THE DIOPHANTINE EQUATION

$$ax^2 + 2bxy - 4ay^2 = \pm 1$$

LIONEL BAPOUNGUÉ

Received 30 April 2002

We discuss, with the aid of arithmetical properties of the ring of the Gaussian integers, the solvability of the Diophantine equation $ax^2 + 2bxy - 4ay^2 = \pm 1$, where a and b are nonnegative integers. The discussion is relative to the solution of Pell's equation $v^2 - (4a^2 + b^2)w^2 = -4$.

2000 Mathematics Subject Classification: 11D09.

1. Introduction. The objective of this paper is the expansion and also the extension of [1, Section 2]. More precisely, it deals with the complete treatment of the solvability of the Diophantine equation

$$ax^2 + 2bxy - 4ay^2 = \pm 1, \tag{1.1}$$

where a and b are positive integers. From [2, Proposition 1], (1.1) is always solvable if $a = 1$. Hence, we may assume that $a > 1$. Moreover, we restrict oneself to $\gcd(a, 2b) = 1$. In the opposite case, (1.1) is insolvable.

We denote by $\delta = 4a^2 + b^2$ the discriminant of the quadratic form $ax^2 + 2bxy - 4ay^2$.

If $a > 1$, $b \geq 0$, and $\gcd(a, 2b) = 1$, [2, Theorem 1] shows that (1.1) is insolvable if δ is a square in $\mathbb{Z}$. Hence, we will assume also that $\delta = 4a^2 + b^2$ is a nonsquare in $\mathbb{Z}$, which requires b to be odd. Then δ verifies $\delta \equiv 5 \pmod{8}$. Thus, (cf. [4]) the algebraic integers of $\mathbb{Q}(\sqrt{\delta})$ are the numbers $(1/2)(v + w\sqrt{\delta})$ with $v, w \in \mathbb{Z}$ of the same parity. Consequently, if $(1/2)(v + w\sqrt{\delta})$ is a unit of $\mathbb{Q}(\sqrt{\delta})$, we must have

$$v^2 - \delta w^2 = \pm 4. \tag{1.2}$$

Conversely, if (v, w) is an integer solution of (1.2), then $(1/2)(v + w\sqrt{\delta})$ is an integer of $\mathbb{Q}(\sqrt{\delta})$ (its trace is v and its norm, by (1.2), is ± 1) and, hence, a unit of $\mathbb{Q}(\sqrt{\delta})$. Writing $(1/2)(v_0 + w_0\sqrt{\delta})$ for the fundamental unit of $\mathbb{Q}(\sqrt{\delta})$, we see that the solutions in pairs of natural numbers (v, w) of (1.2) comprise the values of the sequence (v_n, w_n) ($n \geq 1$) defined by setting

$$\frac{1}{2}(v_n + w_n\sqrt{\delta}) = \left(\frac{v_0 + w_0\sqrt{\delta}}{2} \right)^n. \tag{1.3}$$

Hence, we remark easily (cf. [2]) that if (x, y) is a solution of (1.1), then

$$v = -bx^2 + 8axy + 4by^2, \quad w = x^2 + 4y^2 \quad (1.4)$$

verify with $\delta = 4a^2 + b^2$ ($a > 1$, $b \geq 1$, and $\gcd(a, 2b) = 1$) the Pell's equation

$$v^2 - \delta w^2 = -4 \quad \text{with } v, w \text{ odd.} \quad (1.5)$$

Hence, our study will be based on (1.5). Thus, assuming its solvability, we give in Section 2 a necessary and sufficient condition for (1.1) to be solvable (Theorem 2.3) by methods using the arithmetic of the order $\mathbb{Z}[2i]$ of index 2, included in the principal ring $\mathbb{Z} + \mathbb{Z}[i]$. In the remainder of Section 2, we establish that if (1.1) is solvable, then $\pm a$ is the norm of an element of $\mathbb{Z}[\sqrt{\delta}]$ (Proposition 2.6). Next, we prove in Section 3 that when δ is given, among all the pairs of positive coprime odd integers (a, b) satisfying $\delta = 4a^2 + b^2$, there is exactly one pair for which (1.1) is solvable (Theorem 3.1). That unique pair will be constructed (Theorem 4.1) in Section 4 with the aid of the following result proved in [5].

