TWISTINGS, CROSSED COPRODUCTS, AND HOPF-GALOIS COEXTENSIONS

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Let H be a Hopf algebra. Ju and Cai (2000) introduced the notion of twisting of an H -module coalgebra. In this paper, we study the relationship between twistings, crossed coproducts, and Hopf-Galois coextensions. In particular, we show that a twisting of an H -Galois coextension remains H -Galois if the twisting is invertible.

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1. Introduction. A fundamental result in Hopf-Galois theory is the normal basis theorem, stating that, for a finitely generated cocommutative Hopf algebra H over a commutative ring k , the set of isomorphism classes of Galois H -objects that are isomorphic to H as an H -comodule is a group, and this group is isomorphic to the second Sweedler cohomology group $H^2(H, k)$ (see [14]). The Galois object corresponding to a 2-cocycle is then given by a crossed product construction. The crossed product construction can be generalized to arbitrary Hopf algebras and plays a fundamental role in the theory of extensions of Hopf algebras (see [3, 11]). Also in this more general situation, it turns out that there is a close relationship between crossed products on one side and Hopf-Galois extensions and cleft extensions on the other side (cf. [3, 4, 11]). A survey can be found in [13]. An alternative way to deform the multiplication on an H -comodule algebra A has been proposed in [1], using the so-called twisting of A , and it was shown that the crossed product construction can be viewed as a special case of the twisting construction. The relation between twistings and H -Galois extensions was studied in [2].

Now, there exists a coalgebra version of the normal basis theorem (see [6]). In this situation, one tries to deform the comultiplication on a commutative Hopf algebra H , using this time a Harrison cocycle instead of a Sweedler cocycle. Crossed coproducts, cleft coextensions, and Hopf-Galois coextensions have been introduced and studied in [8, 10]. Ju and Cai [12] have introduced the notion of twisting of an H -module coalgebra, which can be viewed as a dual version of the twistings introduced in [1]. The aim of this paper is to study the relationship between twistings, crossed coproducts, and Hopf-Galois coextensions. Our main result is the fact that the twisting of a Hopf-Galois coextension by an invertible twist map is again a Hopf-Galois coextension (and vice versa).

Our paper is set up as follows. In [Section 2.1](#), we recall the twistings introduced in [\[12\]](#), and in [Section 2.2](#) we recall the definition of a Harrison cocycle and the crossed coproduct construction from [\[8, 10\]](#). In [Section 3](#), we introduce an alternative version of 2-cocycles, called twisted 2-cocycles, and discuss the relation with Harrison cocycles ([Proposition 3.3](#)). In [Section 4](#), we introduce an equivalence relation on the set of twistings of an H -module coalgebra and we show that a twisting in an equivalence class is invertible if and only if all the other twistings in this equivalence class are invertible ([Theorem 4.4](#)). Two twistings are equivalent if and only if their corresponding crossed coproducts are isomorphic ([Proposition 2.1](#)). In [Section 5](#), the relationship between twistings and Hopf-Galois coextensions is investigated.

For the general theory of Hopf algebras, we refer to the literature (see, e.g., [\[9, 13, 15\]](#)).

2. Notation and preliminary results. We work over a field k . All maps are assumed to be k -linear. For the comultiplication on a k -coalgebra C , we use the Sweedler-Heyneman notation

$$\Delta_C(c) = c_1 \otimes c_2 \quad (2.1)$$

with the summation implicitly understood. We use a similar notation for a (right) coaction of a coalgebra on a comodule:

$$\rho(m) = m_0 \otimes m_1 \in M \otimes C. \quad (2.2)$$

Let A be a k -algebra, then $\text{Hom}(C, A)$ is also an algebra, with convolution product

$$(f * g)(c) = f(c_1)g(c_2). \quad (2.3)$$

We will denote by $\text{Reg}(C, A)$ the set of convolution invertible elements in $\text{Hom}(C, A)$, and $\mathcal{M}_A^C$ will be the category of modules with a right A -action and a right C -coaction, such that the C -coaction is A -linear.

2.1. Twistings of a coalgebra. We recall some definitions and results from [\[12\]](#). Let H be a Hopf algebra over a field k , with bijective antipode S . The composition inverse of the antipode will be denoted by $\bar{S}$.

Recall that a right H -module coalgebra is a coalgebra C which is also a right H -module such that

$$\Delta(c \cdot h) = c_1 \cdot h_1 \otimes c_2 \cdot h_2, \quad \varepsilon_C(c \cdot h) = \varepsilon_C(c)\varepsilon_H(h) \quad (2.4)$$

for all $c \in C$ and $h \in H$.

