IJMMS 2003:31, 1973–1979
PII. S0161171203207122
http://ijmms.hindawi.com

COMPOSITION OPERATORS FROM THE BLOCH SPACE INTO THE SPACES Q_T

PENGCHENG WU and HASI WULAN

Received 10 July 2002

Suppose that $\varphi(z)$ is an analytic self-map of the unit disk Δ. We consider the boundedness of the composition operator C_φ from Bloch space $\mathcal{B}$ into the spaces Q_T ($Q_{T,0}$) defined by a nonnegative, nondecreasing function $T(r)$ on $0 \le r < \infty$.

2000 Mathematics Subject Classification: 30D50, 47B38, 46E15.

1. Introduction. Let $\Delta = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$ and let $H(\Delta)$ be the space of all analytic functions in Δ. For $a \in \Delta$, Green's function with logarithmic singularity at $a \in \Delta$ is denoted by $g(z,a) = \log|(1-\bar{a}z)/(a-z)|$. For $0 < p < \infty$, the space Q_p consists of all functions f analytic in Δ for which

$$\sup_{a\in\Delta} \iint_\Delta |f'(z)|^2 (g(z,a))^p \, dA(z) < \infty, \tag{1.1}$$

where $dA(z)$ is the Euclidean area element on Δ.

Q_p-spaces have been investigated by many authors (cf. [1, 2, 3, 9]). We know that $Q_1 = \mathrm{BMOA}$, the space of all analytic functions of bounded mean oscillation (cf. [4]). Further, the spaces Q_p are the same for each $p \in (1,\infty)$, and each space equals to the Bloch space $\mathcal{B}$, which is a Banach space with the norm

$$\|f\|_{\mathcal{B}} := |f(0)| + \|f\|_b := |f(0)| + \sup_{z\in\Delta} (1-|z|^2)|f'(z)|. \tag{1.2}$$

Recently, we introduced a new space Q_T (cf. [5, 10]) by a nondecreasing function $T(r)$ on $0 \le r < \infty$ as follows.

Definition 1.1. Let $T(r) \not\equiv 0$ be a nonnegative, nondecreasing function on $0 \le r < \infty$. A function $f \in H(\Delta)$ is said to belong to Q_T if

$$\|f\|_{Q_T}^2 := \sup_{a\in\Delta} \iint_\Delta |f'(z)|^2 T(g(z,a)) dA(z) < \infty. \tag{1.3}$$

If

$$\lim_{|a|\to 1} \iint_\Delta |f'(z)|^2 T(g(z,a)) dA(z) = 0, \tag{1.4}$$

then f is said to belong to $Q_{T,0}$.

For $0 < p < \infty$, if we take $T(r) = r^p$, the space Q_T coincides with the space Q_p. We note that $Q_T \subset \mathcal{B}$ for all nondecreasing functions T. We have previously shown that $Q_T = Q_p$ under certain growth conditions on $T(r)$ (cf. [10]).

In the present paper, first we give some basic properties of Q_T spaces, some of which are also new for the special case $Q_T = Q_p$. For example, Q_T is a Banach space with the norm $\|f\|_T$ defined by

$$\|f\|_T := |f(0)| + \|f\|_{Q_T}. \tag{1.5}$$

Then we investigate the boundedness of the composition operators from the Bloch space $\mathcal{B}$ into Q_T or $Q_{T,0}$. These results extend some previously known results (cf. [6, 8]).

2. Basic properties of Q_T spaces. We give the following propositions.

Proposition 2.1. *The space Q_T is a subspace of the Bloch space $\mathcal{B}$.*

The proof of Proposition 2.1 can be found in [10].