THEOREM 1.1 (Thue). *If α and δ are integers satisfying $\delta > 1$, $\gcd(\alpha, \delta) = 1$, and m the least integer greater than $\sqrt{\delta}$, there exist x and y in $]0, m[$ such that $\alpha y \equiv \pm x \pmod{\delta}$.*

When solutions exist, we show using any of them in Section 5 that (1.1) possesses an infinity of solutions (Theorem 5.1); afterwards, we describe using it a family (Proposition 5.2). We give the conclusion of our paper in Section 6 with some numerical examples.

2. The case $a > 1$, δ odd nonsquare, and (1.5) solvable with v, w odd

2.1. Preliminaries. Let v, w be odd integers greater than or equal to 1 such that $v^2 - \delta w^2 = -4$. It is clear that

$$\gcd(v, w) = 1 \quad (2.1)$$

because if v and w have a common prime factor d , then d divides $v^2 - \delta w^2 = -4$ and, therefore, d divides also 2. Write (1.5) in the form

$$\delta w^2 = (v + 2i)(v - 2i). \quad (2.2)$$

The two factors of the right-hand side of (2.2) are relatively prime in $\mathbb{Z}[i]$ since any common divisor would divide $4i$, but w is odd, hence, $\gcd(w, 4i) = 1$. Hence, in $\mathbb{Z}[i]$, we have

$$\gcd(v + 2i, v - 2i) = 1. \quad (2.3)$$

Moreover, since (1.5) is written in the form (2.2), we will manipulate the elements of the nonmaximal order $\mathbb{Z}[2i]$ of index 2, for which we have shown

in [3] that the half group F defined by

$$F = \{v + 2i \in \mathbb{Z}[2i] : \gcd(N(v + 2i), 2) = 1\} \quad (2.4)$$

is factorial, where $N(\alpha)$ denotes the norm of α . Thus, the remark of [2, Proposition 4] applied to F enables us to state the following definition.

DEFINITION 2.1. An odd solution $(v, w) \in \mathbb{Z}^2$ of (1.5) is said to be

- (i) violain if, in F , $b + 2ai$ divides $v + 2i$ or $v - 2i$;
- (ii) monic if, in F , $b + 2ai = \gcd(v + 2i, \delta)$ or $\gcd(v - 2i, \delta)$.

PROPOSITION 2.2. Any odd violain solution $(v, w) \in \mathbb{Z}^2$ of (1.5) is monic.

2.2. One criterion of solvability for (1.1). We prove the following theorem.

THEOREM 2.3. If $a \geq 3$ and $b \geq 1$ are odd integers with $\gcd(a, 2b) = 1$ and $\delta = 4a^2 + b^2$ nonsquare in $\mathbb{Z}$, the following statements are equivalent:

- (i) (1.1) has a solution $(x, y) \in \mathbb{Z}^2$;
- (ii) (1.5) has an odd violain solution $(v, w) \in \mathbb{Z}^2$;
- (iii) the odd minimal solution $(v_0, w_0) \in \mathbb{Z}^2$ ($v_0 > 0$, $w_0 > 0$) of (1.5) is monic.

PROOF. (i) $\Rightarrow$ (ii). Let $(x, y) \in \mathbb{Z}^2$ be a solution of (1.1). We set

$$\begin{aligned} \varepsilon &= \operatorname{sgn}(ax^2 + 2bxy - 4ay^2), \\ v &= \varepsilon(bx^2 - 8axy - 4by^2), \quad w = x^2 + 4y^2. \end{aligned} \quad (2.5)$$

As b and x are odd, v and w are also odd. Then we have

$$v + 2i = \varepsilon(bx^2 - 8axy - 4by^2) + 2\varepsilon(ax^2 + 2bxy - 4ay^2)i \quad (2.6)$$

so that

$$v + 2i = \varepsilon(b + 2ai)(x + 2iy)^2, \quad (2.7)$$

where we see that $(v, w) \in \mathbb{Z}^2$ is an odd violain solution of (1.5). Further, taking the norm of the two sides of (2.7), we obtain

$$v^2 + 4 = (b^2 + 4a^2)(x^2 + 4y^2)^2 = \delta w^2 \quad (2.8)$$

so that (v, w) is an odd integer solution of (1.5).

