THE ADDITIVE APPROXIMATION ON A FOUR-VARIATE JENSEN-TYPE OPERATOR EQUATION

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Received 27 October 2002

We study the Hyers-Ulam stability theory of a four-variate Jensen-type functional equation by considering the approximate remainder \( \phi \) and obtain the corresponding error formulas. We bring to light the close relation between the \( \beta \)-homogeneity of the norm on \( F^{*} \)-spaces and the approximate remainder \( \phi \), where we allow \( p, q, r, \) and \( s \) to be different in their Hyers-Ulam-Rassias stability.

2000 Mathematics Subject Classification: 47A62, 39B72.

1. Introduction. Throughout this paper, we denote by \( G \) a linear space and by \( E \) a real or complex Hausdorff topological vector space. By \( \mathbb{N} \) and \( \mathbb{R} \) we denote the sets of positive integers and of reals, respectively. Let \( f \) be a mapping from \( G \) into \( E \). We refer to the equations

\[
2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \theta, \tag{1.1}
\]

\[
4f\left(\frac{x+y+z+w}{4}\right) + 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x+w}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+w}{2}\right) \\
-3f\left(\frac{x+y+z}{3}\right) - 3f\left(\frac{y+z+w}{3}\right) - 3f\left(\frac{z+w+x}{3}\right) - 3f\left(\frac{w+x+y}{3}\right) = \theta \tag{1.2}
\]

as a Jensen equation and a four-variate Jensen-type functional equation, respectively. The approximate remainder \( \phi \) is defined by

\[
4f\left(\frac{x+y+z+w}{4}\right) + 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x+w}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+w}{2}\right) \\
-3f\left(\frac{x+y+z}{3}\right) - 3f\left(\frac{y+z+w}{3}\right) - 3f\left(\frac{z+w+x}{3}\right) - 3f\left(\frac{w+x+y}{3}\right) \\
= \phi(x, y, z, w) \tag{1.3}
\]

for all \( x, y, z, w \in G \).
In 1940, the following problem was proposed (see Ulam [11]): let $G$ be a group and let $E$ be a metric group with the metry $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G \to E$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $H : G \to E$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G$?

In 1941, Hyers [2] answered this question in the affirmative when $G$ and $E$ are Banach spaces. In 1978, Rassias [6] generalized the result of Hyers. The result was further generalized by Rassias [7], Rassias and Šemrl [9], and Găvruţa [1].

The stability problems of Jensen equations can be found in [3, 4, 5].

The author [12] considered Hyers-Ulam-Rassias stability of several functional equations under the assumption that $G$ and $E$ are a power-associative groupoid and a sequentially complete topological vector space, respectively. In the following, we introduce [12, Theorem 4].

**Theorem 1.1.** The approximate remainder $\phi : G \times G \to E$ of Jensen equation (1.1) satisfies

$$\lim_{n \to \infty} \frac{\phi(3^n x, 3^n y)}{3^n} = 0 \quad \forall x, y \in G,$$

$$\sum_{k=1}^{\infty} \frac{\phi(3^{k-1} x, -3^{k-1} x) - \phi(-3^{k-1} x, 3^k x)}{3^k} = \eta(x) \in E \quad \forall x \in G$$

if and only if the limit $T(x) = \lim_{n \to \infty} f(3^n x) / 3^n$ exists for all $x \in G$, and $T$ is additive, where $G$ is a real linear space and $E$ is a real Hausdorff topological vector space. In addition,

$$T(x) - f(x) + f(\theta) = \eta(x) \quad \forall x \in G.$$

Trif [10] investigated the Hyers-Ulam-Rassias stability of the three-variate Jensen-type functional equation

$$3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) = 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+x}{2}\right)$$

under the assumption that $G$ and $E$ are a real normed linear space and a real Banach space, respectively.

In this paper, we investigate the Hyers-Ulam stability of (1.2) by considering the approximate remainders under the assumption that $G$ and $E$ are a real linear space and a certain kind of $F^\ast$-space, respectively. First we solve (1.2) in Section 2. Second, in Section 3, still using the direct method, we obtain some theorems of the Hyers-Ulam stability of (1.2). Finally, we give an example that the Hyers-Ulam-Rassias stability of (1.2) does not hold.
2. Solutions of (1.2). From now we let $G$ be a real linear space and $E$ a real Hausdorff topological vector space, unless otherwise specified. In this section, we claim that (1.2) is equivalent to (1.1). It is well known that if $G$ and $E$ are real linear spaces, then a function $f : G \to E$ satisfying $f(\theta) = \theta$ is a solution of (1.1) if and only if it is additive.

**Theorem 2.1.** A function $f : G \to E$ satisfies (1.2) for all $x, y, z, w \in G$ if and only if there exist a constant element $C \in E$ and a unique additive mapping $T : G \to E$ such that

$$f(x) = T(x) + C \quad \forall x \in G. \quad (2.1)$$

**Proof.** The proof of the sufficiency is straightforward, so we will show only the necessity. Set $C = f(\theta)$ and $T(x) = f(x) - C$ for each $x \in G$. Then $T(\theta) = \theta$ and

$$4T\left(\frac{x + y + z + w}{4}\right) + 2T\left(\frac{x + y}{2}\right) + 2T\left(\frac{x + w}{2}\right) + 2T\left(\frac{y + z}{2}\right) + 2T\left(\frac{z + w}{2}\right)$$

$$= 3T\left(\frac{x + y + z}{3}\right) + 3T\left(\frac{y + z + w}{3}\right) + 3T\left(\frac{z + w + x}{3}\right) + 3T\left(\frac{w + x + y}{3}\right) \quad (2.2)$$

for any $x, y, z, w \in G$. We will show that $T$ is additive. Let $x \in G$. Put $y = x$ and $z = w = -x$ in (2.2) to yield

$$T(x) + T(-x) = 3\left[T\left(\frac{x}{3}\right) + T\left(\frac{-x}{3}\right)\right]. \quad (2.3)$$

