w, p, [x_1, y_1]))^p \, dw \right\}. \tag{3.13}
\]

On using Lemma 2.1, then interchanging integration in \( x \) and \( w \), and lastly, using the fact \( \omega(f^{(r-1)}, w, p, [x_1, y_1]) = O(\omega^\beta) \), we find

\[
J_2 = O\left( n^{-\frac{(r+\beta)}{2}} \right), \quad n \to \infty. \tag{3.14}
\]
Combining the estimates of $J_1, J_2$, and $J_3$, we obtain (3.6). The proof of (3.6) shows that

$$\omega(f, \tau, p, I_2) = O(\tau^\alpha), \quad \alpha < 2k + 2, \quad \alpha \neq 2, 3, \ldots, 2k + 1. \quad (3.15)$$

The above statement implies that it is true for integer values $2, 3, \ldots, 2k + 1$ also. To prove this, let $\alpha = \tau$, where $r$ takes values from $2, 3, \ldots, 2k + 1$. Since (3.15) is true for $(r, r + 1)$, it follows that

$$\omega_{2k+2}(f, \tau, p, I_2) = O(\tau^{r+\theta}) = O(\tau^r), \quad 0 < \theta < 1. \quad (3.16)$$

This completes the proof of the theorem. □

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**REFERENCES**


Vijay Gupta: School of Applied Sciences, Netaji Subhas Institute of Technology, Sector 3 Dwarka, Azad Hind Fauj Marg, New Delhi 110045, India

E-mail address: vijay@nsit.ac.in

Prerna Maheshwari: Department of Mathematics, Khandelwal College of Management Sciences and Technology, Bareilly 243122, India

E-mail address: prerna_m2002@rediffmail.com

V. K. Jain: Department of Mathematics, Bareilly College, Bareilly 243001, India
Submit your manuscripts at http://www.hindawi.com