Consider the category $\mathcal{M}_H^C$, whose objects are right H -modules and right C -comodules M such that the following compatibility relation is satisfied:

$$\rho(m \cdot h) = m_0 \cdot h_1 \otimes m_1 \cdot h_2. \quad (2.5)$$

Recall from [12] that we have the following associative multiplication on $\text{Hom}(C, H \otimes C)$:

$$\tau * \lambda = (m_H \otimes \text{id}_C) \circ (\text{id}_H \otimes \lambda) \circ \tau \quad (2.6)$$

for all $\tau, \lambda \in \text{Hom}(C, H \otimes C)$. The unit of this multiplication is the map $\sigma : C \rightarrow H \otimes C$, $\sigma(c) = 1 \otimes c$.

Remark that we have an algebra isomorphism

$$\alpha : \text{Hom}(C, H \otimes C) \longrightarrow_H \text{End}(H \otimes C)^{\text{op}}, \quad (2.7)$$

where α is defined as follows. Take $\tau : C \rightarrow H \otimes C$, and write $\tau(c) = c_{-1} \otimes c_0$ (summation is understood). Then $\alpha(\tau) = f_\tau : H \otimes C \rightarrow H \otimes C$ is given by

$$f_\tau(h \otimes c) = h\tau(c) = hc_{-1} \otimes c_0. \quad (2.8)$$

Assume that τ satisfies the following normality conditions:

$$(1 \otimes \varepsilon_C)\tau(c) = \varepsilon(c)1_H, \quad (\varepsilon_H \otimes 1)\tau(c) = c \quad (2.9)$$

or

$$c_{-1}\varepsilon_C(c_0) = \varepsilon_C(c)1_H, \quad \varepsilon_H(c_{-1})c_0 = c. \quad (2.10)$$

We can then define a new (in general noncoassociative) comultiplication Δ_τ on C as follows:

$$\Delta_\tau(c) = c_1 \cdot c_{2,-1} \otimes c_{2,0} \quad \text{or} \quad \Delta_\tau = (\psi_H \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ \Delta, \quad (2.11)$$

where $\psi_H : C \otimes H \rightarrow C$ is the right H -action on C , and where we used the Sweedler-type notation $(\text{id} \otimes \tau)(\Delta c) = c_1 \otimes c_{2,-1} \otimes c_{2,0}$. Let C^τ be equal to C as a right H -module, with comultiplication $\Delta_\tau(c)$. A similar construction applies to $M \in \mathcal{M}_H^C : F_\tau(M) = M^\tau$ as a right H -comodule, with

$$\rho^\tau(m) = m_0 \cdot m_{1,-1} \otimes m_{1,0}, \quad (2.12)$$

where τ is called a twisting if and only if C^τ is a right H -module coalgebra, and $M^\tau \in \mathcal{M}_H^{C^\tau}$ for all $M \in \mathcal{M}_H^C$. It is shown in [12, Theorem 1.1] that $\tau : C \rightarrow H \otimes C$,

satisfying (2.9), is a twisting if and only if for all $h \in H$ and $c \in C$,

$$c_{-1}h_1 \otimes c_0 \cdot h_2 = h_1(c \cdot h_2)_{-1} \otimes (c \cdot h_2)_0, \quad (2.13)$$

$$c_{-1} \otimes c_{0,1} \cdot c_{0,2,-1} \otimes c_{0,2,0} = c_{1,-1}c_{2,-1,1} \otimes c_{1,0} \cdot c_{2,-1,2} \otimes c_{2,0}, \quad (2.14)$$

where (2.13) is equivalent to

$$S(h_1)c_{-1}h_2 \otimes c_0 \cdot h_3 = (c \cdot h)_{-1} \otimes (c \cdot h)_0. \quad (2.15)$$

If τ has an inverse λ , then the functor F is an equivalence of categories.

Left-hand twistings are defined in a similar way. Consider the vector space isomorphism

$$\text{Hom}(C, C \otimes H) \cong \text{End}_H(C \otimes H^{\text{op}}, C \otimes H^{\text{op}}). \quad (2.16)$$