(ii) $\Rightarrow$ (iii). Let $(v, w) \in \mathbb{Z}^2$ be an odd integer violain solution of (1.5). Then from equality (2.7), we have

$$\gcd(v + 2i, \delta) = (b + 2ai) \gcd((x + 2iy)^2, b - 2ai). \quad (2.9)$$

Now, we show that

$$\gcd((x + 2iy)^2, b - 2ai) = 1. \quad (2.10)$$

If there exists $\alpha \in \mathbb{F}$, α is not a unit, that is, $\alpha \neq \pm 1$ such that

$$\alpha | x + 2iy, \quad \alpha | b - 2ai, \quad (2.11)$$

then as (2.7) implies

$$v = \varepsilon(bx^2 - 8axy - 4by^2), \quad w = x^2 + 4y^2, \quad (2.12)$$

we deduced that, in $\mathbb{F}$,

$$\begin{aligned} v &\equiv \varepsilon[b(-2iy)^2 - 8a(-2iy)y - 4by^2] \\ &\equiv -8\varepsilon y^2(b - 2ai) \equiv 0 \pmod{\alpha}, \\ w &\equiv 0 \pmod{\alpha}, \end{aligned} \quad (2.13)$$

that is, α is also a divisor of v and w , contradicting the fact that $\gcd(v, w) = 1$ according to (2.1). Hence, we have

$$\gcd(v + 2i, \delta) = b + 2ai. \quad (2.14)$$

Then we show that (2.14) is true for v_0 arising from the odd minimal solution of (1.5). As (v, w) is an odd solution of (1.5), we have by the theory of the Pellian equation

$$\frac{v + w\sqrt{\delta}}{2} = \begin{cases} \left(\frac{v_0 + w_0\sqrt{\delta}}{2}\right)^{2n+1}, & \text{if } v > 0, \\ -\left(\frac{v_0 - w_0\sqrt{\delta}}{2}\right)^{2n+1}, & \text{if } v < 0, \end{cases} \quad (2.15)$$

for some integer $n \geq 0$. Developing (2.15), we obtain

$$4^n v = \begin{cases} v_0^{2n+1} + \binom{2n+1}{2} v_0^{2n-1} w_0^2 \delta + \dots, & \text{if } v > 0, \\ -v_0^{2n+1} - \binom{2n+1}{2} v_0^{2n-1} w_0^2 \delta - \dots, & \text{if } v < 0, \end{cases} \quad (2.16)$$

where the terms are all divisible by δ except v_0^{2n+1} . Hence, as $v_0^2 \equiv -4 \pmod{\delta}$, we have

$$v \equiv \begin{cases} (-1)^n v_0 \pmod{\delta}, & \text{if } v > 0, \\ (-1)^{n+1} v_0 \pmod{\delta}, & \text{if } v < 0. \end{cases} \quad (2.17)$$

From (2.14) and (2.17), we deduce that

$$b + 2ai = \gcd(v + 2i, \delta) = \gcd(\pm v_0 + 2i, \delta) = \gcd(v_0 \pm 2i, \delta) \quad (2.18)$$

as required. This proves that (v_0, w_0) is a monic solution of (1.5).

(iii) $\Rightarrow$ (i). Suppose that (v_0, w_0) is a monic solution of (1.5). The equality

$$v_0^2 - \delta w_0^2 = -4 \quad (2.19)$$

may be expressed in the form

$$\left(\frac{v_0 + 2i}{b + 2ai} \right) \left(\frac{v_0 - 2i}{b - 2ai} \right) = w_0^2, \quad (2.20)$$

where, from (2.3), $(v_0 + 2i)/(b + 2ai)$ and $(v_0 - 2i)/(b - 2ai)$ are coprime in F . Hence, for some unit $\varepsilon = \pm 1$ and integers x, y , we have

$$\frac{v_0 + 2i}{b + 2ai} = \varepsilon(x + 2iy)^2, \quad w_0 = x^2 + 4y^2. \quad (2.21)$$

Taking

$$v_0 + 2i = \varepsilon(b + 2ai)(x + 2iy)^2 \quad (2.22)$$

and equating coefficients of i on both sides of (2.22), we obtain

$$ax^2 + 2bxy - 4ay^2 = \varepsilon, \quad (2.23)$$

showing that $(x, y) \in \mathbb{Z}^2$ is a solution of (1.1). $\square$

REMARK 2.4. The proof above also confirms the following result.

