ON HOPF DEMEYER-KANZAKI GALOIS EXTENSIONS

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Received 15 October 2002

Let $H$ be a finite-dimensional Hopf algebra over a field $k$, $B$ a left $H$-module algebra, and $H^*$ the dual Hopf algebra of $H$. For an $H^*$-Azumaya Galois extension $B$ with center $C$, it is shown that $B$ is an $H^*$-DeMeyer-Kanzaki Galois extension if and only if $C$ is a maximal commutative separable subalgebra of the smash product $B\#H$. Moreover, the characterization of a commutative Galois algebra as given by S. Ikehata (1981) is generalized.

2000 Mathematics Subject Classification: 16W30, 16H05.

1. Introduction. Let $H$ be a finite-dimensional Hopf algebra over a field $k$, $B$ a left $H$-module algebra, and $H^*$ the dual Hopf algebra of $H$. In [7], the class of Azumaya Galois extensions of a ring as studied in [1, 2] was generalized to $H^*$-Azumaya Galois extensions. An $H^*$-Azumaya Galois extension $B$ was characterized in terms of the smash product $B\#H$ see [7, Theorem 3.4]. Observing that the commutator $V_B(B^H)$ of $B^H$ in $B$ is also an $H^*$-Azumaya Galois extension (see [7, Lemma 4.1]), in the present paper, we will give a characterization of an $H^*$-Azumaya Galois extension $B$ in terms of $V_B(B^H)$. Moreover, we will investigate the class of $H^*$-Azumaya Galois extensions $B$ such that $V_B(B^H) = C$, where $C$ is the center of $B$. We note that when $H = kG$, where $G$ is a finite automorphism group of $B$, such a $B$ is precisely a DeMeyer-Kanzaki Galois extension with Galois group $G$ [3, 6, 8, 9]. Several equivalent conditions are then given for an $H^*$-Azumaya Galois extension being an $H^*$-DeMeyer-Kanzaki Galois extension, and the characterization of a commutative Galois algebra as given by Ikehata [5, Theorem 2] is generalized to an $H^*$-DeMeyer-Kanzaki Galois extension.

2. Basic definitions and notation. Throughout, $H$ denotes a finite-dimensional Hopf algebra over a field $k$ with comultiplication $\Delta$ and counit $\varepsilon$, $H^*$ the dual Hopf algebra of $H$, $B$ a left $H$-module algebra, $C$ the center of $B$, $B^H = \{b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H\}$, and $B\#H$ the smash product of $B$ with $H$, where $B\#H = B \otimes_k H$ such that, for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1b')\#h_2h'$, where $\Delta(h) = \sum h_1 \otimes h_2$.

For a subring $A$ of $B$ with the same identity 1, we denote the commutator subring of $A$ in $B$ by $V_B(A)$. We call $B$ a separable extension of $A$ if there
exist \{a_i, b_i \mid i = 1, 2, \ldots, m\} such that \( \sum a_i b_i = 1 \) and 
\( \sum b_i a_i \otimes b_i = \sum a_i \otimes b_i b \) for all \( b \) in \( B \) where \( \otimes \) is over \( A \). An Azumaya algebra is a separable extension of its center. A ring \( B \) is called a Hirata separable extension of \( A \) if \( B \otimes_A B \) is isomorphic to a direct summand of a finite direct sum of \( B \) as a \( B \)-bimodule. A ring \( B \) is called an \( H^* \)-Galois extension of \( B^H \) if \( B \) is a right \( H^* \)-comodule algebra with structure map \( \rho : B \to B \otimes_k H^* \) such that 
\[ \beta : B \otimes_B H \to B \otimes_k H^* \] is a bijection where \( \beta(a \otimes b) = (a \otimes 1) \rho(b) \). An \( H^* \)-Galois extension \( B \) is called an \( H^* \)-Azumaya Galois extension if \( B \) is separable over \( B^G \) which is an Azumaya algebra over \( C^G \), and an \( H^* \)-DeMeyer-Kanzaki Galois extension if \( B \) is an \( H^* \)-Azumaya Galois extension and \( V_B(B^H) = C \).