The composition on the right-hand side is transported into the following associative multiplication on $\text{Hom}(C, C \otimes H)$:

$$\tau \times \lambda = T \circ (T \circ \lambda * T \circ \tau). \quad (2.17)$$

Here, T is the usual twist map. The unit σ' on $\text{Hom}(C, C \otimes H)$ is given by $\sigma'(c) = c \otimes 1$. If $\lambda \in \text{Hom}(C, C \otimes H)$ satisfies the normalizing conditions

$$(1 \otimes \varepsilon_H)\lambda(c) = c, \quad (\varepsilon_C \otimes 1)\lambda(c) = \varepsilon_C(c)1_H, \quad (2.18)$$

then we can twist the comultiplication on C as follows. Write $\lambda(c) = c_0 \otimes c_1$, and define ${}^\lambda\Delta$ by

$${}^\lambda\Delta(c) = c_{1,0} \otimes c_2 \cdot c_{1,1}. \quad (2.19)$$

Let ${}^\lambda C$ be equal to C as a right H -module, equipped with the comultiplication ${}^\lambda\Delta$. The C -coaction $M \in {}^C\mathcal{M}_H$ can also be twisted as follows:

$${}^\lambda\rho(m) = m_{-1,0} \otimes m_0 m_{-1,1}, \quad (2.20)$$

where λ is called a left-hand twisting if ${}^\lambda C$ is an H -module coalgebra, and ${}^\lambda M \in {}^\lambda\mathcal{M}_H$ for every $M \in {}^C\mathcal{M}_H$. The map $\lambda : C \rightarrow C \otimes H$ satisfying (2.18) is a left-hand twisting if and only if for all $h \in H$ and $c \in C$,

$$c_0 \cdot h_1 \otimes c_1 h_2 = (c \cdot h_1)_0 \otimes h_2(c \cdot h_1)_1, \quad (2.21)$$

$$c_{0,1,0} \otimes c_{0,2} \cdot c_{0,1,1} \otimes c_1 = c_{1,0} \otimes c_{2,0} \cdot c_{1,1,1} \otimes c_{2,1} c_{1,1,2}. \quad (2.22)$$

Equation (2.21) is equivalent to

$$c_0 \cdot h_1 \otimes \bar{S}(h_3) c_1 h_2 = \sum (c \cdot h)_0 \otimes (c \cdot h)_1. \quad (2.23)$$

For $\tau \in \text{Hom}(C, H \otimes C)$ with inverse λ , we write

$$\tau(c) = c_{-1} \otimes c_0, \quad \lambda(c) = c_{(-1)} \otimes c_{(0)}. \quad (2.24)$$

We then have

$$c_{-1} c_{0,(-1)} \otimes c_{0,(0)} = c_{(-1)} c_{(0),-1} \otimes c_{(0),0} = 1 \otimes c. \quad (2.25)$$

For $\gamma \in \text{Hom}(C, C \otimes H)$ with inverse μ , we write

$$\gamma(c) = c_0 \otimes c_1, \quad \mu(c) = c_{(0)} \otimes c_{(1)}. \quad (2.26)$$

Let $\mathcal{T}(C)$ and $\mathcal{L}(C)$ be the sets of twistings and left-hand twistings of C , respectively, and $U(\mathcal{T}(C))$ and $U(\mathcal{L}(C))$ the sets of invertible twistings and left-hand twistings, respectively.

PROPOSITION 2.1. *Take $\tau \in U(\mathcal{T}(C))$ with inverse λ . Define $\ell(\tau) : C \rightarrow C \otimes H$ by*

$$\ell(\tau)(c) = c_{0,(0)} \cdot \bar{S}(c_{0,(-1)}) \bar{S}(c_{-1})_1 \otimes \bar{S}(c_{-1})_2. \quad (2.27)$$

Take $\gamma \in U(\mathcal{L}(C))$, with inverse μ . Define $r(\gamma) : C \rightarrow H \otimes C$ by

$$r(\gamma)(c) = S(c_1)_1 \otimes c_{0,(0)} \cdot S(c_{0,(1)}) S(c_1)_2. \quad (2.28)$$

Then $\ell : U(\mathcal{T}(C)) \rightarrow U(\mathcal{L}(C))$ is a bijection with inverse r . Furthermore, $\ell(\sigma) = \sigma'$ and $r(\sigma') = \sigma$.

PROOF. It is shown in [12] that $\ell(\tau) \in U(\mathcal{L}(C))$ with an inverse given by

$$\ell(\tau)'(c) = c_{0,(0)} \cdot \bar{S}(c_{0,(-1)})_1 \bar{S}(c_{-1})_1 \otimes \bar{S}(\bar{S}(c_{-1})_3) \bar{S}(c_{0,(-1)})_2 \bar{S}(c_{-1})_2. \quad (2.29)$$