Take $y = -x$ and $z = w = \theta$ in (2.2) to get

$$2\left[T\left(\frac{x}{2}\right) + T\left(\frac{-x}{2}\right)\right] = 3\left[T\left(\frac{x}{3}\right) + T\left(\frac{-x}{3}\right)\right]. \quad (2.4)$$

From (2.3) and the last equality, we obtain

$$T(x) + T(-x) = 2\left[T\left(\frac{x}{2}\right) + T\left(\frac{-x}{2}\right)\right]. \quad (2.5)$$

Putting $y = x$, $z = -2x$, and $w = \theta$ in (2.2) gives

$$2[T(x) + T(-x)] + 2\left[T\left(\frac{x}{2}\right) + T\left(\frac{-x}{2}\right)\right] = 6T\left(\frac{-x}{3}\right) + 3T\left(\frac{2x}{3}\right). \quad (2.6)$$
From (2.5) and the last equality, we have
\[
T(x) + T(-x) = 2T\left(-\frac{x}{3}\right) + T\left(\frac{2x}{3}\right). \tag{2.7}
\]

Put \(y = z = x\) and \(w = -3x\) in (2.2) to conclude that
\[
T(x) + 4T(-x) = 9T\left(-\frac{x}{3}\right). \tag{2.8}
\]
Replacing \(x\) by \(-x\) in the above equality, we have
\[
T(-x) + 4T(x) = 9T\left(\frac{x}{3}\right). \tag{2.9}
\]
Adding the last two formulas together produces
\[
5[T(x) + T(-x)] = 9\left[T\left(\frac{x}{3}\right) + T\left(-\frac{x}{3}\right)\right]. \tag{2.10}
\]

Hence, from (2.3) and the last equality, we conclude that
\[
T(x) + T(-x) = \theta, \quad \text{that is,} \quad T(-x) = -T(x). \tag{2.11}
\]

It follows from (2.7), (2.9), and (2.11) that
\[
T\left(\frac{x}{3}\right) = \frac{1}{3}T(x), \quad T\left(\frac{2x}{3}\right) = 2T\left(\frac{x}{3}\right). \tag{2.12}
\]
Replacing \(x/3\) by \(x\) in the last equality, we obtain
\[
T(2x) = 2T(x), \quad \text{that is,} \quad T\left(\frac{x}{2}\right) = \frac{1}{2}T(x), \tag{2.13}
\]
and so, \(T(x/4) = (1/4)T(x)\). Substituting
\[
T\left(\frac{x}{2}\right) = \frac{1}{2}T(x), \quad T\left(\frac{x}{3}\right) = \frac{1}{3}T(x), \quad T\left(\frac{x}{4}\right) = \frac{1}{4}T(x) \tag{2.14}
\]
into (2.2) supplies
\[
T(x + y + z + w) + T(x + y) + T(x + w) + T(y + z) + T(z + w)
= T(x + y + z) + T(y + z + w) + T(z + w + x) + T(w + x + y). \tag{2.15}
\]
Finally, we take \(z = -x - y\) and \(w = \theta\) in the above equality to get from (2.11) that \(T(x + y) = T(x) + T(y)\), and so, \(T\) is additive in terms of the arbitrariness of \(x\) and \(y\).
3. Hyers-Ulam-Rassias stability of (1.2). Next we are interested in the
Hyers-Ulam stability of (1.2). For convenience, we set \( \varphi(x, y) = \phi(x, y, x, y) \)
for all \( x, y \in G \), where \( \phi \) is of (1.3).

**Theorem 3.1.** The map \( \varphi : G \times G \to E \) satisfies

\[
\lim_{n \to \infty} \frac{\varphi(3^n x, 3^n y)}{3^n} = \theta \quad \forall x, y \in G, \tag{3.1}
\]

\[
\frac{1}{2} \sum_{k=1}^{\infty} \frac{\varphi(3^k x, -3^k x) - \varphi(-3^{k-1}(5x), 3^{k-1}(7x))}{3^{k+1}} = \eta(x) \in E \quad \forall x \in G \tag{3.2}
\]

if and only if the limit \( T(x) = \lim_{n \to \infty} f(3^n x)/3^n \) exists for all \( x \in G \), and \( T \) is
additive. In this case (1.5) holds.

**Proof.** We omit the easy proof of sufficiency and, like Theorem 2.1, we will
show the necessity only. Let any \( x, y \in G \). Putting \( z = x \) and \( w = y \) in (1.3), we get

\[
2f\left(\frac{x+y}{2}\right) - f\left(\frac{2x+y}{3}\right) - f\left(\frac{x+2y}{3}\right) = \frac{1}{6} \varphi(x, y). \tag{3.3}
\]

Let \( u, v \in G \), \( x = 2u - v \), and \( y = -u + 2v \). Then \( u = (2x+y)/3, v = (x + 2y)/3 \), and \( x + y = u + v \), and so we have

\[
2f\left(\frac{u+v}{2}\right) - f(u) - f(v) = \Phi(u, v), \tag{3.4}
\]

where \( \Phi(u, v) \xrightarrow{\text{def}} (1/6)\varphi(2u-v, -u+2v) \).

