WEAKLY PERIODIC AND SUBWEAKLY PERIODIC RINGS

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Our objective is to study the structure of subweakly periodic rings with a particular emphasis on conditions which imply that such rings are commutative or have a nil commutator ideal. Related results are also established for weakly periodic (as well as periodic) rings.

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Throughout, R represents a ring and $\mathcal{C}(R)$ denotes the commutator ideal of R . For any x, y in R , $[x, y] = xy - yx$ is the usual commutator. A *word* $w(x, y)$ is a product in which each factor is x or y . The *empty word* is defined to be 1. We now state formally the definition of a *subweakly periodic* ring.

DEFINITION 1. A ring R is called *subweakly periodic* if every x in $R \setminus J$ can be written in the form $x = a + b$, where a is nilpotent and b is potent.

In the preparation for the proofs of the main theorems, we first prove the following lemmas.

LEMMA 2. Suppose that R is any ring with the property that for all x, y in R , there exist words $w(x, y)$, $w'(x, y)$ depending on x and y such that

$$w(x, y)[xy, yx] = 0 = [xy, yx]w'(x, y) \quad (x, y \in R). \quad (1)$$

Then the commutator ideal $\mathcal{C}(R)$ is contained in J . In particular, if R is also semisimple, then R is commutative.

PROOF. The semisimple ring R/J is isomorphic to a subdirect sum of primitive rings R_i , $i \in \Gamma$, each of which clearly satisfies (1).

CASE 1. Suppose that R_i is a division ring. We claim that R_i must be commutative. Suppose not. Let $x, y \in R_i$ be such that $[x, y] \neq 0$. As $x \neq 0$ and $y \neq 0$, we must have $w(x, y) \neq 0$, and so by (1), $[xy, yx] = 0$. Since $[x, y] \neq 0$ implies that $[x, y + 1] \neq 0$, we may repeat the above argument, with $y + 1$ playing the role of y , to conclude that $[x(y + 1), (y + 1)x] = 0$, and hence $[xy + x, yx + x] = 0$. Combining this with $[xy, yx] = 0$, we obtain $[xy, x] + [x, yx] = 0$, and thus $[xy - yx, x] = 0$, that is, $[x, y]$ commutes with x . Interchanging x and y in the above argument, we see that $[y, x]$ commutes with y , and hence

$$[x, y] \text{ commutes with both } x \text{ and } y. \quad (2)$$

Observe that $[xy, x] = x[y, x]$, and hence

$$x = [xy, x][y, x]^{-1}. \quad (3)$$

By (2), $[xy, x]$ commutes with both x and xy , and hence $[xy, x]$ commutes with $x^{-1}(xy) = y$. Again, by (2), $[y, x]$ commutes with y , and hence $[y, x]^{-1}$ also commutes with y . The net result is that both factors of the right-hand side of (3) commute with y , and hence $[x, y] = 0$, contradiction. This contradiction proves that the division ring R_i is commutative.

CASE 2. The primitive ring R_i is not a division ring. Since (1) is inherited by all subrings and all homomorphic images of the ground ring R , it follows, by Jacobson's density theorem [4, page 33], that for some $n > 1$ and some division ring D , the complete matrix ring D_n of all $n \times n$ matrices over D satisfies (1). This, however, is false as can be seen by taking $x = E_{11}$, $y = E_{11} + E_{12}$ ($x, y \in D_n$). Indeed, in this case, any word $w(x, y)$ must be x or y (since $x^2 = x$, $y^2 = y$, $xy = y$, and $yx = x$), which implies that

$$w(x, y)[xy, yx] = w(x, y)[y, x] = x[y, x] = x - y \neq 0 \quad (4)$$

or

$$w(x, y)[xy, yx] = w(x, y)[y, x] = y[y, x] = x - y \neq 0. \quad (5)$$

This contradiction shows that each R_i must be a division ring, and hence must be commutative (by Case 1). Therefore, R/J is commutative, and hence $\mathcal{C}(R) \subseteq J$. This proves Lemma 2. $\square$

LEMMA 3. *Suppose that R is a ring which satisfies the "word" hypothesis (1) of Lemma 2. Suppose, further, that J is commutative. Then the commutator ideal of R is nil, and hence N is an ideal of R .*