THEOREM 2.5. *If $a \geq 3$ and $b \geq 1$ are odd integers with $\gcd(a, 2b) = 1$ and $\delta = 4a^2 + b^2$ nonsquare in $\mathbb{Z}$, the following statements are equivalent:*

- (i) (1.1) has a solution $(x, y) \in \mathbb{Z}^2$;
- (ii) (1.5) has an odd violain solution $(v, w) \in \mathbb{Z}^2$;
- (iii) (1.5) has an odd monic solution $(v, w) \in \mathbb{Z}^2$.

We have also the following proposition.

PROPOSITION 2.6. *Let $a \geq 3$ and $b \geq 1$ be odd integers with $\gcd(a, 2b) = 1$ and $\delta = 4a^2 + b^2$ nonsquare in $\mathbb{Z}$. If the Diophantine equation (1.1) has any solution $(x, y) \in \mathbb{Z}^2$, then*

$$a = \pm N(y\sqrt{\delta} + \mu) \quad \text{or} \quad \pm \frac{1}{4}N(x\sqrt{\delta} + \sigma), \quad \mu, \sigma \in \mathbb{Z}. \quad (2.24)$$

In other words, $\pm a$ (resp., $\pm 4a$) is the norm of an element of $\mathbb{Z}[\sqrt{\delta}]$.

PROOF. We suppose that $(x, y) \in \mathbb{Z}^2$ is any solution of (1.1). Then the equation

$$at^2 + 2bty - 4ay^2 - \varepsilon = 0 \quad (\varepsilon = \pm 1) \quad (2.25)$$

has an integer root, hence its discriminant is a square in $\mathbb{Z}$:

$$b^2y^2 + 4a^2y^2 - \varepsilon a = \mu^2, \quad \mu \in \mathbb{Z}, \quad (2.26)$$

whence

$$\varepsilon a = \gamma^2(b^2 + 4a^2) - \mu^2 = \delta \gamma^2 - \mu^2, \quad \mu \in \mathbb{Z}, \quad (2.27)$$

so that

$$\varepsilon a = N(\gamma\sqrt{\delta} + \mu), \quad \mu \in \mathbb{Z}. \quad (2.28)$$

Exchanging the roles of x and y , we obtain also

$$4\varepsilon a = N(x\sqrt{\delta} + \sigma), \quad \sigma \in \mathbb{Z}. \quad (2.29)$$

□

3. Uniqueness of the pair (a, b) , δ given. We assume in this section that δ is given and can be factorized into several sums of two squares in $\mathbb{Z}$. Then we use [Theorem 2.3](#) to show that among all the pairs of positive integers (a, b) , there is exactly one pair for which (1.1) is solvable.

THEOREM 3.1. *Let δ be an odd nonsquare positive integer for which (1.5) has an odd solution $(v, w) \in \mathbb{Z}^2$. Then among all the pairs of odd positive coprime integers (a, b) satisfying $\delta = 4a^2 + b^2$, there is exactly one pair $(a, b) = (A, B)$ such that (1.1) is solvable.*

PROOF. Let $(v, w) \in \mathbb{Z}^2$ be any odd violain solution of (1.5). We define positive integers A and B as follows:

$$A = |a|, \quad B = b. \quad (3.1)$$

Let $g = \gcd(a, b)$. Then

$$v + 2i = g(\alpha + 2i\beta) \implies 1 = g\beta, \quad (3.2)$$

hence $g = 1$. Since $(v, w) \in \mathbb{Z}^2$ is an odd solution of (1.5), we have $\delta \equiv 5 \pmod{8}$, and thus a and b are odd. Hence, we have

$$\gcd(A, B) = 1 \quad (3.3)$$

with both A and B all odd. Then we show that $\delta = 4A^2 + B^2$.