Set $g = \bar{S}(c_{0,(-1)})$, $h = \bar{S}(c_{-1})$. Then $\ell(\tau)(c) = c_{0,(0)} \cdot g h_1 \otimes h_2$, so

$$\begin{aligned}
r(\ell(\tau))(c) &= S(h_2)_1 \otimes (c_{0,(0)} \cdot gh_1)_{0,(0)} \cdot \bar{S}((c_{0,(0)} \cdot gh_1)_{0,(-1)})_1 \\
&\quad \cdot \bar{S}((c_{0,(0)} \cdot gh_1)_{-1})_1 S(\bar{S}(\bar{S}((c_{0,(0)} \cdot gh_1)_{-1})_3)) \\
&\quad \cdot \bar{S}((c_{0,(0)} \cdot gh_1)_{0,(-1)})_2 \bar{S}((c_{0,(0)} \cdot gh_1)_{-1})_2 S(h_2)_2 \\
&= S(h_2)_1 \otimes (c_{0,(0)} \cdot gh_1)_{0,(0)} \bar{S}((c_{0,(0)} \cdot gh_1)_{0,(-1)})_1 \\
&\quad \cdot \bar{S}((c_{0,(0)} \cdot gh_1)_{-1})_1 S(\bar{S}((c_{0,(0)} \cdot gh_1)_{-1})_2) \\
&\quad \cdot S(\bar{S}((c_{0,(0)} \cdot gh_1)_{0,(-1)})_2) \bar{S}((c_{0,(0)} \cdot gh_1)_{-1})_3 S(h_2)_2 \\
&= S(h_2)_1 \otimes (c_{0,(0)} \cdot gh_1)_{0,(0)} \\
&\quad \cdot \varepsilon((c_{0,(0)} \cdot gh_1)_{0,(-1)}) \bar{S}((c_{0,(0)} \cdot gh_1)_{-1}) S(h_2)_2 \quad (2.30) \\
&= S(h_2)_1 \otimes (c_{0,(0)} \cdot gh_1)_0 \cdot \bar{S}((c_{0,(0)} \cdot gh_1)_{-1}) S(h_2)_2 \\
&= S(h_2)_1 \otimes c_{0,(0),0} \cdot (gh_1)_3 \bar{S}((gh_1)_2) \bar{S}(c_{0,(0),-1}) \\
&\quad \cdot \bar{S}(S(gh_1)_1) S(h_2)_2 \\
&= S(h_3) \otimes c_{0,(0),0} \cdot \bar{S}(c_{0,(0),-1}) gh_1 S(h_2) \\
&= S(h) \otimes c_{0,(0),0} \cdot \bar{S}(c_{0,(0),-1}) g \\
&= S(\bar{S}(c_{-1})) \otimes c_{0,(0),0} \cdot \bar{S}(c_{0,(-1)} c_{0,(0),-1}) \\
&= c_{-1} \otimes c_0 = \tau(c).
\end{aligned}$$

In [12], it is also shown that $r(\gamma) \in U(\mathcal{T}(C))$, and it is straightforward to verify that the $*$ -inverse of $r(\gamma)$ is given by

$$r(\gamma)'(c) = S(S(c_1)_1) S(c_{0,(1)})_1 S(c_1)_2 \otimes c_{0,(0)} \cdot S(c_{0,(1)})_2 S(c_1)_3. \quad (2.31)$$

Then a routine verification similar to the above one shows that

$$\ell(r(\gamma))(c) = \gamma(c) \quad (2.32)$$

for all $c \in C$. It is easy to show that $\ell(\sigma) = \sigma'$ and $r(\sigma') = \sigma$. $\square$

2.2. The crossed coproduct. We recall the following definitions from [8, 10].

DEFINITION 2.2. Let C be a coalgebra and H a Hopf algebra. It is said that H coacts weakly on C if there is a k -linear map $\rho : C \rightarrow H \otimes C$; $\rho(c) = c_{[-1]} \otimes c_{[0]}$ satisfying the following conditions for all $c \in C$:

$$\begin{aligned}
c_{[-1]} \otimes c_{[0]1} \otimes c_{[0]2} &= c_{1[-1]} c_{2[-1]} \otimes c_{1[0]} \otimes c_{2[0]}, \\
\varepsilon_C(c_{[0]}) c_{[-1]} &= \varepsilon(c) 1_H, \\
\varepsilon_H(c_{[-1]}) c_{[0]} &= c.
\end{aligned} \quad (2.33)$$

