Suppose that $\varphi(z)$ is an analytic self-map of the unit disk $\Delta$. We consider the boundedness of the composition operator $C_{\varphi}$ from Bloch space $\mathcal{B}$ into the spaces $Q_T (Q_T, 0)$ defined by a nonnegative, nondecreasing function $T(r)$ on $0 \leq r < \infty$.

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1. Introduction. Let $\Delta = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$ and let $H(\Delta)$ be the space of all analytic functions in $\Delta$. For $a \in \Delta$, Green’s function with logarithmic singularity at $a \in \Delta$ is denoted by $g(z, a) = \log \frac{|(1 - \overline{a}z)/(a - z)|}{1 - |z|^2}$. For $0 < p < \infty$, the space $Q_p$ consists of all functions $f$ analytic in $\Delta$ for which

$$\sup_{a \in \Delta} \int_{\Delta} |f'(z)|^2 (g(z, a))^p \, dA(z) < \infty,$$

(1.1)

where $dA(z)$ is the Euclidean area element on $\Delta$.

$Q_p$-spaces have been investigated by many authors (cf. [1, 2, 3, 9]). We know that $Q_1 = \text{BMOA}$, the space of all analytic functions of bounded mean oscillation (cf. [4]). Further, the spaces $Q_p$ are the same for each $p \in (1, \infty)$, and each space equals to the Bloch space $\mathcal{B}$, which is a Banach space with the norm

$$\|f\|_\mathcal{B} := |f(0)| + \|f\|_p := |f(0)| + \sup_{z \in \Delta} (1 - |z|^2) |f'(z)|.$$

(1.2)

Recently, we introduced a new space $Q_T$ (cf. [5, 10]) by a nondecreasing function $T(r)$ on $0 \leq r < \infty$ as follows.

**Definition 1.1.** Let $T(r) \neq 0$ be a nonnegative, nondecreasing function on $0 \leq r < \infty$. A function $f \in H(\Delta)$ is said to belong to $Q_T$ if

$$\|f\|_{Q_T}^2 := \sup_{a \in \Delta} \int_{\Delta} |f'(z)|^2 T(g(z, a)) \, dA(z) < \infty.$$  

(1.3)

If

$$\lim_{|a| \to 1} \int_{\Delta} |f'(z)|^2 T(g(z, a)) \, dA(z) = 0,$$

(1.4)

then $f$ is said to belong to $Q_{T, 0}$. 

**Proof.**
For $0 < p < \infty$, if we take $T(r) = r^p$, the space $Q_T$ coincides with the space $Q_p$. We note that $Q_T \subset B$ for all nondecreasing functions $T$. We have previously shown that $Q_T = Q_p$ under certain growth conditions on $T(r)$ (cf. [10]).

In the present paper, first we give some basic properties of $Q_T$ spaces, some of which are also new for the special case $Q_T = Q_p$. For example, $Q_T$ is a Banach space with the norm $\|f\|_T$ defined by

$$\|f\|_T := |f(0)| + \|f\|_{Q_T}.$$  \hfill (1.5)

Then we investigate the boundedness of the composition operators from the Bloch space $B$ into $Q_T$ or $Q_{T,0}$. These results extend some previously known results (cf. [6, 8]).

2. Basic properties of $Q_T$ spaces. We give the following propositions.

**Proposition 2.1.** The space $Q_T$ is a subspace of the Bloch space $B$.

The proof of Proposition 2.1 can be found in [10].

**Proposition 2.2.** The space $Q_T$ is a Banach space with the norm defined in (1.5).

**Proof.** For $f \in Q_T$ and $a \in \Delta$, define

$$I^2(f, a) := \iint_{\Delta} |f'(z)|^2 T(g(z,a))dA(z).$$  \hfill (2.1)

Let $f_1, f_2 \in Q_T$. It follows from Schwarz’s inequality that

$$\iint_{\Delta} |f_1(z)f_2(z)||T(g(z,a))|dA(z) \leq I(f_1, a)I(f_2, a),$$  \hfill (2.2)

and then

$$I^2(f_1 + f_2, a) \leq I^2(f_1, a) + 2I(f_1, a)I(f_2, a) + I^2(f_2, a)$$
$$= \left(I(f_1, a) + I(f_2, a)\right)^2.$$  \hfill (2.3)