From the definition of A and B , we see that $B + 2Ai \mid v + 2i$ or $v - 2i$. Hence, we may assume, for example, that $B + 2Ai \mid v + 2i$. Then (1.5) may be expressed in the form

$$\frac{v + 2i}{B + 2Ai}(v - 2i) = \frac{\delta}{B + 2Ai}w^2, \quad (3.4)$$

where $(v + 2i)/(B + 2Ai)$ and $\delta/(B + 2Ai)$ are coprime elements of $\mathbb{F}$ (since v , δ , and B are odd). Equation (3.4) shows that $\delta/(B + 2Ai)$ divides $v - 2i$, but

$\delta/(B+2Ai)$ also divides δ , therefore $\delta/(B+2Ai)$ divides

$$\gcd(v-2i, \delta) = B-2Ai, \quad (3.5)$$

in $\mathbb{F}$, and so

$$\delta | B^2 + 4A^2. \quad (3.6)$$

On the other hand, since $B+2Ai | \delta$, taking conjugates, we obtain $B-2Ai | \delta$. Let $\pi \in \mathbb{Z}[i]$ be any prime factor of $B+2Ai$ and $B-2Ai$. Then we have

$$\frac{B^2 + 4A^2}{\pi} \Big| \delta. \quad (3.7)$$

Since (v, w) is any odd violain solution of (1.5), we have

$$\pi \Big| \frac{v+2i}{B+2Ai}, \quad \pi \Big| \frac{v-2i}{B-2Ai}, \quad (3.8)$$

and as $v \equiv B \pmod{2}$, [2, Lemma 2] applied to $\mathbb{Z}[i]$ shows that $\pi = 1$, then the relation (3.7) becomes

$$B^2 + 4A^2 | \delta. \quad (3.9)$$

Thus, $\delta = 4A^2 + B^2$ follows from (3.6) and (3.9). Hence, $\delta = 4A^2 + B^2$ is a decomposition of δ which satisfies statement (ii) of Theorem 2.3. So, by Theorem 2.3, the equation

$$Ax^2 + 2Bxy - 4Ay^2 = \pm 1 \quad (3.10)$$

is solvable. A and B are unique. $\square$

Applying Theorems 2.3 (or 2.5) and 3.1, we obtain the following corollary.

COROLLARY 3.2. *If $\delta \equiv 5 \pmod{8}$ is a prime number for which (1.5) is solvable, d, e denote integers such that $\delta = 4d^2 + e^2$ (they are odd, unique, and positive), then the Diophantine equation*

$$dx^2 + 2exy - 4dy^2 = \pm 1 \quad (3.11)$$

is solvable.

PROOF. This results from the fact that any prime number δ of the form $4m+1$ may be represented as the sum of two squares (cf. [5]). $\square$

4. Construction of (A, B) , δ given. We show in this section how the pair (A, B) can be constructed.

THEOREM 4.1. *Let δ be an odd nonsquare positive integer such that (1.5) is solvable in odd integers $(v, w) \in \mathbb{Z}^2$. Then there exists a unique pair of coprime odd integers (a, b) satisfying*

$$\begin{aligned} b \pm av &\equiv 0 \pmod{\delta}, \\ 0 < a < \sqrt{\delta}, \quad 0 < b < \sqrt{\delta}, \\ \delta &= 4a^2 + b^2. \end{aligned} \tag{4.1}$$

Then, for that unique pair (a, b) , (1.1) is solvable in $(x, y) \in \mathbb{Z}^2$.

PROOF. Taking $\alpha = v$ in Theorem 1.1, we see that there exist integers $a > 0$ and $b > 0$ such that

$$b \pm av \equiv 0 \pmod{\delta}, \quad a < \sqrt{\delta}, \quad b < \sqrt{\delta}. \tag{4.2}$$

Since $(v, \delta) = 1$, we have

$$b^2 + 4a^2 \equiv a^2 v^2 + 4a^2 \equiv -4a^2 + 4a^2 \equiv 0 \pmod{\delta}. \tag{4.3}$$

But $0 < b^2 + 4a^2 < 5\delta$, hence the equations

$$b^2 + 4a^2 = 2\delta, 3\delta, 4\delta \tag{4.4}$$

are insolvable in $\mathbb{Z}$ since, modulus 4, the first and the second congruences $b^2 \equiv 2, 3$ are impossible and the third imposes b to be even. Hence, we have

$$\delta = 4a^2 + b^2, \tag{4.5}$$

which verifies $\delta \equiv 5 \pmod{8}$ since $(v, w) \in \mathbb{Z}^2$ is an odd solution of (1.5), and thus a and b are both odd.