Let \( P \) be a finitely generated and projective module over a commutative ring \( R \). Then for a prime ideal \( p \) of \( R \), \( R_p \) (= \( R \otimes_k R_p \)) is a free module over \( R_p \) (= the local ring of \( R \) at \( p \)), and the rank of \( R_p \) over \( R_p \) is the number of copies of \( R_p \) in \( R_p \), that is, \( \text{rank}_{R_p}(R_p) = m \) for some integer \( m \). It is known that the rank\( R_p(P) \) is a continuous function (\( \text{rank}_{R_p}(P) = \text{rank}_{R_p}(P_p) = m \)) from \( \text{Spec}(R) \) to the set of nonnegative integers with the discrete topology (see [4, Corollary 4.11, page 31]). We will use the rank\( R(p) \)-function for a finitely generated and projective module \( P \) over a commutative ring \( R \).

3. \( H^* \)-Azumaya Galois extensions. In this section, keeping all notations as given in Section 2, we will characterize an \( H^* \)-Azumaya Galois extension \( B \) in terms of the commutator \( V_B(B^H) \) of \( B^H \) in \( B \).

**Theorem 3.1.** If \( B = B^H \cdot V_B(B^H) \), then \((V_B(B^H))^H = C^H\).

**Proof.** Since \( C \subset V_B(B^H) \), \( C^H \subset (V_B(B^H))^H \). Conversely, since \( V_B(B^H) \subset B \), \((V_B(B^H))^H \subset B^H \). Hence \((V_B(B^H))^H \subset B^H \cap V_B(B^H) \subset \text{center of } V_B(B^H) \). But \( B = B^H \cdot V_B(B^H) \), so the center of \( V_B(B^H) \) is \( C \). Thus, \((V_B(B^H))^H \subset C^H \). \( \square \)

**Theorem 3.2.** A ring \( B \) is an \( H^* \)-Azumaya Galois extension of \( B^H \) if and only if \( B = B^H \cdot V_B(B^H) \) such that \( V_B(B^H) \) is an \( H^* \)-Azumaya Galois extension of \( C^H \) and \( B^H \) is an Azumaya \( C^H \)-algebra.

**Proof.** (\( \Rightarrow \)) Since \( B \) is an \( H^* \)-Azumaya Galois extension of \( B^H \), then \( V_B(B^H) \) is an \( H^* \)-Azumaya Galois extension of \( (V_B(B^H))^H \) (see [7, Lemma 4.1]) and \( B^H \) is an Azumaya \( C^H \)-algebra (see [7, Theorem 3.4]). Moreover, by the proof of [7, Lemma 4.1], \( B \# H \) is an Azumaya \( C^H \)-algebra such that \( B \# H \cong B^H \otimes_{C^H} (V_B(B^H) \# H) \cong B^H(V_B(B^H) \# H) \), where \( B^H \) and \( V_B(B^H) \# H \) are Azumaya \( C^H \)-algebras. But \( H \) is a finite-dimensional Hopf algebra over a field \( k \), so \( B \cong B^H \otimes_{C^H} V_B(B^H) \) from the isomorphism \( B \# H \cong B^H \otimes_{C^H} (V_B(B^H) \# H) \), and so \( B = B^H \cdot V_B(B^H) \). Hence \((V_B(B^H))^H = C^H \) by Theorem 3.1. Thus \( V_B(B^H) \) is an \( H^* \)-Azumaya Galois \( C^H \)-algebra.