PROOF. By Lemma 2, every commutator $[x, y]$ is in J . Since, by hypothesis, J is commutative,

$$[[x, y], [z, w]] = 0 \quad \forall x, y, z, w \in R. \quad (6)$$

Observe that (6) represents a polynomial identity which is satisfied by all the elements of the ground ring R . Moreover, the greatest common divisor of all the coefficients of this polynomial is 1. Furthermore, (6) is not satisfied by any 2×2 matrix ring over $\text{GF}(p)$ for any prime p , as a consideration of the following commutators shows:

$$[x, y] = [E_{11}, E_{12}], \quad [z, w] = [E_{22}, E_{21}]. \quad (7)$$

It follows from [1] that the commutator ideal $\mathcal{C}(R)$ is nil, and hence N is an ideal of R . This proves Lemma 3. $\square$

LEMMA 4. *Suppose that R is a subweakly periodic ring which satisfies all the hypotheses of Lemma 3. Then for any x in R ,*

$$x \in J \quad \text{or} \quad x - x^n \in N \quad \text{for some integer } n > 1. \quad (8)$$

PROOF. Let $x \in R$, $x \notin J$. Then, by Definition 1,

$$x = a + b, \quad a \in N, \quad b^n = b, \quad \text{where } n = n(b) > 1. \quad (9)$$

Now, by Lemma 3, N is an ideal of R , and since $a \in N$,

$$x^n = (a + b)^n = a_0 + b^n, \quad a_0 \in N. \quad (10)$$

Thus,

$$x - x^n = a + b - a_0 - b^n = a - a_0 \quad (\text{since } b^n = b), \quad (11)$$

and hence $x - x^n \in N$. This proves Lemma 4. $\square$

LEMMA 5. *Suppose that R is a subweakly periodic ring which satisfies all the hypotheses of Lemma 4. Suppose that $\sigma : R \rightarrow S$ is a homomorphism of R onto a ring S . Then the set N' of nilpotents of S is contained in $\sigma(J)$, and hence N' is a commutative set.*

PROOF. Suppose that $s \in N'$ with $s^k = 0$. Let $d \in R$ such that $\sigma(d) = s$. If $d \in J$, then $s = \sigma(d) \in \sigma(J)$, and the lemma follows. So, suppose that $d \notin J$. Then by Lemma 4,

$$d - d^n \in N \quad \text{for some integer } n > 1. \quad (12)$$

It is readily verified that

$$d - d^{k+1}d^{k(n-2)} = (d - d^n) + d^{n-1}(d - d^n) + \cdots + (d^{n-1})^{k-1}(d - d^n). \quad (13)$$

Combining (12), (13), and the fact that N is an ideal of R , we see that

$$d - d^{k+1}d^{k(n-2)} \in N, \quad (14)$$

and hence

$$s - s^{k+1}s^{k(n-2)} = \sigma(d - d^{k+1}d^{k(n-2)}) \in \sigma(N). \quad (15)$$

This implies (since $s^k = 0$) that $s \in \sigma(N)$. Thus, $N' \subseteq \sigma(N)$. By Lemma 3, N is an ideal of R , and hence $N \subseteq J$, which implies $N' \subseteq \sigma(N) \subseteq \sigma(J)$. Finally, since J is commutative, $\sigma(J)$ is commutative, and thus N' is a commutative set. $\square$

LEMMA 6. *Suppose that R is any ring which satisfies the “word” hypothesis (1) of Lemma 2. Then the set E of all idempotents of R is contained in the center of R .*

PROOF. Suppose that $e^2 = e \in R$, $x \in R$, and $f = e + ex - exe$. Then

$$f^2 = f, \quad ef = f, \quad fe = e, \quad [e, f] = ex - exe. \quad (16)$$

By the “word” hypothesis (1) of Lemma 2, there exists a word $w(e, f)$ such that

$$w(e, f)[ef, fe] = 0. \quad (17)$$

By (16), we see that $w(e, f) = e$ or $w(e, f) = f$, and hence (17) is equivalent to

$$e[ef, fe] = 0 \quad \text{or} \quad f[ef, fe] = 0. \quad (18)$$

These two equations, in turn, are equivalent to $e - f = 0$, and hence

$$ex = exe. \quad (19)$$

Now, let $f' = e + xe - exe$. Again, by the second part of the “word” hypothesis (1) of Lemma 2, there exists a word $w'(e, f')$ such that

$$[ef', f'e]w'(e, f') = 0. \quad (20)$$

An argument similar to the one above shows that (20) is equivalent to $e - f' = 0$, and hence

$$xe = exe. \quad (21)$$

Combining (19) and (21), we obtain the lemma. $\square$

LEMMA 7. *Suppose that R is a subweakly periodic ring which satisfies all the hypotheses of Lemma 4. Then, for any x in R , x is in J or $x^q = x^q e$ for some idempotent e and some $q \geq 1$.*