On the one hand, clearly,

\[
\lim_{n \to \infty} \frac{\Phi(3^n u, 3^n v)}{3^n} - \frac{1}{6} \lim_{n \to \infty} \frac{\varphi(3^n (2u-v), 3^n (-u+2v))}{3^n}. \tag{3.5}
\]

This yields from assumption (3.1) that

\[
\lim_{n \to \infty} \frac{\Phi(3^n u, 3^n v)}{3^n} = \theta. \tag{3.6}
\]

On the other hand, using the definition of \( \Phi(u, v) \), we compute

\[
\Phi(3^{k-1} u, 3^{k-1} u) = \frac{1}{6} \varphi(3^{k-1} u, -3^{k-1} u),
\]

\[
\Phi(-3^{k-1} u, 3^k u) = \frac{1}{6} \varphi(-3^{k-1}(5u), 3^{k-1}(7u)), \tag{3.7}
\]
then we conclude from (3.2) that
\[
\sum_{k=1}^{\infty} \frac{\Phi(3^{k-1}u, -3^{k-1}u) - \Phi(-3^{k-1}u, 3^k u)}{3^k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\varphi(3^k u, -3^k u) - \varphi(-3^{k-1}(5u), 3^{k-1}(7u))}{3^{k+1}} = \eta(u) \in E.
\]

(3.8)

Thus, by Theorem 1.1, the limit \( T(u) = \lim_{n \to \infty} f(3^n u)/3^n \) exists, \( T \) is additive, and the equality \( T(u) - f(u) + f(\theta) = \eta(u) \) holds for each \( u \in G \).

The proof is complete. \( \square \)

For abbreviation, we set
\[
B(x, -x) = \text{co} \left( \{ \theta \} \cup \{ \varphi(3^i x, -3^i x) \}_{i=1}^{\infty} \right) \quad \forall \, x \in G,
\]
\[
B(-5x, 7x) = \text{co} \left( \{ \theta \} \cup \{ \varphi(-3^{i-1}(5x), 3^{i-1}(7x)) \}_{i=1}^{\infty} \right) \quad \forall \, x \in G.
\]

(3.9)

By Theorem 3.1 and [12, Corollary 6], we conclude the following corollary.

**Corollary 3.2.** Let \( E \) be sequentially complete and let (3.1) hold. If \( B(x, -x) \) and \( B(-5x, 7x) \) are bounded for any \( x \in G \), then there exists a unique additive mapping \( T: G \to E \) such that
\[
T(x) - f(x) + f(\theta) \in \frac{1}{6} [\overline{B^s}(x, -x) - \overline{B^s}(-5x, 7x)] \quad \forall \, x \in G,
\]

(3.10)

where \( \text{co}(A) \) is the convex hull of a set \( A \), and \( \overline{A} \) denotes the sequential closure of set \( A \). If \( E \) is also locally convex, then the boundedness of \( \{ \varphi(3^i x, -3^i x) \}_{i=1}^{\infty} \) and \( \{ \varphi(-3^{i-1}(5x), 3^{i-1}(7x)) \}_{i=1}^{\infty} \) ensures the boundedness of \( B(x, -x) \) and \( B(-5x, 7x) \), respectively.

Next we derive the Hyers-Ulam-Rassias stability of (1.2), which is an application of Theorem 3.1. Note that it is close correlative with the \( \beta \)-homogeneity of the norm on \( F^* \)-spaces. Simultaneously, we allow \( p, q, r, \) and \( s \) to be different.

Let \( X \) be a linear space. A nonnegative-valued function \( \| \cdot \| \) defined on \( X \) is called an \( F \)-norm if it satisfies the following conditions:

(\text{n1}) \( \| x \| = 0 \) if and only if \( x = 0 \);
(\text{n2}) \( \| ax \| = |a| \| x \| \) for all \( a, \ |a| = 1 \);
(\text{n3}) \( \| x + y \| \leq \| x \| + \| y \| \);
(\text{n4}) \( \| a_n x \| \to 0 \) provided \( a_n \to 0 \);
(\text{n5}) \( \| ax_n \| \to 0 \) provided \( x_n \to 0 \).

A space \( X \) with an \( F \)-norm is called an \( F^* \)-space. An \( F \)-pseudonorm (\( \| x \| = 0 \) does not necessarily imply that \( x = 0 \) in (\text{n1})) is called \( \beta \)-homogeneous \((\beta > 0)\) if \( \| tx \| = |t|^\beta \| x \| \) for all \( x \in X \) and all \( t \in \mathbb{R} \). A complete \( F^* \)-space is said to be an \( F \)-space.
Corollary 3.3. Suppose that $G$ is an $F^*$-space and $E$ a $β$-homogeneous $F$-space ($0 < β ≤ 1$). Given $ε_1, ε_2, ε_3, ε_4, δ ≥ 0$ and $0 ≤ p, q, r, s < β$, if $φ$ satisfies

$$
\|φ(x, y, z, w)\| ≤ δ + ε_1 \|x\|^p + ε_2 \|y\|^q + ε_3 \|z\|^r + ε_4 \|w\|^s \quad ∀x, y, z, w ∈ G,
$$

(3.11)

then there exists a unique additive mapping $T : G → E$ such that

$$
\|T(x) − f(x) + f(θ)\| ≤ Aδ + ε_1 B_1 \|x\|^p + ε_2 B_2 \|x\|^q + ε_3 B_3 \|x\|^r + ε_4 B_4 \|x\|^s
$$

(3.12)

for all $x ∈ G$, where

$$
A = \frac{2}{6β(3β - 1)}, \quad B_1 = \frac{(3^p + 5^p)}{6β(3β - 3p)}, \quad B_2 = \frac{(3^q + 7^q)}{6β(3β - 3q)},
$$

$$
B_3 = \frac{(3^r + 5^r)}{6β(3β - 3r)}, \quad B_4 = \frac{(3^s + 7^s)}{6β(3β - 3s)}.
$$

(3.13)