Next, we show that if (a, b) satisfies (4.2) and (4.5), then $\gcd(a, b) = 1$.

Let $g = \gcd(a, b)$, and set $a = ga'$, $b = gb'$. Then (4.5) becomes

$$(b')^2 + 4(a')^2 = \delta_1 \tag{4.6}$$

with $\delta_1 = \delta/g^2$. Relations (4.2) show that there exists $\lambda \in \mathbb{Z}$ such that $b = \pm av + \lambda\delta$, and thus $b' = \pm a'v + \lambda g\delta_1$. Replacing b' in (4.6), we obtain

$$(\pm a'v + \lambda g\delta_1)^2 + 4(a')^2 = \delta_1, \tag{4.7}$$

and, using (1.5), we deduce from it that

$$g(w^2(a')^2 \pm 2a'\lambda v + \lambda^2 g\delta_1) = 1, \tag{4.8}$$

proving that $g = 1$.

Now, we show that (a, b) is unique. We suppose that (a_1, b_1) is another solution of (4.2). Then from congruences

$$b + av \equiv b_1 + a_1v \equiv 0 \pmod{\delta} \quad (4.9)$$

or

$$b - av \equiv b_1 - a_1v \equiv 0 \pmod{\delta}, \quad (4.10)$$

we see that

$$bb_1 + 4aa_1 \equiv 0 \pmod{\delta}, \quad ab_1 - a_1b \equiv 0 \pmod{\delta}. \quad (4.11)$$

From the product of the two following expressions:

$$b^2 + 4a^2 = \delta, \quad b_1^2 + 4a_1^2 = \delta, \quad (4.12)$$

we deduce that

$$(bb_1 + 4aa_1)^2 + 4(ab_1 - a_1b)^2 = \delta^2 \quad (4.13)$$

so that (dividing by δ^2)

$$\left(\frac{bb_1 + 4aa_1}{\delta} \right)^2 + 4 \left(\frac{ab_1 - a_1b}{\delta} \right)^2 = 1, \quad (4.14)$$

which gives

$$bb_1 + 4aa_1 = \pm \delta, \quad ab_1 - a_1b = 0. \quad (4.15)$$

Relations (4.15) impose

$$(a_1, b_1) = \pm(a, b). \quad (4.16)$$

Thus, there exists a unique solution of (4.2) satisfying $a > 0, b > 0$.

Finally, our last assertion is to prove that (a, b) defined by (4.1) satisfies Theorem 2.3(ii). We suppose that

$$b \pm av \equiv 0 \pmod{\delta}. \quad (4.17)$$

As $v^2 \equiv -4 \pmod{\delta}$, multiplying (4.17) by v , we obtain

$$bv \equiv \pm av^2 \equiv \pm 4a \pmod{\delta}, \quad (4.18)$$

and so

$$\frac{v \pm 2i}{b + 2ai} = \frac{bv \pm 4a}{\delta} - 2 \left(\frac{\pm b + av}{\delta} \right) i \quad (4.19)$$

is an element of F . Thus

$$b + 2ai | v \pm 2i. \quad (4.20)$$

This proves that (v, w) is an odd violain solution of (1.5). So, from Theorem 2.3, (1.1) is solvable in integers $(x, y) \in \mathbb{Z}^2$. $\square$

REMARK 4.2. Denoting by (v_0, w_0) the odd minimal solution of (1.5), we can easily determine A and B , such that $B + 2Ai$ is the $\gcd(v_0 \pm 2i, \delta)$, using the following algorithm:

- (1) (i) factorize δ in $\mathbb{Z}$;
 (ii) calculate the norm of $v_0 + 2i$ and factorize it in $\mathbb{Z}$;
- (2) factorize the prime factors obtained in F ;
- (3) deduce from (2) the common divisors of δ and $v_0 + 2i$.