PROOF. Let $x \in R$. By Lemma 4,

$$x \in J \quad \text{or} \quad x - x^n \in N \quad \text{for some integer } n > 1. \quad (22)$$

Suppose that $x \notin J$. Then, $(x - x^n)^q = 0$ for some positive integer q , and hence

$$x^q = x^{q+1}g(x) \quad \text{for some polynomial } g(\lambda) \in \mathbb{Z}[\lambda]. \quad (23)$$

By reiterating, we see that

$$x^q = x^q(xg(x)) = x^q(xg(x))^2 = \cdots = x^q(xg(x))^q. \quad (24)$$

Let $e = (xg(x))^q$. It is readily verified that $e^2 = e$, and thus by (24), the lemma is proved. $\square$

The following three lemmas are well known and are stated without proofs.

LEMMA 8. *If $[x, y]$ commutes with x , then for all positive integers k ,*

$$[x^k, y] = kx^{k-1}[x, y]. \quad (25)$$

LEMMA 9. *Let R be a subdirectly irreducible ring. Then the only central idempotent elements of R are 0 and 1 (if $1 \in R$).*

LEMMA 10. *Let R be a weakly periodic ring. Then the Jacobson radical J of R is $\text{nil}(J \subseteq N)$.*

This lemma was proved in [2].

We are now in a position to prove our main theorems.

THEOREM 11. *Let R be a subweakly periodic ring such that the following two conditions hold:*

- (i) *for all x, y in R , there exist words $w(x, y)$ and $w'(x, y)$ depending on x and y such that $w(x, y)[xy, yx] = 0 = [xy, yx]w'(x, y)$;*
- (ii) *the Jacobson radical J is commutative.*

Then R is commutative. (In particular, this theorem holds if R is weakly periodic (or periodic).)

PROOF. By Lemma 7, we have for any x in R ,

$$x \in J \quad \text{or} \quad x^q = x^q e, \quad e^2 = e, \quad q \geq 1. \quad (26)$$

As is well known, the ground ring R can be written as

$$R \cong \text{a subdirect sum of subdirectly irreducible rings } R_i \quad (i \in \Gamma). \quad (27)$$

Let $\sigma_i : R \rightarrow R_i$ be the natural homomorphism of R onto R_i . Let $x_i \in R_i$, and suppose that $x \in R$ such that $\sigma_i(x) = x_i$. By (26), we see that

$$x_i \in \sigma_i(J) \quad \text{or} \quad x_i^q = x_i^q e_i \quad \text{with } e_i = \sigma_i(e), \quad e^2 = e \in R, \quad q \geq 1. \quad (28)$$

Moreover, by the proof of Lemma 7, we see that we can take $e = (xg(x))^q$ for some polynomial $g(\lambda) \in \mathbb{Z}[\lambda]$. Also, by Lemma 6, e is a central idempotent of R , and hence

$$e_i = (x_i g(x_i))^q \text{ is a central idempotent of } R_i. \quad (29)$$

We now distinguish two cases.

CASE 1. The ring R_i does not have an identity. Suppose that there is an $x_i \notin \sigma_i(J)$. Then by (28), (29), and Lemma 9, $x_i^q = 0$ for some $q \geq 1$. Thus, x_i is nilpotent, and hence by Lemma 5, $x_i \in \sigma_i(J)$. This contradiction shows that

$x_i \in \sigma_i(J)$ for all $x_i \in R_i$. By hypothesis, J is commutative, and consequently, $R_i = \sigma_i(J)$ is commutative as well.