Thus, $I(f_1 + f_2, a) \leq I(f_1, a) + I(f_2, a)$ for all $a \in \Delta$. Hence

$$\|f_1 + f_2\|_{Q_T} \leq \|f_1\|_{Q_T} + \|f_2\|_{Q_T}.$$  \hfill (2.4)

Therefore,

$$\|f_1 + f_2\|^2_T = \left(\|f_1(0) + f_2(0)\| + \|f_1 + f_2\|_{Q_T}\right)^2$$
$$\leq \left(\|f_1(0)\| + |f_2(0)| + \|f_1\|_{Q_T} + \|f_2\|_{Q_T}\right)^2$$
$$= \left(\|f_1\|_T + \|f_2\|_T\right)^2.$$  \hfill (2.5)
that is, \( \|f_1 + f_2\|_T \leq \|f_1\|_T + \|f_2\|_T \). On the other hand, it is obvious that \( \|f\|_T \geq 0 \) for each \( f \in Q_T \) and that \( \|f\|_T = 0 \) if and only if \( f \equiv 0 \). It is obvious that \( \|cf\|_T = |c|\|f\|_T \) for any constant \( c \). Thus, \( Q_T \) is a normed space.

Let \( f \in Q_T \) and let \( \phi_a(w) = (a - w)/(1 - \bar{a}w) \), \( a \in \Delta \). Then by changing a variable \( w = \phi_a(z) \), we obtain

\[
\|f\|^2_{Q_T} \geq \int_\Delta |f'(z)|^2 T(g(z,a))dA(z) = \int_\Delta |(f \circ \phi_a)'(w)|^2 T\left(\log \frac{1}{|w|}\right)dA(w) \\
\geq T\left(\log \frac{1}{r}\right) \int_{|w|<r} |(f \circ \phi_a)'(w)|^2 dA(w) \\
\geq \pi r^2 T\left(\log \frac{1}{r}\right)(1-|a|^2)^2 |f'(a)|^2.
\]

(2.6)

For \( r_0, 0 < r_0 < 1 \), such that \( T(\log (1/r_0)) \neq 0 \), we have

\[
\|f\|_b \leq \frac{\|f\|_{Q_T}}{r_0(\pi T(\log 1/r_0))^{1/2}}.
\]

(2.7)

Since \( f \in Q_T \subset \mathbb{B} \), we have for \( z \in \Delta \),

\[
|f(z)| \leq |f(0)| + \frac{\|f\|_b}{2} \log \frac{1 + |z|}{1 - |z|} \\
\leq |f(0)| + \frac{\|f\|_{Q_T}}{2r_0(\pi T(\log 1/r_0))^{1/2}} \log \frac{1 + |z|}{1 - |z|} \\
\leq \|f\|_T \left(1 + \frac{1}{2r_0(\pi T(\log 1/r_0))^{1/2}}\right) \log \frac{1 + |z|}{1 - |z|}.
\]

(2.8)

Suppose \( \{f_n\} \) is a Cauchy sequence in \( Q_T \). Then there is a constant \( M > 0 \) such that

\[
\|f_n\|_T \leq M, \quad n = 1, 2, \ldots
\]

(2.9)

By the estimate (2.8) for a fixed \( r_0 \in (0, 1) \), we obtain that

\[
|f_n(z)| \leq M \left(1 + \frac{1}{2r_0(\pi T(\log 1/r_0))^{1/2}}\right) \log \frac{1 + |z|}{1 - |z|}
\]