(\( \Leftarrow \)) Since \( V_B(B^H) \) is an \( H^* \)-Azumaya Galois algebra over \( C^H \), \( V_B(B^H) \# H \) is an Azumaya \( C^H \)-algebra [7, Theorem 3.4]. By hypothesis, \( B^H \) is an Azumaya \( C^H \)-algebra, so \( B^H \otimes_{C^H} (V_B(B^H) \# H) \cong B^H V_B(B^H) \# H = B \# H \) which is an Azumaya...
Thus \( B \# H \) is a Hirata separable extension of \( B \) (see [5, Theorem 1]). Moreover, \( V_B(B^H) \) is a separable \( C^H \)-algebra (see [7, Theorem 3.4]) and \( B^H \) is an Azumaya \( C^H \)-algebra by hypothesis, so \( B^H : V_B(B^H) (\equiv B) \) is also a separable \( C^H \)-algebra. Thus \( B \) is an \( H^* \)-Azumaya Galois extension of \( B^H \) [7, Theorem 3.4].

Next we generalize the characterization of a commutative Galois algebra as given by Ikehata (see [5, Theorem 2]) to a commutative \( H^* \)-Galois algebra.

**Lemma 3.3.** If \( C \) is a commutative \( H^* \)-Galois algebra over \( C^H \), then \( C \) is a maximal commutative subalgebra of \( C \# H \).

**Proof.** Since \( C \) is a commutative \( H^* \)-Galois algebra over \( C^H \), \( C \# H \equiv \text{Hom}_{C^H}(C, C) \) [6, Theorem 1.7]. Hence it suffices to show that \( V_{\text{Hom}_{C^H}(C, C)}(C_L) = C_L \) where \( C_L = \{c_L, \text{ the left multiplication map induced by } c \in C \}. \) In fact, \( C_L \subset V_{\text{Hom}_{C^H}(C, C)}(C_L) \) is clear. Conversely, let \( f \in V_{\text{Hom}_{C^H}(C, C)}(C_L) \). Then, for each \( c \in C \), \((c f)(x) = (f c)(x) \) for all \( x \in C \). Hence \( c f(x) = f(c x) \), and so \( c f(1) = f(c) \) for all \( c \in C \). Thus \( f(c) = d f(c) \) for all \( c \in C \), where \( d_f = f(1) \in C \), that is, \( f = (d_f)_L \in C_L \).

**Theorem 3.4.** Let \( C \) be a commutative separable \( C^H \)-algebra containing \( C^H \) as a direct summand as a \( C^H \)-module. Then, \( C \) is a commutative \( H^* \)-Galois algebra over \( C^H \) if and only if \( C \otimes_{C^H} (C \# H) \equiv M_n(C) \), the matrix algebra over \( C \) of order \( n \) where \( n \) is the dimension of \( H \) over \( k \).

**Proof.** \((\Rightarrow)\) Since \( C \) is an \( H^* \)-Galois algebra over \( C^H \), \( C \# H \equiv \text{Hom}_{C^H}(C, C) \) such that \( C \) is finitely generated and projective over \( C^H \) [6, Theorem 1.7]. Hence \( C \# H \) is an Azumaya \( C^H \)-algebra and \( C \) is a maximal commutative subalgebra of the Azumaya \( C^H \)-algebra \( C \# H \) by **Lemma 3.3**. By hypothesis, \( C \) is also a separable \( C^H \)-algebra, so \( C \) is a splitting ring for the Azumaya \( C^H \)-algebra \( C \# H \) such that \( C \otimes_{C^H} (C \# H) \equiv \text{Hom}_{C}(C \# H, C \# H) \) (see the proof of [4, Theorem 5.5, page 64]). Noting that \( C \# H = C \otimes_k H \) which is a free \( C \)-module of rank \( n \) where \( n = \dim_k(H) \), we have that \( C \otimes_{C^H} (C \# H) \equiv M_n(C) \).

\((\Leftarrow)\) Since \( C \otimes_{C^H} (C \# H) \equiv M_n(C) \), \( C \otimes_{C^H} (C \# H) \) is an Azumaya \( C \)-algebra. By hypothesis, \( C^H \) is a direct summand of \( C \) as a \( C^H \)-module, so \( C \# H \) is an Azumaya \( C^H \)-algebra [4, Corollary 1.10, page 45]. Hence \( C \# H \) is a Hirata separable extension of \( C \). But \( C \) is a separable \( C^H \)-algebra by hypothesis, so \( C \) is an \( H^* \)-Galois algebra over \( C^H \) [7, Theorem 3.4].