Proof. Let any $x, y ∈ G$. Firstly, put $z = x$ and $w = y$ in (3.11) to get according to the definition of $φ$ that

$$
\|φ(x, y)\| = \|φ(x, y, x, y)\| ≤ δ + ε_1 \|x\|^p + ε_2 \|y\|^q + ε_3 \|x\|^r + ε_4 \|y\|^s \quad ∀x, y ∈ G.
$$

(3.14)

It follows from $p, q, r, s < β$ that

$$
\lim_{n → ∞} \left[ \frac{δ}{3^n} + \frac{ε_1}{3^n(β - p)} \|x\|^p + \frac{ε_2}{3^n(β - q)} \|y\|^q + \frac{ε_3}{3^n(β - r)} \|x\|^r + \frac{ε_4}{3^n(β - s)} \|y\|^s \right] = 0.
$$

(3.15)

Secondly, in light of the triangle inequality of $F$-norm and $p, q, r, s ≥ 0$, we have, for any $i ∈ N$,

$$
\|φ(3^i x, 3^i y)\| ≤ δ + ε_1 3^{i p} \|x\|^p + ε_2 3^{i q} \|y\|^q + ε_3 3^{i r} \|x\|^r + ε_4 3^{i s} \|y\|^s,
$$

$$
\|φ(-3^{i-1}(5x), 3^{i-1}(7x))\| ≤ δ + ε_1 3^{i p} 5^p \|x\|^p + ε_2 3^{i q} 7^q \|y\|^q + ε_3 3^{i r} 5^r \|x\|^r + ε_4 3^{i s} 7^s \|y\|^s.
$$

(3.16)

As in the proof of [12, Theorem 3], we infer from (3.4) that

$$
\frac{1}{3^i n} f(3^n x) = \sum_{k=1}^{n} \Psi_k(3^{k-1} x) \frac{1}{3^k}
$$

holds for any $n ∈ N$, where $Ψ(x) = Φ(x, -x) − Φ(-x, 3x) − 2f(θ)$. 

(3.17)
Consequently, for any \( n \in \mathbb{N} \),
\[
\frac{1}{3^n} f(3^n x) - f(x) + 2 \sum_{k=1}^{n} \frac{f(\theta)}{3^k}
= \sum_{k=1}^{n} \frac{\Phi(3^{k-1}x, -3^{k-1}x) - \Phi(-3^{k-1}x, 3^k x)}{3^k}
= \frac{1}{2} \sum_{k=1}^{n} \varphi(3^k x, -3^k x) - \varphi(-3^{k-1}(5x), 3^{k-1}(7x))
\]
\[(3.18)\]

It is easy to see that
\[
\frac{1}{2} \sum_{k=1}^{\infty} \varphi(3^k x, -3^k x) - \varphi(-3^{k-1}(5x), 3^{k-1}(7x))
\]
\[(3.19)\]
exists for every \( x \in G \). Indeed, from the above, we conclude that
\[
f(3^m x) - f(3^n x)
= \frac{1}{3^n} \left[ f(3^{m-n}(3^n x)) - f(3^n x) \right]
= \frac{1}{3^n} \sum_{k=1}^{m-n} \frac{\psi(3^{n+k-1} x)}{3^k}
= \sum_{k=n+1}^{m} \frac{\psi(3^{k-1} x)}{3^k}
= \frac{1}{2} \sum_{k=n+1}^{m} \varphi(3^k x, -3^k x) - \varphi(-3^{k-1}(5x), 3^{k-1}(7x)) - 2 \sum_{k=n+1}^{m} \frac{f(\theta)}{3^k}
\]
\[(3.20)\]
for any \( m > n \), where \( m, n \in \mathbb{N} \), and so
\[
\left\| \frac{f(3^m x) - f(3^n x)}{3^m} \right\|
\leq \frac{1}{2^\beta} \sum_{k=n+1}^{m} 2 \delta + \epsilon_1 (3^{k(p + (k-1)p) 5^p}) \|x\|_p + \epsilon_2 (3^{k(q + (k-1)q) 7^q}) \|x\|_q
+ \frac{1}{2^\beta} \sum_{k=n+1}^{m} \epsilon_3 (3^{k r + (k-1) r 5^r}) \|x\|_r + \epsilon_4 (3^{k s + (k-1) s 7^s}) \|x\|_s
+ 2 \|f(\theta)\| \sum_{k=n+1}^{m} \frac{1}{3^k}
\leq \sum_{k=n+1}^{m} \frac{2^{1-\beta} \delta}{3^{(k+1)\beta}} \sum_{k=n+1}^{m} \left[ \frac{1}{3^\beta} 3^{k(p-\beta)} + \frac{5^p}{3^{2\beta}} 3^{(k-1)(p-\beta)} \right] \|x\|_p
+ \frac{\epsilon_2}{2^\beta} \sum_{k=n+1}^{m} \left[ \frac{1}{3^\beta} 3^{k(q-\beta)} + \frac{7^q}{3^{2\beta}} 3^{(k-1)(q-\beta)} \right] \|x\|_q
for any $m > n$, where $m, n \in \mathbb{N}$. Since $p, q, r, s < \beta$, \( \{ f(3^n x)/3^n \} \) is a Cauchy sequence of $E$. By the completeness of $E$, \( \{ f(3^n x)/3^n \} \) converges to an element of $E$.

Thus, by Theorem 3.1, \( T(x) = \lim_{n \to \infty} (f(3^n x)/3^n) \) and it is additive. In addition, from (3.18), inequality (3.12) holds for all $x \in G$.