Proposition 2.2. *The space Q_T is a Banach space with the norm defined in (1.5).*

Proof. For $f \in Q_T$ and $a \in \Delta$, define

$$I^2(f,a) := \iint_\Delta |f'(z)|^2 T(g(z,a))\,dA(z). \tag{2.1}$$

Let $f_1, f_2 \in Q_T$. It follows from Schwarz's inequality that

$$\iint_\Delta |f_1'(z) f_2'(z)|\, T(g(z,a))\,dA(z) \le I(f_1,a) I(f_2,a), \tag{2.2}$$

and then

$$\begin{aligned} I^2(f_1+f_2,a) &\le I^2(f_1,a) + 2I(f_1,a)I(f_2,a) + I^2(f_2,a) \\ &= \big(I(f_1,a) + I(f_2,a)\big)^2. \end{aligned} \tag{2.3}$$

Thus, $I(f_1+f_2,a) \le I(f_1,a) + I(f_2,a)$ for all $a \in \Delta$. Hence

$$\|f_1+f_2\|_{Q_T} \le \|f_1\|_{Q_T} + \|f_2\|_{Q_T}. \tag{2.4}$$

Therefore,

$$\begin{aligned} \|f_1+f_2\|_T^2 &= \Big(|f_1(0)+f_2(0)| + \|f_1+f_2\|_{Q_T}\Big)^2 \\ &\le \Big(|f_1(0)| + |f_2(0)| + \|f_1\|_{Q_T} + \|f_2\|_{Q_T}\Big)^2 \\ &= \Big(\|f_1\|_T + \|f_2\|_T\Big)^2, \end{aligned} \tag{2.5}$$

that is, $\|f_1+f_2\|_T \le \|f_1\|_T+\|f_2\|_T$. On the other hand, it is obvious that $\|f\|_T \ge 0$ for each $f \in Q_T$ and that $\|f\|_T = 0$ if and only if $f \equiv 0$. It is obvious that $\|cf\|_T = |c|\|f\|$ for any constant c. Thus, Q_T is a normed space.

Let $f \in Q_T$ and let $\phi_a(w) = (a-w)/(1-\bar{a}w)$, $a \in \Delta$. Then by changing a variable $w = \phi_a(z)$, we obtain

$$\begin{aligned}\|f\|_{Q_T}^2 &\ge \iint_\Delta |f'(z)|^2 T(g(z,a))\,dA(z)\\ &= \iint_\Delta |(f\circ\phi_a)'(w)|^2 T\left(\log\frac{1}{|w|}\right)dA(w)\\ &\ge T\left(\log\frac{1}{r}\right)\iint_{|w|<r} |(f\circ\phi_a)'(w)|^2 dA(w)\\ &\ge \pi r^2 T\left(\log\frac{1}{r}\right)(1-|a|^2)^2|f'(a)|^2.\end{aligned} \tag{2.6}$$

For r_0, $0<r_0<1$, such that $T(\log(1/r_0)) \ne 0$, we have

$$\|f\|_b \le \frac{\|f\|_{Q_T}}{r_0(\pi T(\log 1/r_0))^{1/2}}. \tag{2.7}$$

Since $f \in Q_T \subset \mathcal{B}$, we have for $z \in \Delta$,

$$\begin{aligned}|f(z)| &\le |f(0)| + \frac{\|f\|_b}{2}\log\frac{1+|z|}{1-|z|}\\ &\le |f(0)| + \frac{\|f\|_{Q_T}}{2r_0(\pi T(\log(1/r_0)))^{1/2}}\log\frac{1+|z|}{1-|z|}\\ &\le \|f\|_T\left(1+\frac{1}{2r_0(\pi T(\log 1/r_0))^{1/2}}\right)\log\frac{1+|z|}{1-|z|}.\end{aligned} \tag{2.8}$$

Suppose $\{f_n\}$ is a Cauchy sequence in Q_T. Then there is a constant $M>0$ such that

$$\|f_n\|_T \le M, \quad n=1,2,\ldots. \tag{2.9}$$

By the estimate (2.8) for a fixed $r_0 \in (0,1)$, we obtain that

$$|f_n(z)| \le M\left(1+\frac{1}{2r_0(\pi T(\log 1/r_0))^{1/2}}\right)\log\frac{1+|z|}{1-|z|} \tag{2.10}$$

holds for all integral numbers $n=1,2,\ldots$. Hence, there exist a subsequence $\{f_{n_j}(z)\}$ of $\{f_n(z)\}$ and an analytic function f defined on the unit disk Δ such that both $\{f_{n_j}(z)\}$ and $\{f'_{n_j}(z)\}$ converge uniformly to f and f', respectively. The conditions here are such that both the sequence of functions and the sequence of derivatives converge since we know that $\{f_n(z)\}$ is bounded on