We remark that the necessity does not need the hypothesis that \( C^H \) is a direct summand of \( C \).

**4. \( H^* \)-DeMeyer-Kanzaki Galois extensions.** We recall that \( B \) is an \( H^* \)-DeMeyer-Kanzaki Galois extension of \( B^H \) if \( B \) is an \( H^* \)-Azumaya Galois extension of \( B^H \) and \( V_B(B^H) = C \). In this section, we characterize an \( H^* \)-DeMeyer-Kanzaki Galois extension in terms of the smash product \( V_B(B^H) \# H \) and prove that \( C \) is a splitting ring for the Azumaya \( C^H \)-algebras \( V_B(B^H) \# H \) and \( B \# H \).
**Theorem 4.1.** Let $B$ be an $H^*$-Azumaya Galois extension of $B^H$. Then the following statements are equivalent:

1. $B$ is an $H^*$-DeMeyer-Kanzaki Galois extension of $B^H$;
2. $\text{rank}_{CH}(V_B(B^H)) = \text{rank}_{CH}(C)$;
3. $C$ is a maximal commutative separable subalgebra of $V_B(B^H)\#H$.

**Proof.**

$(1)\Rightarrow(2)$. It is clear.

$(2)\Rightarrow(1)$. Since $B$ is an $H^*$-Azumaya Galois extension of $B^H$, $V_B(B^H)$ is an $H^*$-Azumaya Galois algebra over $C^H$ by Theorem 3.2 such that $V_B(B^H)$ is a separable and finitely generated projective module over $C^H$ (see [7, Theorem 3.4]). Hence the rank function $\text{rank}_{CH}(V_B(B^H))$ is defined and $V_B(B^H)$ is an Azumaya algebra over its center [4, Theorem 3.8, page 55]. But $B = B^H \cdot V_B(B^H)$ by Theorem 3.2, so the center of $V_B(B^H)$ is $C$. Thus $V_B(B^H)$ is an Azumaya $C$-algebra; and so $C$ is a direct summand $V_B(B^H)$ as a $C$-module. This implies that $C$ is a direct summand $V_B(B^H)$ as a $C^H$-module. Therefore the rank function $\text{rank}_{C^H}(C)$ is also defined. Now by hypothesis, $\text{rank}_{CH}(V_B(B^H)) = \text{rank}_{CH}(C)$, so $V_B(B^H) = C$, that is, $B$ is an $H^*$-DeMeyer-Kanzaki Galois extension of $B^H$.

$(1)\Rightarrow(3)$. Since $B$ is an $H^*$-DeMeyer-Kanzaki Galois extension of $B^H$, $B$ is an $H^*$-Azumaya Galois extension such that $V_B(B^H) = C$. Hence $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{C^H} C$ such that $C$ is an $H^*$-Galois algebra over $C^H$ by Theorem 3.2, and so $C$ is a separable $C^H$-algebra containing $C^H$ as a direct summand as a $C^H$-module [7, Theorem 3.4]. Hence $C$ is a maximal commutative separable subalgebra of $C^H$ where $C = V_B(B^H)$ by Lemma 3.3.

$(3)\Rightarrow(2)$. Since $B$ is an $H^*$-Azumaya Galois extension of $B^H$, $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{C^H} V_B(B^H)$ such that $V_B(B^H)$ is an $H^*$-Azumaya Galois algebra over $C^H$ by Theorem 3.2. Hence $V_B(B^H)\#H$ is an Azumaya $C^H$-algebra and $V_B(B^H)$ is an Azumaya $C$-algebra [7, Theorem 3.4]. By hypothesis, $C$ is a maximal commutative separable subalgebra of $V_B(B^H)\#H$, so

$$C \otimes_{C^H} (V_B(B^H)\#H) \cong \text{Hom}_C(V_B(B^H)\#H, V_B(B^H)\#H)$$

(4.1)