5. Complete set of solutions of (1.1). First of all, we prove the following theorem.

THEOREM 5.1. *Under the conditions of Theorem 2.3, the Diophantine equation (1.1) has an infinity of solutions in $\mathbb{Z}$.*

PROOF. We assume that (1.1) has a solution $(x_0, y_0) \in \mathbb{Z}^2$. Then we have $\gcd(x_0, y_0) = 1$, hence there exists $(a, b) \in \mathbb{Z}^2$ such that

$$ax_0 + by_0 = 1. \quad (5.1)$$

We set

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 & -\beta \\ y_0 & \alpha \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad (5.2)$$

$$g(X, Y) = f(x, y) = ax^2 + 2bxy - 4ay^2.$$

Then g and f are two equivalent quadratic forms. Further

$$g(X, Y) = \varepsilon X^2 - 2BXY - CY^2, \quad B, C \in \mathbb{Z}, \quad (5.3)$$

with

$$\begin{aligned} \varepsilon &= ax_0^2 + 2bx_0y_0 - 4ay_0^2, \\ B &= (a\beta - b\alpha)x_0 + (b\beta + 4a\alpha)y_0, \\ C &= a\beta^2 + 2b\alpha\beta + 4a\alpha^2. \end{aligned} \quad (5.4)$$

But the equations

$$f(x, y) = \varepsilon, \quad g(X, Y) = \varepsilon \quad (5.5)$$

are equivalent, hence as

$$g(X, Y) = \varepsilon N(X - \theta Y), \quad (5.6)$$

where θ is a root of the equation

$$t^2 - \varepsilon Bt + \varepsilon C = 0, \quad (5.7)$$

we conclude that if $f(x, y) = \varepsilon$ has a solution in $\mathbb{Z}$, it has an infinity of solutions in $\mathbb{Z}$. $\square$

Now, we describe the family of solutions of (1.1).

PROPOSITION 5.2. *Under the conditions of Theorem 2.3, let (x_0, y_0) be a particular solution of (1.1). Then the set of solutions (x, y) of (1.1) is given by*

$$ax + by + y\sqrt{\delta} = \pm \left(\frac{v_0 + w_0\sqrt{\delta}}{2} \right)^{3n} (ax_0 + by_0 + y_0\sqrt{\delta}) \quad (5.8)$$

in which (v_0, w_0) is the minimal solution of (1.5) and $n \in \mathbb{Z}$.

PROOF. Let (x_0, y_0) be a particular solution of (1.1). We show how all the solutions (x, y) of (1.1) may be obtained in terms of (x_0, y_0) and the minimal solution (v_0, w_0) of (1.5). If (x, y) is any solution of (1.1) and if we set

$$J = \frac{ax + by + y\sqrt{\delta}}{ax_0 + by_0 + y_0\sqrt{\delta}}, \quad (5.9)$$

the norm of J is

$$\frac{(ax + by)^2 - \delta y^2}{(ax_0 + by_0)^2 - \delta y_0^2} = \frac{a(ax^2 + 2bxy - 4ay^2)}{a(ax_0^2 + 2bx_0y_0 - 4ay_0^2)}, \quad (5.10)$$

that is, ± 1 .

Moreover, J is of the form $D + E\sqrt{\delta}$, where D and E are integers given by

$$\begin{aligned} D &= axx_0 + b(xy_0 + x_0y) - 4ayy_0, \\ E &= x_0y - xy_0. \end{aligned} \quad (5.11)$$

Hence, by the theory of the Pellian equation, we have

$$J = \pm \left(\frac{v_0 + w_0\sqrt{\delta}}{2} \right)^{3n}, \quad (5.12)$$

where $n \in \mathbb{Z}$. Thus, we have shown the existence of an integer n such that we have (5.8).

Conversely, let x and y be defined by (5.8) for some $n \in \mathbb{Z}$. Taking norms of both sides of (5.8), we see that x and y verify (1.1). It remains to show that they are both integers.