In order to prove the uniqueness of $T$, suppose that $U : G \to E$ is another additive mapping which satisfies

\[
\|U(x) - f(x) + f(\theta)\| \leq A\delta + \varepsilon_1 B_1 \|x\|^p + \varepsilon_2 B_2 \|x\|^q + \varepsilon_3 B_3 \|x\|^r + \varepsilon_4 B_4 \|x\|^s
\]  

(3.22)

for all $x \in G$. On account of the last two inequalities, we conclude that, for all $x \in G$,

\[
\|U(x) - T(x)\| = \frac{1}{n^\beta} \|U(nx) - T(nx)\|
\]

\[
= \frac{1}{n^\beta} \|U(nx) - f(nx) + f(\theta) - T(nx) + f(n x) - f(\theta)\|
\]

\[
\leq \frac{1}{n^\beta} \left( \|U(nx) - f(nx) + f(\theta)\| + \|T(nx) - f(nx) + f(\theta)\| \right)
\]

(3.23)

\[
\leq \frac{2}{n^\beta} \left( A\delta + \varepsilon_1 B_1 \|nx\|^p + \varepsilon_2 B_2 \|nx\|^q + \varepsilon_3 B_3 \|nx\|^r + \varepsilon_4 B_4 \|nx\|^s \right)
\]

\[
= 2 \left[ \frac{A\delta}{n^\beta} + \frac{\varepsilon_1 B_1}{n^{\beta - p}} \|x\|^p + \frac{\varepsilon_2 B_2}{n^{\beta - q}} \|x\|^q + \frac{\varepsilon_3 B_3}{n^{\beta - r}} \|x\|^r + \frac{\varepsilon_4 B_4}{n^{\beta - s}} \|x\|^s \right],
\]

and so, $\|U(x) - T(x)\| \to 0$ as $n \to \infty$ since $p, q, r, s < \beta$. As a consequence, $U(x) = T(x)$ for all $x \in G$.

Therefore, the result holds.

In order to show that Corollary 3.3 is valid in the case that $p, q, r, s > 1/\beta$, we need the following theorem, which can be proved in the same manner as Theorem 1.1.
Theorem 3.4. The approximate remainder $\phi : G \times G \to E$ of (1.1) satisfies

$$\lim_{n \to \infty} 3^n \phi(3^{-n}x, 3^{-n}y) = \theta \quad \forall x, y \in G,$$

$$\sum_{k=1}^{\infty} 3^{k-1} [\phi(3^{-k}x, -3^{-k}x) - \phi(-3^{-k}x, 3^{-k+1}x)] = \eta(x) \in E \quad \forall x \in G$$  \hfill (3.24)

if and only if the limit $T(x) = \lim_{n \to \infty} 3^n [f(3^{-n}x) - f(\theta)]$ exists for all $x \in G$, and $T$ is additive. In this case (1.5) holds.

Proof. Note that if set $g(x) = f(x) - f(\theta)$ for any $x \in G$, then $g(\theta) = 0$ and the approximate remainders $\phi_g$ and $\phi_f$ of (1.1) with respect to $g$ and $f$, respectively, are equal. We still write it as $\phi$. As in the proof of Theorem 1.1, we can conclude that, for every $x \in G$ with $x \neq 0$ and every $n \in \mathbb{N}$,

$$g(x) - 3^n (3^{-n}x) = \sum_{k=1}^{n} [\phi(3^{-k}x, -3^{-k}x) - \phi(-3^{-k}x, 3^{-k+1}x)].$$  \hfill (3.25)

We may see that it is possible that $T(x) = \lim_{n \to \infty} 3^n [f(3^{-n}x) - f(\theta)]$ exists, in particular, if $f$ is differentiable at $\theta$ in $G$.

Corollary 3.5. Suppose that $G$ is a $\beta$-homogeneous $F^*$-space ($0 < \beta \leq 1$) and $E$ an $F$-space with a nondecreasing $F$-norm. Given $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in [0, +\infty)$ and $p, q, r, s \in (1/\beta, +\infty)$, if $\phi$ satisfies

$$||\phi(x, y, z, w)|| \leq \epsilon_1 ||x||^p + \epsilon_2 ||y||^q + \epsilon_3 ||z||^r + \epsilon_4 ||w||^s \quad \forall x, y, z, w \in G,$$  \hfill (3.26)

then there exists a unique additive mapping $T : G \to E$ such that

$$||T(x) - f(x) + f(\theta)|| \leq \epsilon_1 B_1 ||x||^p + \epsilon_2 B_2 ||x||^q + \epsilon_3 B_3 ||x||^r + \epsilon_4 B_4 ||x||^s \quad \forall x \in G,$$  \hfill (3.27)

where

$$B_1 \overset{\text{def}}{=} \frac{(3p^\beta + 5p^\beta)}{(3p^\beta - 3)}, \quad B_2 \overset{\text{def}}{=} \frac{(3q^\beta + 7q^\beta)}{(3q^\beta - 3)},$$

$$B_3 \overset{\text{def}}{=} \frac{(3r^\beta + 5r^\beta)}{(3r^\beta - 3)}, \quad B_4 \overset{\text{def}}{=} \frac{(3s^\beta + 7s^\beta)}{(3s^\beta - 3)}.$$  \hfill (3.28)
**Proof.** Let \( g(x) = f(x) - f(\theta) \) for any \( x \in G \). Using Theorem 3.4, as in the proofs of Theorem 3.1 and Corollary 3.3, we can achieve that

\[
\frac{1}{6} \sum_{k=1}^{\infty} 3^{k-1} [\phi(3^{-k+1}x, -3^{-k+1}x) - \phi(-3^{-k}(5x), 3^{-k}(7x))] \tag{3.29}
\]

exists for every \( x \in G \) and

\[
g(x) - 3^n g(3^{-n}x) = \frac{1}{6} \sum_{k=1}^{n} 3^{k-1} [\phi(3^{-k+1}x, -3^{-k+1}x) - \phi(-3^{-k}(5x), 3^{-k}(7x))]. \tag{3.30}
\]

Finally, we can evaluate the error formula.