CASE 2. The ring R_i has an identity 1. Let $x_i \notin \sigma_i(J)$. By (28) and (29), $x_i^q = x_i^q e_i$, where $e_i = (x_i g(x_i))^q$ is a central idempotent of R_i . Now, by Lemma 9, $e_i = 0$ or $e_i = 1$. If $e_i = 1$, then $(x_i g(x_i))^q = 1$, which implies that x_i is a unit of R_i . On the other hand, if $e_i = 0$, then $x_i^q = 0$, and hence, by Lemma 5, $x_i \in \sigma_i(J)$. The net result is

$$\forall x_i \in R_i, x_i \in \sigma_i(J) \quad \text{or} \quad x_i \text{ is a unit in } R_i. \quad (30)$$

Next, we claim that the set N_i of nilpotents of R_i is an ideal of R_i . To prove this, first recall that by Lemma 5, N_i is a commutative set, and hence N_i is closed with respect to subtraction. Now, suppose that $a_i \in N_i$, $x_i \in R_i$. By Lemma 5, $a_i \in \sigma_i(J)$, and therefore, $a_i = \sigma_i(j)$ for some $j \in J$. Suppose that $x \in R$ is such that $x_i = \sigma_i(x)$. Then $a_i x_i = \sigma_i(j) \sigma_i(x) = \sigma(jx) \in \sigma_i(J)$ since $j \in J$. Since J is commutative, $\sigma_i(J)$ is commutative, and hence

$$[a_i x_i, a_i] = 0 \quad (a_i \in N_i, x_i \in R_i). \quad (31)$$

An easy induction shows that (31) implies that $(a_i x_i)^q = a_i^q x_i^q$ for all positive integers q , and hence $a_i x_i$ is nilpotent. Similarly, $x_i a_i \in N_i$, and thus

$$N_i \text{ is a commutative ideal of } R_i. \quad (32)$$

Our next goal is to show that

$$[[a_i, u_i], u_i] = 0 \quad \text{for all units } u_i \text{ in } R_i \text{ and all } a_i \in \sigma_i(J). \quad (33)$$

To prove this, suppose that $a_i = \sigma_i(j)$ for some $j \in J$. Then j is quasiregular, and hence $\sigma_i(j)$ is quasiregular as well, that is, a_i is quasiregular. Therefore, $u_i' = 1 + a_i$ is a unit in R_i . By hypothesis (i), there exists a word $w(u_i, u_i')$ such that $w(u_i, u_i')[u_i u_i', u_i' u_i] = 0$. Since both u_i and u_i' are units in R_i , $w(u_i, u_i')$ must also be a unit, and hence $[u_i u_i', u_i' u_i] = 0$, which is equivalent to

$$[u_i(1 + a_i), (1 + a_i)u_i] = 0. \quad (34)$$

Note that $\sigma_i(J)$ is a commutative ideal of R_i , and hence $[u_i a_i, a_i u_i] = 0$. So, (34) implies that $[u_i, a_i u_i] + [u_i a_i, u_i] = 0$, which is equivalent to $[u_i, [a_i, u_i]] = 0$. This proves (33). Next, we prove that

$$\sigma_i(J) \text{ is contained in the center of } R_i. \quad (35)$$

Suppose not. Then

$$[a_i, b_i] \neq 0 \quad \text{for some } a_i \in \sigma_i(J), b_i \in R_i. \quad (36)$$

By (30), $b_i \in \sigma_i(J)$ or b_i is a unit of R_i . As $\sigma_i(J)$ is commutative, (36) implies that $b_i \notin \sigma_i(J)$. Therefore,

$$b_i \text{ is a unit of } R_i. \quad (37)$$

Moreover, by (36), we cannot have both $[a_i, 2b_i] = 0$ and $[a_i, 3b_i] = 0$. We assume, without loss of generality, that $[a_i, 2b_i] \neq 0$. Letting $2b_i$ play the role of b_i in the argument which led to (37), we see that

$$2b_i \text{ is a unit of } R_i. \quad (38)$$

Let $b \in R$ such that $\sigma_i(b) = b_i$. By Lemma 4, $b \in J$ or $b - b^n \in N$ for some integer $n > 1$. Since $b_i \notin \sigma_i(J)$, $b \notin J$, and hence

$$b - b^n \in N \quad \text{for some integer } n > 1. \quad (39)$$

Applying a similar argument to $2b_i$ yields

$$2b - (2b)^m \in N \quad \text{for some integer } m > 1. \quad (40)$$

Now, by (39) and (40),

$$b_i - b_i^n \in N_i, \quad 2b_i - (2b_i)^m \in N_i. \quad (41)$$

For any $x_i \in R_i$, let $\bar{x}_i = x_i + N_i \in R_i/N_i$. Then (41) implies that

$$(\bar{b}_i)^n = \bar{b}_i, \quad (2\bar{b}_i)^m = 2\bar{b}_i, \quad (n > 1, m > 1). \quad (42)$$