(2.10)

holds for all integral numbers \( n = 1, 2, \ldots \). Hence, there exist a subsequence \( \{f_{n_j}(z)\} \) of \( \{f_n(z)\} \) and an analytic function \( f \) defined on the unit disk \( \Delta \) such that both \( \{f_{n_j}(z)\} \) and \( \{f'_{n_j}(z)\} \) converge uniformly to \( f \) and \( f' \), respectively. The conditions here are such that both the sequence of functions and the sequence of derivatives converge since we know that \( \{f_n(z)\} \) is bounded on
compact subsets of $\Delta$ by inequality (2.10). By Fatou’s lemma, we get that
\[
\int_{\Delta} |f'(z)|^2 T(g(z,a))dA(z) = \int_{\Delta} \lim_{j \to \infty} |f_{nj}'(z)|^2 T(g(z,a))dA(z)
\leq \liminf_{j \to \infty} \int_{\Delta} |f_{nj}'(z)|^2 T(g(z,a))dA(z)
\leq \liminf_{j \to \infty} \|f_{nj}\|^2_{QT} \leq M^2
\]
holds for all $a \in \Delta$, so that $f \in QT$. By a similar reasoning, we can prove that
$\|f_n - f\|_T \to 0$ as $n \to \infty$. The proof of Proposition 2.2 is complete. \hfill \square

3. Boundedness of composition operators. Let $\varphi(z)$ be an analytic self-map of the unit disk $\Delta$. Let the composition operator $C_{\varphi}$ induced by $\varphi$ from $H(\Delta)$ to itself be defined by $C_{\varphi}(f) = f \circ \varphi$ for $f \in H(\Delta)$. The boundedness of composition operators from $B$ to itself and from $B$ to $Q_p$ have been studied in [6, 8], respectively. In this paper, we consider the same problems for the general spaces $QT$.

**Theorem 3.1.** Let $T(r) \not\equiv 0$ be a nonnegative, nondecreasing function on $0 \leq r < \infty$ and let $\varphi$ be an analytic self-map of $\Delta$. Then $C_{\varphi} : B \to QT$ is bounded if and only if
\[
\sup_{a \in \Delta} \int_{\Delta} \frac{|\varphi'(z)|^2}{1 - |\varphi(z)|^2} T(g(z,a))dA(z) < \infty. \tag{3.1}
\]

**Proof.** Let (3.1) hold and let $K_1^2(K_1 > 0)$ be the supremum in (3.1). If $f \in B$, then for all $a \in \Delta$, we have
\[
\int_{\Delta} |(C_{\varphi}f)'(z)|^2 T(g(z,a))dA(z) = \int_{\Delta} |f'(\varphi(z))| |\varphi'(z)|^2 T(g(z,a))dA(z)
\leq \|f\|^2_b \int_{\Delta} \frac{|\varphi'(z)|^2}{1 - |\varphi(z)|^2} T(g(z,a))dA(z)
\leq K_1^2 \|f\|^2_b. \tag{3.2}
\]
Consequently, $\|C_{\varphi}f\|^2_{QT} \leq K_1 \|f\|^2_b$. Since $f(z) \in B$, we obtain
\[
\|C_{\varphi}f\|^2_T = \left( |f \circ \varphi (0)| + \|C_{\varphi}f\|^2_{QT} \right)^2
\leq \left( |f(0)| + \frac{\|f\|^2_b \log \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} + K_1 \|f\|^2_b} \right)^2
\leq K^2 \left( |f(0)| + \|f\|^2_b \right)^2. \tag{3.3}
\]
where \( K = \max\{1, K_1 + (1/2) \log(1 + |\varphi(0)|)/(1 - |\varphi(0)|)\} \). Thus, \( \|C_{\varphi}f\|_T \leq K\|f\|_{\mathcal{B}} \), which shows that \( C_{\varphi} : \mathcal{B} \rightarrow Q_T \) is bounded.

Conversely, assume that \( C_{\varphi} : \mathcal{B} \rightarrow Q_T \) is bounded, there exists a constant \( K > 0 \) such that for each \( f \in \mathcal{B} \), we have

\[
\|C_{\varphi}f\|_T \leq K\|f\|_{\mathcal{B}}. \tag{3.4}
\]

On the other hand, by a result in [7], there exist \( f_1, f_2 \in \mathcal{B} \) such that

\[
\sum_{\{z \in \Delta\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} \leq 2 \left( (f_1 \circ \varphi)'(z)^2 + (f_2 \circ \varphi)'(z)^2 \right) \tag{3.5}
\]

holds for all \( z \in \Delta \), so that

\[
\sum_{\{z \in \Delta\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} \leq 2 \left( (f_1 \circ \varphi)'(z)^2 + (f_2 \circ \varphi)'(z)^2 \right). \tag{3.6}
\]