We may also deal with the Hyers-Ulam stability of (1.2) as usual.

**Theorem 3.6.** The approximate remainder \( \phi \) satisfies

\[
\lim_{n \to \infty} \frac{\phi(3^n x, 3^n y, 3^n z, 3^nw)}{3^n} = \theta \quad \forall x, y, z, w \in G, \tag{3.31}
\]

\[
\sum_{k=1}^{\infty} \frac{\psi(3^k x)}{3^k} = \eta(x) \in E \quad \forall x \in G \tag{3.32}
\]

if and only if the limit \( T(x) = \lim_{n \to \infty} f(3^n x)/3^n \) exists for all \( x \in G \), and \( T \) is additive. Moreover, (1.5) holds, where

\[
\psi(x) \overset{\text{def}}{=} \frac{1}{4} \phi(x, x, -x, -x) + \frac{1}{6} [\phi(-x, -x, -x, 3x) - \phi(x, x, x, -3x)]. \tag{3.33}
\]

**Proof.** It is enough to show the necessity. Define \( g \) as above. Let any \( x \in G \). Put \( y = x \) and \( z = w = -x \) in (1.3) to yield

\[
g(x) + g(-x) - 3 \left[ g \left( \frac{x}{3} \right) + g \left( -\frac{x}{3} \right) \right] = \frac{1}{2} \phi(x, x, -x, -x). \tag{3.34}
\]

Put \( y = z = x \) and \( w = -3x \) in (1.3) to give

\[
g(x) + 4g(-x) - 9g \left( -\frac{x}{3} \right) = \phi(x, x, x, -3x). \tag{3.35}
\]
Replacing \( x \) by \(-x\) in the above equality, we have
\[
g(-x) + 4g(x) - 9g\left(\frac{x}{3}\right) = \phi(-x,-x,-x,3x). \tag{3.36}
\]

Adding the last two formulas together, we conclude that
\[
5[g(x) + g(-x)] - 9\left[g\left(\frac{x}{3}\right) + g\left(-\frac{x}{3}\right)\right] = \phi(x,x,x,-3x) + \phi(-x,-x,-x,3x). \tag{3.37}
\]

Hence, from (3.34) and the above equality, we know that
\[
g(x) + g(-x) = \frac{1}{2}\left[\phi(x,x,x,-3x) + \phi(-x,-x,-x,3x)\right] - \frac{3}{4}\phi(x,x,-x,-x). \tag{3.38}
\]

It follows from (3.36) and (3.38) that
\[
g(x) - 3g\left(\frac{x}{3}\right) = \frac{1}{4}\phi(x,x,-x,-x)
+ \frac{1}{6}\left[\phi(-x,-x,-x,3x) - \phi(x,x,x,-3x)\right]
= \psi(x). \tag{3.39}
\]

With \(3x\) in place of \(x\) in the above equality and dividing by 3, we obtain
\[
\frac{1}{3}g(3x) - g(x) = \frac{1}{3}\psi(3x). \tag{3.40}
\]

We will prove by induction that
\[
\frac{1}{3^n}g(3^nx) - g(x) = \sum_{k=1}^{n} \psi\left(\frac{3^kx}{3^k}\right) \forall n \in \mathbb{N}. \tag{3.41}
\]

For \(n = 1\) this is trivial according to (3.40). Suppose that (3.41) holds for a certain \(m - 1\). Then (3.40) and the induction hypothesis imply that
\[
\frac{1}{3m}g(3^mx) - g(x) = \frac{1}{3}\left[\frac{1}{3^{m-1}}g(3^{m-1}(3x)) - g(3x)\right] + \frac{1}{3}g(3x) - g(x)
= \frac{1}{3} \sum_{k=1}^{m-1} \psi\left(\frac{3^k(3x)}{3^k}\right) + \frac{1}{3} \psi(3x) = \sum_{k=1}^{m} \psi\left(\frac{3^kx}{3^k}\right),
\]
that is, (3.41) holds for \(n = m\).
We define \( T(x) = \lim_{n \to \infty} g(3^n x)/3^n \). Obviously, \( T(x) = \lim_{n \to \infty} f(3^n x)/3^n \), and so, by (3.32) and (3.41), \( T(x) \) exists and

\[
T(x) - g(x) = \eta(x). \tag{3.43}
\]

Substituting the definition of \( g \) into the last equality implies that

\[
T(x) - f(x) + f(\theta) = \eta(x). \tag{3.44}
\]

Finally, we verify that \( T \) is additive. Indeed, the definition of \( T \) implies that

\[
T(\theta) = \lim_{n \to \infty} g(3^n \theta)/3^n = \theta. \tag{3.45}
\]

Because of (3.31), \( T \) is a solution of (1.2). Hence \( T(x) = T^*(x) + T(\theta) = T^*(x) \) by Theorem 2.1, where \( T^* \) is additive. It follows that \( T \) is additive.

To show the following corollary, we may use a manner analogous to that used in Corollary 3.3.

**Corollary 3.7.** Keeping all the hypotheses of Corollary 3.3, there exists a unique additive mapping \( T : G \to E \) such that (3.12) holds, where

\[
A \defeq \frac{3^\beta + 2^{\beta+1}}{12^\beta(3^\beta - 1)}, \quad B_1 \defeq \frac{3^p (3^\beta + 2^{\beta+1})}{12^\beta(3^\beta - 3^p)}, \quad B_2 \defeq \frac{3^q (3^\beta + 2^{\beta+1})}{12^\beta(3^\beta - 3^q)},
\]

\[
B_3 \defeq \frac{3^r (3^\beta + 2^{\beta+1})}{12^\beta(3^\beta - 3^r)}, \quad B_4 \defeq \frac{3^s (3^\beta + 2^{\beta+1} 3^s)}{12^\beta(3^\beta - 3^s)}. \tag{3.46}
\]

If there exists at least one of \( p, q, r, \) and \( s \) such that it is strictly less than 0, it is supposed that (3.11) holds for all \( x, y, z, w \in G \setminus \{\theta\} \). Then the domain of \( T \) is \( G \setminus \{\theta\} \) instead of \( G \).