Observe that by (42),

$$\begin{aligned} (2\bar{b}_i)^{(m-1)(n-1)+1} &= (2\bar{b}_i)^m = 2\bar{b}_i, \\ (2\bar{b}_i)^{(m-1)(n-1)+1} &= 2^{(m-1)(n-1)+1} (\bar{b}_i)^n = 2^{(m-1)(n-1)+1} \bar{b}_i. \end{aligned} \quad (43)$$

Hence,

$$(2^{(m-1)(n-1)+1} - 2)\bar{b}_i = \bar{0}. \quad (44)$$

Also, by (37), $\bar{b}_i$ is a unit of R_i/N_i , and hence the above equation implies that

$$(2^{(m-1)(n-1)+1} - 2)\bar{1} = \bar{0}. \quad (45)$$

Therefore, $(2^{(m-1)(n-1)+1} - 2) \cdot 1 \in R_i$ is nilpotent, and hence R_i is not of characteristic zero. Since R_i is subdirectly irreducible, we have

$$\text{characteristic of } R_i \text{ is } p^k, \quad p \text{ prime}, \quad k \geq 1. \quad (46)$$

Now, using (37), (42), and (46), we see that the subring $\langle \bar{b}_i \rangle$ generated by the unit $\bar{b}_i$ is a finite commutative ring with identity which has no nonzero nilpotents, and hence

$$\langle \bar{b}_i \rangle \cong \bigoplus_{j=1}^t \text{GF}(p^{k_j}), \quad t \text{ finite, each } k_j \geq 1. \quad (47)$$

Let $\alpha = k_1 k_2 \cdots k_t$, then by (47), $(\bar{b}_i)^{p^{k\alpha}} = \bar{b}_i$. Thus

$$b_i^{p^{k\alpha}} - b_i \in N_i \subseteq \sigma_i(J). \quad (48)$$

Now, since $a_i \in \sigma_i(J)$ and $\sigma_i(J)$ is commutative, (48) yields $[b_i^{p^{k\alpha}} - b_i, a_i] = 0$. Thus

$$[b_i^{p^{k\alpha}}, a_i] = [b_i, a_i]. \quad (49)$$

Combining (37), (33), Lemma 8, and (46), we see that (49) implies that $[b_i, a_i] = 0$, which contradicts (36). This contradiction proves (35). To complete the proof of the theorem, let $x_i \in R_i$ and let $x \in R$ be such that $\sigma_i(x) = x_i$. By Lemma 3, N is an ideal of R , and therefore $N \subseteq J$. Now, by Lemma 4, $x \in J$, in which case, $x - x^n \in J$ for any integer n or $x - x^n \in N \subseteq J$ for some integer $n > 1$, and hence, by (35), $x_i - x_i^n \in \sigma_i(J) \subseteq \text{Center of } R_i$. Thus, by a well-known Herstein [3, Theorem 21], R_i is commutative. Therefore, R is commutative, and this proves the theorem. $\square$

THEOREM 12. *Let R be a subweakly periodic ring such that the following two conditions hold:*

- (i) *for all x and y in R , there exist relatively prime positive integers $m = m(x, y)$ and $n = n(x, y)$ such that*

$$(xy)^m - (yx)^m \in C, \quad (xy)^n - (yx)^n \in C, \quad (50)$$

where C is the center of R ;

- (ii) *the Jacobson radical J is commutative.*

Then R is commutative. (In particular, this theorem holds if R is weakly periodic (or periodic).)