compact subsets of Δ by inequality (2.10). By Fatou's lemma, we get that

$$\begin{aligned}
&\iint_{\Delta} |f'(z)|^2 T(g(z,a))\,dA(z) \\
&\quad = \iint_{\Delta} \lim_{j\to\infty} |f'_{n_j}(z)|^2 T(g(z,a))\,dA(z) \\
&\quad \le \liminf_{j\to\infty} \iint_{\Delta} |f'_{n_j}(z)|^2 T(g(z,a))\,dA(z) \\
&\quad \le \liminf_{j\to\infty} \|f_{n_j}\|^2_{Q_T} \le M^2
\end{aligned} \tag{2.11}$$

holds for all $a \in \Delta$, so that $f \in Q_T$. By a similar reasoning, we can prove that $\|f_n - f\|_T \to 0$ as $n \to \infty$. The proof of Proposition 2.2 is complete. □

3. Boundedness of composition operators. Let $\varphi(z)$ be an analytic self-map of the unit disk Δ. Let the composition operator C_φ induced by φ from $H(\Delta)$ to itself be defined by $C_\varphi(f) = f \circ \varphi$ for $f \in H(\Delta)$. The boundednesses of composition operators from $\mathcal{B}$ to itself and from $\mathcal{B}$ to Q_p have been studied in [6, 8], respectively. In this paper, we consider the same problems for the general spaces Q_T.

Theorem 3.1. *Let $T(r) \not\equiv 0$ be a nonnegative, nondecreasing function on $0 \le r < \infty$ and let φ be an analytic self-map of Δ. Then $C_\varphi : \mathcal{B} \to Q_T$ is bounded if and only if*

$$\sup_{a\in\Delta} \iint_{\Delta} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) < \infty. \tag{3.1}$$

Proof. Let (3.1) hold and let $K_1^2 (K_1 > 0)$ be the supremum in (3.1). If $f \in \mathcal{B}$, then for all $a \in \Delta$, we have

$$\begin{aligned}
&\iint_{\Delta} |(C_\varphi f)'(z)|^2 T(g(z,a))\,dA(z) \\
&\quad = \iint_{\Delta} |f'(\varphi(z))|^2 |\varphi'(z)|^2 T(g(z,a))\,dA(z) \\
&\quad \le \|f\|_b^2 \iint_{\Delta} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) \\
&\quad \le K_1^2 \|f\|_b^2.
\end{aligned} \tag{3.2}$$

Consequently, $\|C_\varphi f\|_{Q_T} \le K_1 \|f\|_b$. Since $f(z) \in \mathcal{B}$, we obtain

$$\begin{aligned}
\|C_\varphi f\|_T^2 &= \left(|f \circ \varphi(0)| + \|C_\varphi f\|_{Q_T}\right)^2 \\
&\le \left(|f(0)| + \frac{\|f\|_b}{2} \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} + K_1 \|f\|_b\right)^2 \\
&\le K^2 \left(|f(0)| + \|f\|_b\right)^2 = K^2 \|f\|_{\mathcal{B}}^2,
\end{aligned} \tag{3.3}$$

where $K = \max\{1, K_1 + (1/2)\log(1+|\varphi(0)|)/(1-|\varphi(0)|)\}$. Thus, $\|C_\varphi f\|_T \le K\|f\|_{\mathcal{B}}$, which shows that $C_\varphi : \mathcal{B} \to Q_T$ is bounded.

Conversely, assume that $C_\varphi : \mathcal{B} \to Q_T$ is bounded, there exists a constant $K > 0$ such that for each $f \in \mathcal{B}$, we have

$$\|C_\varphi f\|_T \le K\|f\|_{\mathcal{B}}. \tag{3.4}$$

On the other hand, by a result in [7], there exist $f_1, f_2 \in \mathcal{B}$ such that

$$\frac{1}{1-|z|^2} \le |f_1'(z)| + |f_2'(z)| \tag{3.5}$$

holds for all $z \in \Delta$, so that

$$\frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} \le 2|(f_1\circ\varphi)'(z)|^2 + 2|(f_2\circ\varphi)'(z)|^2. \tag{3.6}$$