Thus, the inequalities

\[
\int_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z, a)) dA(z) 
\leq 2 \int_{\Delta} \left( (f_1 \circ \varphi)'(z)^2 + (f_2 \circ \varphi)'(z)^2 \right) T(g(z, a)) dA(z) 
\leq 2K^2 (\sum_{\{z \in \Delta\}} \|f_1\|_{\mathcal{B}}^2 + \sum_{\{z \in \Delta\}} \|f_2\|_{\mathcal{B}}^2) \tag{3.7}
\]

hold for all \( z, a \in \Delta \), which establishes (3.1). The proof of Theorem 3.1 is completed.

\textbf{Remark 3.2.} Note that if \( C_{\varphi} : \mathcal{B} \rightarrow \mathcal{B} \), then (3.1) holds for any increasing function \( T \) satisfying \( Q_T = \mathcal{B} \). Indeed, we know that \( Q_T = \mathcal{B} \) (see [5]) if and only if

\[
\int_0^1 T\left( \log \left( \frac{1}{r} \right) \right) (1 - r^2)^{-2} r \, dr < \infty. \tag{3.8}
\]

The Schwarz-Pick lemma guarantees that

\[
\left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right) |\varphi'(z)| \leq 1,
\]

so that (3.8) leads easily to (3.1). It means that \( C_{\varphi} : \mathcal{B} \rightarrow \mathcal{B} \) is always bounded (cf. [6]).

\textbf{Remark 3.3.} If one considers the composition operator \( C_{\varphi} \) from the Bloch space to the Dirichlet space

\[
\mathcal{D} = \left\{ f \in H(\Delta) : \int_{\Delta} |f'(z)|^2 \, dA(z) < \infty \right\}, \tag{3.9}
\]

then \( C_{\varphi} : \mathcal{B} \rightarrow \mathcal{D} \) is bounded if and only if

\[
\int_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} dA(z) < \infty. \tag{3.10}
\]
For the spaces $Q_{T,0}$, we have the following results.

**Theorem 3.4.** Let $T(r)$ be a nonnegative, nondecreasing function on $0 \leq r < \infty$ and let $\varphi$ be an analytic self-map of $\Delta$. Then $C_\varphi : \mathbb{B} \rightarrow Q_{T,0}$ is bounded if and only if

$$
\lim_{|a| \to 1} \int_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a)) dA(z) = 0. \quad (3.11)
$$

**Proof.** Suppose $C_\varphi : \mathbb{B} \rightarrow Q_{T,0}$ is bounded. Using a way similar to the proof of Theorem 3.1, we choose functions $f_1, f_2 \in \mathbb{B}$ such that

$$
\frac{1}{1-|z|^2} \leq |f_1'(z)| + |f_2'(z)| \quad (3.12)
$$

for all $z \in \Delta$. Then $C_\varphi f_1$ and $C_\varphi f_2$ belong to $Q_{T,0}$. Therefore,

$$
\lim_{|a| \to 1} \int_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a)) dA(z)
\leq 2 \lim_{|a| \to 1} \int_{\Delta} \left( |(f_1 \circ \varphi)'(z)|^2 + |(f_2 \circ \varphi)'(z)|^2 \right) T(g(z,a)) dA(z) = 0,
$$

which shows that (3.11) holds.

Conversely, by Theorem 3.1, we know that $C_\varphi : \mathbb{B} \rightarrow Q_T$ is bounded since condition (3.11) implies that

$$
\sup_{a \in \Delta} \int_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a)) dA(z) < \infty. \quad (3.14)
$$

We need only to prove that $C_\varphi f \in Q_{T,0}$ for each $f \in \mathbb{B}$, and this follows from the inequality

$$
\int_{\Delta} |(C_\varphi f)'(z)|^2 T(g(z,a)) dA(z)
= \int_{\Delta} |f'(\varphi(z))|^2 |\varphi'(z)|^2 T(g(z,a)) dA(z)
\leq \|f\|_p^2 \int_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a)) dA(z). \quad (3.15)
$$

The proof of Theorem 3.4 is completed.

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