As earlier, we consider the case of \( p, q, r, s > 1/\beta \).

**Theorem 3.8.** The approximate remainder \( \phi \) satisfies

\[
\lim_{n \to \infty} 3^n \phi(3^{-n} x, 3^{-n} y, 3^{-n} z, 3^{-n} w) = \theta \quad \forall x, y, z, w \in G,
\]

\[
\sum_{k=1}^{\infty} 3^{k-1} \psi(3^{-(k-1)} x) = \eta(x) \in E \quad \forall x \in G \tag{3.47}
\]
if and only if the limit \( T(x) = \lim_{n \to \infty} 3^n[f(3^{-n}x) - f(\theta)] \) exists for all \( x \in G \), and \( T \) is additive, where \( \psi \) is as above. Moreover,

\[
T(x) - f(x) + f(\theta) = \eta(x) \quad \forall x \in G.
\] (3.48)

**Proof.** Let \( g(x) = f(x) - f(\theta) \). Note that, by virtue of (3.39), we conclude by induction that

\[
g(x) - 3^ng(3^{-n}x) = \sum_{k=1}^{n} 3^{k-1}\psi(3^{-(k-1)}x) \quad \forall x \in G, \ n \in \mathbb{N}.
\] (3.49)

**Corollary 3.9.** Keeping all the hypotheses of Corollary 3.5, then there exists a unique additive mapping \( T: G \to E \) such that

\[
\|T(x) - f(x)\| \leq \varepsilon_1 B_1 \|x\|^{p} + \varepsilon_2 B_2 \|x\|^{q}
+ \varepsilon_3 B_3 \|x\|^{r} + \varepsilon_4 B_4 \|x\|^{s} \quad \forall x \in G,
\] (3.50)

where

\[
B_1 = \frac{3^{p\beta+1}}{(3^{p\beta} - 3)}, \quad B_2 = \frac{3^{q\beta+1}}{(3^{q\beta} - 3)},
B_3 = \frac{3^{r\beta+1}}{(3^{r\beta} - 3)}, \quad B_4 = \frac{3^{s\beta}(1+2(3^{s\beta}))}{(3^{s\beta} - 3)}.
\] (3.51)

We still mention the following immediate consequence of Corollary 3.3.

**Remark 3.10.** Let \( E \) be a \( \beta \)-homogeneous \( F \)-space (\( 0 < \beta \leq 1 \)). If \( \phi \) satisfies the property that there exists \( \delta \in [0, \infty) \) such that \( \|\phi(x,y,z,w)\| \leq \delta \) for any \( x,y,z,w \in G \), then there exists a unique additive mapping \( T: G \to E \) such that

\[
\|T(x) - f(x) + f(\theta)\| \leq \frac{2\delta}{6^{\beta}(3^{\beta} - 1)} \quad \forall x \in G.
\] (3.52)

As in [13], in the last of this section we give an example by means of Rassias and Šemrl [8] who constructed a function \( f: \mathbb{R} \to \mathbb{R} \) (\( f(x) = x \log_2(1 + |x|) \)) to show that (1.2) does not have Hyers-Ulam-Rassias stability property if \( p, q, r, \) and \( s \) satisfy any one condition of (\( \Delta_1 \)) \( p = q = r = s = \beta \), (\( \Delta_2 \)) \( p = q = r = s = 1/\beta \), and (\( \Delta_3 \)) \( \beta \leq p = q = r = s = 1 \leq 1/\beta \) (\( 0 < \beta \leq 1 \)). What if \( p, q, r, \) and \( s \) satisfy that \( \beta \leq p, q, r, s \leq 1/\beta \), where \( p \neq 1, q \neq 1, r \neq 1, \) and \( s \neq 1 \) under the assumption that \( G \) and \( E \) are \( \beta \)-homogeneous \( F \)-space (\( 0 < \beta < 1 \))?
THEOREM 3.11. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) \overset{\text{def}}{=} x \log_2(1 + |x|)$ satisfies the inequality

$$|\phi(x, y, z, w)| \leq 14(|x| + |y| + |z| + |w|) \quad \forall x, y, z, w \in \mathbb{R}, \quad (3.53)$$

but

$$\sup \left\{ \left| \frac{f(x) - T(x)}{x} \right| : x \in \mathbb{R} \setminus \{0\} \right\} = \infty \quad (3.54)$$

for each additive mapping $T : \mathbb{R} \to \mathbb{R}$.