Thus, the inequalities

$$\begin{aligned}
&\iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) \\
&\quad \le 2\iint_\Delta \left(|(f_1\circ\varphi)'(z)|^2 + |(f_2\circ\varphi)'(z)|^2\right) T(g(z,a))\,dA(z) \\
&\quad \le 2K^2\left(\|f_1\|_{\mathcal{B}}^2 + \|f_2\|_{\mathcal{B}}^2\right)
\end{aligned} \tag{3.7}$$

hold for all $z, a \in \Delta$, which establishes (3.1). The proof of Theorem 3.1 is completed. □

Remark 3.2. Note that if $C_\varphi : \mathcal{B} \to \mathcal{B}$, then (3.1) holds for any increasing function T satisfying $Q_T = \mathcal{B}$. Indeed, we know that $Q_T = \mathcal{B}$ (see [5]) if and only if

$$\int_0^1 T\left(\log\left(\frac{1}{r}\right)\right)(1-r^2)^{-2} r\,dr < \infty. \tag{3.8}$$

The Schwarz-Pick lemma guarantees that $((1-|z|^2)/(1-|\varphi(z)|^2))|\varphi'(z)| \le 1$, so that (3.8) leads easily to (3.1). It means that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is always bounded (cf. [6]).

Remark 3.3. If one considers the composition operator C_φ from the Bloch space to the Dirichlet space

$$\mathcal{D} = \left\{ f \in H(\Delta) : \iint_\Delta |f'(z)|^2 dA(z) < \infty \right\}, \tag{3.9}$$

then $C_\varphi : \mathcal{B} \to \mathcal{D}$ is bounded if and only if

$$\iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} dA(z) < \infty. \tag{3.10}$$

For the spaces $Q_{T,0}$, we have the following results.

Theorem 3.4. *Let $T(r)$ be a nonnegative, nondecreasing function on $0 \le r < \infty$ and let φ be an analytic self-map of Δ. Then $C_\varphi : \mathcal{B} \to Q_{T,0}$ is bounded if and only if*

$$\lim_{|a|\to 1} \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) = 0. \tag{3.11}$$

Proof. Suppose $C_\varphi : \mathcal{B} \to Q_{T,0}$ is bounded. Using a way similar to the proof of Theorem 3.1, we choose functions $f_1, f_2 \in \mathcal{B}$ such that

$$\frac{1}{1-|z|^2} \le |f_1'(z)| + |f_2'(z)| \tag{3.12}$$

for all $z \in \Delta$. Then $C_\varphi f_1$ and $C_\varphi f_2$ belong to $Q_{T,0}$. Therefore,

$$\begin{aligned} &\lim_{|a|\to 1} \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) \\ &\quad \le 2 \lim_{|a|\to 1} \iint_\Delta \left(|(f_1\circ\varphi)'(z)|^2 + |(f_2\circ\varphi)'(z)|^2 \right) T(g(z,a))\,dA(z) = 0, \end{aligned} \tag{3.13}$$

which shows that (3.11) holds.

Conversely, by Theorem 3.1, we know that $C_\varphi : \mathcal{B} \to Q_T$ is bounded since condition (3.11) implies that

$$\sup_{a\in\Delta} \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) < \infty. \tag{3.14}$$

We need only to prove that $C_\varphi f \in Q_{T,0}$ for each $f \in \mathcal{B}$, and this follows from the inequality

$$\begin{aligned} &\iint_\Delta |(C_\varphi f)'(z)|^2 T(g(z,a))\,dA(z) \\ &\quad = \iint_\Delta |f'(\varphi(z))|^2 |\varphi'(z)|^2 T(g(z,a))\,dA(z) \\ &\quad \le \|f\|_b^2 \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z). \end{aligned} \tag{3.15}$$

The proof of Theorem 3.4 is completed. □

Acknowledgments. The essential part of this paper was written when the authors were at the University of Joensuu, Finland, in 1998. This research is now also supported in part by grants from the National Natural Science Foundation of China no. 10171058, Guangdong Province no. 010446, and Guizhou Province of China. We would like to thank the referee for helpful comments.

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Pengcheng Wu: Department of Mathematics, Guizhou Normal University, Guiyang 550001, China

E-mail address: wupc@gznu.edu.cn

Hasi Wulan: Department of Mathematics, Shantou University, Shantou, Guangdong 515063, China

E-mail address: wulan@stu.edu.cn