Define integers M and N by

$$M + N\sqrt{\delta} = \pm \left(\frac{v_0 + w_0\sqrt{\delta}}{2} \right)^{3n}. \quad (5.13)$$

Then equating coefficients in (5.8), we obtain

$$\begin{aligned} ax + by &= M(ax_0 + by_0) + \delta Ny_0, \\ y &= My_0 + N(ax_0 + by_0). \end{aligned} \quad (5.14)$$

Clearly, $y \in \mathbb{Z}$. Using $\delta = 4a^2 + b^2$, we obtain

$$x = (M - bN)x_0 + 4aNy_0, \quad (5.15)$$

so that x is also an integer. $\square$

6. Numerical examples

EXAMPLE 6.1. If $a = 19$ and $b = 71$, then $\delta = 4(19)^2 + 71^2 = 6485 \equiv 5 \pmod{8}$ is nonsquare such that $(v_0, w_0) = (1369, 17)$ is the minimal solution of (1.5). In this case, $\gcd(1369 + 2i, 6485) = 71 + 38i$ and Theorem 2.3 shows that (1.1) is solvable; in fact, it is $19x^2 + 142xy - 76y^2 = -1$ ($(x, y) = (1, 2)$ is a solution).

EXAMPLE 6.2. If $a = 3$ and $b = 5$, then $\delta = 4(3)^2 + 5^2 = 61 \equiv 5 \pmod{8}$ is prime such that $(v, w) = (39, 5)$ is a solution of (1.5); Corollary 3.2 shows that (1.1) is solvable; in fact, it is $3x^2 + 10xy - 12y^2 = 1$ ($(x, y) = (1, 1)$ is a solution).

EXAMPLE 6.3. In case $\delta = 2941 = 4(25)^2 + 21^2 = 4(27)^2 + 5^2$, we have $\delta \equiv 5 \pmod{8}$ nonsquare such that $(v_0, w_0) = (705, 13)$ is the minimal solution of (1.5). As $\gcd(705 + 2i, 2941) = 21 - 50i$, Theorems 2.3 and 3.1 show that (1.1) is solvable only in the case when $(a, b) = (25, 21)$; in fact, it is $25x^2 + 42xy - 100y^2 = -1$ ($(x, y) = (-3, 1)$ is a solution).

The equation $27x^2 + 10xy - 108y^2 = \pm 1$ is insolvable.

EXAMPLE 6.4. Take $\delta = 3077$. We have $\delta \equiv 5 \pmod{8}$ nonsquare such that $(v_0, w_0) = (943, 17)$ is the minimal solution of (1.5). The candidates for the unique pair (a, b) satisfying (4.1) must be solutions of $\delta = 4a^2 + b^2$. That is, $(a, b) = \pm(13, 49), \pm(23, 31)$. The only pair satisfying $b + av_0 \equiv 0 \pmod{\delta}$ is $(a, b) = (13, 49)$ so that $(13, 49)$ is the unique pair for which (1.1) is solvable; in fact, it is $13x^2 + 98xy - 52y^2 = 1$ ($(x, y) = (1, 2)$ is a solution).

The equation $23x^2 + 62xy - 92y^2 = \pm 1$ is insolvable.

REFERENCES

- [1] L. Bapoungué, *Sur la résolubilité de l'équation Diophantienne $ax^2 + 2bxy - kay^2 = \pm 1$* [On the solvability of the Diophantine equation $ax^2 + 2bxy -$

- $kay^2 = \pm 1$], C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 5, 235–238 (French).
- [2] ———, *Un critère de résolution pour l'équation Diophantienne $ax^2 + 2bxy - kay^2 = \pm 1$* [A criterion for the solution of the Diophantine equation $ax^2 + 2bxy - kay^2 = \pm 1$], Exposition. Math. **16** (1998), no. 3, 249–262 (French).
- [3] ———, *Factorisation dans un ordre non maximal d'un corps quadratique* [Factorization in a nonmaximal order of a quadratic field], Exposition. Math. **20** (2002), no. 1, 43–57 (French).
- [4] P. Samuel, *Théorie Algébrique des Nombres*, Hermann, Paris, 1967 (French).
- [5] D. Shanks, *Solved and Unsolved Problems in Number Theory. Vol. I*, Spartan Books, District of Columbia, 1962.

Lionel Bapoungué: Département de Mathématiques, École Normale Supérieure, Université de Yaoundé 1, BP 47 Yaoundé, Cameroun
E-mail address: lbapou@uycdc.uninet.cm