(see [4, Theorem 5.5, page 64]). On the other hand, $V_B(B^H)\#H \cong \text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$ (see [7, Theorem 3.4]). Thus

$$C \otimes_{C^H} (V_B(B^H)\#H) \cong C \otimes_{C^H} \text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$$

$$\cong \text{Hom}_C(C \otimes_{C^H} V_B(B^H), C \otimes_{C^H} V_B(B^H));$$

(4.2)

and so $\text{Hom}_C(V_B(B^H)\#H, V_B(B^H)\#H) \cong \text{Hom}_C(C \otimes_{C^H} V_B(B^H), C \otimes_{C^H} V_B(B^H))$. This implies that $V_B(B^H)\#H \cong P \otimes_C (C \otimes_{C^H} V_B(B^H))$ for some finitely generated projective $C$-module $P$ of rank 1, that is, $V_B(B^H)\#H \cong P \otimes_{C^H} V_B(B^H)$. Taking rank$_{C^H}$ ( ) both sides, we have that $n \cdot \text{rank}_{CH}(V_B(B^H)) = (\text{rank}_{CH}(P)) \cdot (\text{rank}_{CH}(V_B(B^H)))$ where $n = \dim_k(H)$. But rank$_{CH}(V_B(B^H))$ is also $n$, so rank$_{CH}(C) = \text{rank}_{CH}(P) = n = \text{rank}_{CH}(V_B(B^H))$.

\[\square\]
Theorem 4.1 implies that the Azumaya $CH$-algebras $V_B(B^H)$ and $B^H$ have a nice splitting ring $C$ which is an $H^*$-Galois algebra over $CH$ and separable over $CH$ such that $C \otimes_{CH} (V_B(B^H))$ and $C \otimes_{CH} (B^H)$ are matrix algebras.

**Corollary 4.2.** If $B$ is an $H^*$-Demeyer-Kanzaki Galois extension of $B^H$, then $C \otimes_{CH} (V_B(B^H)) \cong M_n(C)$, the matrix algebra over $C$ of order $n$ where $n = \dim_k(H)$.

**Proof.** By hypothesis, $B$ is an $H^*$-Demeyer-Kanzaki Galois extension of $B^H$, so $C (= V_B(B^H))$ is an $H^*$-Galois algebra over $CH$ by Theorem 3.2. Hence $C$ is a separable $CH$-algebra and $C^H$ is an Azumaya $CH^*$-algebra [7, Theorem 3.4]. Thus $CH$ is a direct summand of $C$ as a $CH$-module. Therefore, $C \otimes_{CH} (C^H) \cong M_n(C)$ by Theorem 3.4.

**Corollary 4.3.** If $B$ is an $H^*$-Demeyer-Kanzaki Galois extension of $B^H$, then $C \otimes_{CH} (B^H) \cong M_n(B)$, the matrix algebra over $B$ of order $n$ where $n = \dim_k(H)$.

**Proof.** By Corollary 4.2, $C \otimes_{CH} (C^H) \cong M_n(C)$, so

$$B^H \otimes_{CH} C \otimes_{CH} (C^H) \cong B^H \otimes_{CH} M_n(C). \quad (4.3)$$

Since $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{CH} V_B(B^H) = B^H \otimes_{CH} C$, we have that $C \otimes_{CH} (B^H) \cong C \otimes_{CH} ((B^H \otimes_{CH} C)^H)$

$$\cong C \otimes_{CH} B^H \otimes_{CH} (C^H) \cong B^H \otimes_{CH} C \otimes_{CH} (C^H) \cong B^H \otimes_{CH} M_n(C) \cong M_n(B^H \otimes_{CH} C)$$

$$\cong M_n(B). \quad (4.4)$$

**Acknowledgments.** This paper was written under the support of a Caterpillar Fellowship at Bradley University. The authors would like to thank Caterpillar Inc. for the support.

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