PROOF. Since m and n are relatively prime integers, there exist positive integers k and l such that $km - ln = 1$. Let $y = ln$. Then $km = y + 1$. Moreover,

by (50),

$$[xy, (yx)^m] = 0, \quad [xy, (yx)^n] = 0, \quad (51)$$

which implies

$$[xy, (yx)^{km}] = 0, \quad [xy, (yx)^{ln}] = 0. \quad (52)$$

Thus

$$[xy, (yx)^{y+1}] = 0, \quad [xy, (yx)^y] = 0. \quad (53)$$

Hence

$$(yx)^{y+1}xy = xy(yx)^{y+1} = xy(yx)^y(yx) = (yx)^y(xy)(yx), \quad (54)$$

and hence

$$(yx)^{y+1}xy - (yx)^y(xy)(yx) = 0. \quad (55)$$

Thus

$$(yx)^y\{yx \cdot xy - xy \cdot yx\} = 0, \quad (56)$$

and hence

$$(yx)^y[xy, yx] = 0. \quad (57)$$

A similar argument shows that $[xy, yx](yx)^y = 0$, and so

$$(yx)^y[xy, yx] = 0 = [xy, yx](yx)^y. \quad (58)$$

Theorem 12 now follows from (58) and **Theorem 11** as well. $\square$

THEOREM 13. *Let R be a weakly periodic ring and suppose that the following conditions hold:*

- (i) *for all x, y in R , there exist words $w(x, y)$ and $w'(x, y)$ such that $(x, y)[xy, yx] = 0 = [xy, yx]w'(x, y)$;*
- (ii) *every commutator $[a, b]$, with a and b nilpotent, is potent (i.e., $[a, b]^k = [a, b]$ for some integer $k > 1$).*

Then R is commutative. (In particular, this theorem holds if R is periodic.)

PROOF. In view of **Theorem 11**, it suffices to show that J is commutative. By **Lemma 2**, the commutator ideal $\mathcal{C}(R) \subseteq J$. Also, by **Lemma 10**, $J \subseteq N$. Hence, $\mathcal{C}(R) \subseteq N$, and thus N is an ideal of R . This implies that $N \subseteq J$, and hence $J = N$. Suppose that a and b are any elements of N . Then, $[a, b]$ is also in N since N is an ideal. Also, $[a, b]$ is potent by hypothesis (ii), and hence $[a, b] = 0$

(since the only element which is both potent and nilpotent is zero). Therefore, $N = J$ is commutative. The theorem now follows from [Theorem 11](#). $\square$

THEOREM 14. *Let R be a weakly periodic ring such that the following conditions hold:*

- (i) *for all x, y in R , there exist relatively prime positive integers m, n such that $(xy)^n - (yx)^n \in C$ and $(xy)^m - (yx)^m \in C$;*
- (ii) *for all a, b in N , $[a, b]$ is potent.*

Then R is commutative. (In particular, this theorem holds if R is periodic.)

PROOF. A careful examination of the proof of [Theorem 12](#) shows that hypothesis (i) above implies that the “word” hypothesis stated in condition (i) of [Theorem 13](#) holds here as well (see (58)). Thus, all of the hypotheses of [Theorem 13](#) are satisfied, and hence R is commutative. $\square$

We conclude by considering the special case where the “words” involved in the above theorems happen to be the empty words. As an illustration, we consider the status of [Theorem 11](#) when $w(x, y)$ and $w'(x, y)$ are the empty words. The result is the following corollary.

COROLLARY 15. *Let R be a subweakly periodic ring (in particular, R may be chosen to be weakly periodic (or periodic)) satisfying the following conditions:*

- (i) $[xy, yx] = 0$ for all x, y in R ;
- (ii) J is commutative.

Then R is commutative.

A similar corollary may be obtained by taking the “words” in [Theorem 13](#) to be the empty words.

Finally, we have the following corollary of [Corollary 15](#).

COROLLARY 16. *Let R be a subweakly periodic ring (in particular, R may be chosen to be weakly periodic or (periodic)) satisfying the following conditions:*

- (i) $[x, y]$ commutes with x for all x in R ;
- (ii) J is commutative.

Then R is commutative.

PROOF. Let $x, y \in R$. By hypothesis (i) (interchanging x and y), it follows that $[y, x]$ commutes with y , and hence $[x, y]$ commutes with y . Thus, $[x, y]$ commutes with both x and y , and hence with xy . Thus, yx commutes with xy , and [Corollary 16](#) now follows from [Corollary 15](#). $\square$

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