PROOF. For all $x, y, z, w \in \mathbb{R}$, it follows from $|f(x + y) - f(x) - f(y)| \leq |x| + |y|$ in [8] and $|f(x + y + z) - f(x) - f(y) - f(z)| \leq (5/3)(|x| + |y| + |z|)$ in [10] that

$$\phi(x, y, z, w) = \left[ 4f\left( \frac{x + y + z + w}{4} \right) - f(x + y + z + w) \right]$$

$$+ \left[ f(x + y + z + w) - f(x + z) - f(y + w) \right]$$

$$+ \left[ f(x + z) - 2f\left( \frac{x + z}{2} \right) \right] + \left[ f(y + w) - 2f\left( \frac{y + w}{2} \right) \right]$$

$$- \left[ 3f\left( \frac{x + y + z}{3} \right) - f(x + y + z) \right]$$

$$- \left[ f(x + y + z) - f\left( \frac{x + y}{2} \right) - f\left( \frac{y + z}{2} \right) - f\left( \frac{z + x}{2} \right) \right]$$

$$- \left[ 3f\left( \frac{y + z + w}{3} \right) - f(y + z + w) \right]$$

$$- \left[ f(y + z + w) - f\left( \frac{y + z}{2} \right) - f\left( \frac{z + w}{2} \right) - f\left( \frac{w + y}{2} \right) \right]$$

$$- \left[ 3f\left( \frac{z + w + x}{3} \right) - f(z + w + x) \right]$$

$$- \left[ f(z + w + x) - f\left( \frac{z + w}{2} \right) - f\left( \frac{w + x}{2} \right) - f\left( \frac{x + z}{2} \right) \right]$$

$$- \left[ 3f\left( \frac{w + x + y}{3} \right) - f(w + x + y) \right]$$

$$- \left[ f(w + x + y) - f\left( \frac{w + x}{2} \right) - f\left( \frac{x + y}{2} \right) - f\left( \frac{y + w}{2} \right) \right].$$
Furthermore, we evaluate that
\[
\begin{align*}
|\phi(x,y,z,w)| & \leq 8 \left| \frac{x+y+z+w}{4} \right| + |x+z| + |y+w| + 2 \left| \frac{x+z}{2} \right| \\
& + 2 \left| \frac{y+w}{2} \right| + \frac{5}{3} \left| \frac{x+y+z}{3} \right| \\
& + \frac{5}{3} \left[ \left| \frac{x+y}{2} \right| + \left| \frac{y+z}{2} \right| + \left| \frac{x+z}{2} \right| \right] \\
& + \frac{15}{3} \left| \frac{y+z+w}{3} \right| + \frac{5}{3} \left[ \left| \frac{y+z}{2} \right| + \left| \frac{z+w}{2} \right| + \left| \frac{w+y}{2} \right| \right] \\
& + \frac{15}{3} \left| \frac{z+w+x}{3} \right| + \frac{5}{3} \left[ \left| \frac{z+w}{2} \right| + \left| \frac{w+x}{2} \right| + \left| \frac{x+z}{2} \right| \right] \\
& + \frac{15}{3} \left| \frac{w+x+y}{3} \right| + \frac{5}{3} \left[ \left| \frac{w+x}{2} \right| + \left| \frac{x+y}{2} \right| + \left| \frac{y+w}{2} \right| \right] \\
& \leq 14 (|x| + |y| + |z| + |w|)
\end{align*}
\]
for all \(x,y,z,w \in \mathbb{R}\). The rest of the proof has been proved in [10]. \(\square\)

**Remark 3.12.** Let \(f\) be as in Theorem 3.11.

(i) If \(G = (\mathbb{R}, \| \cdot \|_1)\) with the Euclidean metric \(\| \cdot \|_1 = | \cdot |\), and \(E = (\mathbb{R}, \| \cdot \|_2)\) with the \(\beta\)-homogeneous norm \(\| \cdot \|_2 = | \cdot |^\beta\), then
\[
\begin{align*}
\| \phi(x,y,z,w) \|_{L^2} & \leq 14^\beta \left( \| x \|_1^\beta + \| y \|_1^\beta + \| z \|_1^\beta + \| w \|_1^\beta \right) \quad \forall x,y,z,w \in G, \\
\text{but} \\
\sup \left\{ \frac{\| f(x) - T(x) \|_{L^2}}{\| x \|_1^{1/\beta}} : x \in \mathbb{R} \setminus \{0\} \right\} & = \infty
\end{align*}
\]
for each additive mapping \(T : G \to E\).

(ii) If \(G = (\mathbb{R}, \| \cdot \|_1)\) with the \(\beta\)-homogeneous norm \(\| \cdot \|_1 = | \cdot |^\beta\), and \(E = (\mathbb{R}, \| \cdot \|_2)\) with the Euclidean metric \(\| \cdot \|_2 = | \cdot |\), then
\[
\begin{align*}
\| \phi(x,y,z,w) \|_{L^2} & \leq 14 \left( \| x \|_1^{1/\beta} + \| y \|_1^{1/\beta} + \| z \|_1^{1/\beta} + \| w \|_1^{1/\beta} \right) \quad \forall x,y,z,w \in G, \\
\text{but} \\
\sup \left\{ \frac{\| f(x) - T(x) \|_{L^2}}{\| x \|_1^{1/\beta}} : x \in \mathbb{R} \setminus \{0\} \right\} & = \infty
\end{align*}
\]
for each additive mapping \(T : G \to E\).

(iii) If \(G = E = (\mathbb{R}, \| \cdot \|)\) with the \(\beta\)-homogeneous norm \(\| \cdot \| = | \cdot |^\beta\), then
\[
\| \phi(x,y,z,w) \| \leq 14^\beta (\| x \| + \| y \| + \| z \| + \| w \|) \quad \forall x,y,z,w \in G,
\]
but

$$\sup \left\{ \left\| \frac{f(x) - T(x)}{x} \right\| : x \in \mathbb{R} \setminus \{0\} \right\} = \infty$$  \hspace{1cm} (3.62)$$

for each additive mapping $T : G \rightarrow E$.

**Acknowledgments.** I would like to express my deep gratitude to Professor Themistocles M. Rassias for drawing my attention to these kinds of problems and the referee for his many valuable suggestions. This work was supported by the National Science Foundation of China, Grant 10171014, and the Foundation of Fujian Educational Committee, Grant JA02166. This work belongs to the Doctoral Programme Foundation of Institution of Higher Education, Grant 20010055013, and the Programme of National Science Foundation of China, Grant 10271060.

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