Assume that H coacts weakly on C , and let $\alpha : C \rightarrow H \otimes H$, $\alpha(c) = \alpha_1(c) \otimes \alpha_2(c)$, be a linear map. Let $C \bowtie_\alpha H$ be the coalgebra whose underlying vector

space is $C \otimes H$, with comultiplication and counit given by

$$\begin{aligned}\Delta_\alpha(c \bowtie h) &= (c_1 \bowtie c_{2[-1]} \alpha_1(c_3) h_1) \otimes (c_{2[0]} \bowtie \alpha_2(c_3) h_2), \\ \varepsilon_\alpha(c \bowtie h) &= \varepsilon_C(c) \varepsilon_H(h).\end{aligned}\quad (2.34)$$

It was pointed out in [10] that $\varepsilon_\alpha(c \bowtie h)$ satisfies the counit property if and only if

$$(\varepsilon_H \otimes \text{id})\alpha(c) = (\text{id} \otimes \varepsilon_H)\alpha(c) = \varepsilon_C(c) 1_H \quad (2.35)$$

and that Δ_α is coassociative if and only if α satisfies

$$\begin{aligned}c_{1[-1]} \alpha_1(c_2) \otimes \alpha_1(c_{1[0]}) \alpha_2(c_2)_1 \otimes \alpha_2(c_{1[0]}) \alpha_2(c_2)_2 \\ = \alpha_1(c_1) \alpha_1(c_2)_1 \otimes \alpha_2(c_1) \alpha_1(c_2)_2 \otimes \alpha_2(c_2),\end{aligned}\quad (2.36)$$

$$\begin{aligned}c_{1[-1]} \alpha_1(c_2) \otimes c_{1[0][-1]} \alpha_2(c_2) \otimes c_{1[0][0]} \\ = \alpha_1(c_1) c_{2[-1]1} \otimes \alpha_2(c_1) c_{2[-1]2} \otimes c_{2[0]}.\end{aligned}\quad (2.37)$$

In [10], (2.36) is called the cocycle condition and (2.37) is called the twisted comodule condition. Following [7], we call α , satisfying (2.35), (2.36), and (2.37), a Harrison 2-cocycle.

Now, consider two weak H -coactions $\rho, \rho' : C \rightarrow H \otimes C$, and write

$$\rho(c) = c_{[-1]} \otimes c_{[0]}, \quad \rho'(c) = c_{\langle -1 \rangle} \otimes c_{\langle 0 \rangle}. \quad (2.38)$$

Also consider two 2-cocycles $\alpha, \alpha' : C \rightarrow H \otimes H$ corresponding respectively to ρ and ρ' , and write

$$\alpha(c) = \alpha_1(c) \otimes \alpha_2(c), \quad \alpha'(c) = \alpha'_1(c) \otimes \alpha'_2(c). \quad (2.39)$$

Then we can consider the crossed coproducts $C \bowtie_\alpha H$ and $C \bowtie'_{\alpha'} H$. In the next lemma, we discuss when these are isomorphic.

LEMMA 2.3. *Consider a convolution invertible map $u : C \rightarrow H$ satisfying the conditions*

$$c_{\langle -1 \rangle} \otimes c_{\langle 0 \rangle} = u^{-1}(c_1) c_{2[-1]} u(c_3) \otimes c_{2[0]}, \quad (2.40)$$

$$\alpha'(c) = u^{-1}(c_1) c_{2[-1]} \alpha_1(c_3) u(c_4)_1 \otimes u^{-1}(c_{2[0]}) \alpha_2(c_3) u(c_4)_2 \quad (2.41)$$

for all $c \in C$. Then the map

$$\phi : C \bowtie'_{\alpha'} H \rightarrow C \bowtie_\alpha H, \quad \phi(c \bowtie' h) = c_1 \bowtie u(c_2) h \quad (2.42)$$

is a left C -colinear right H -linear coalgebra isomorphism. Every left C -colinear right H -linear coalgebra isomorphism between $C \rtimes_{\alpha} H$ and $C \rtimes_{\alpha'} H$ is of this type.

PROOF. The proof is a dual version of a similar statement for crossed products (see [13]). $\square$

It was shown in [12] that the crossed coproduct construction can be viewed as a special case of the twisting construction from Section 2.1. Let H be a Hopf algebra and C a right H -module coalgebra, and view $C \otimes H$ as a right H -module coalgebra, with the right H -action being induced by the multiplication by H . It was proved in [12] that there is a bijective correspondence between crossed coproduct structures on $C \otimes H$ and twistings of $C \otimes H$. We recall the description of this bijection.

Consider a weak coaction ρ and a 2-cocycle α giving rise to the crossed coproduct $C \rtimes_{\alpha} H$, and write

$$\rho(c) = c_{[-1]} \otimes c_{[0]}, \quad \alpha(c) = \alpha_1(c) \otimes \alpha_2(c). \quad (2.43)$$

The corresponding twisting $\tau : C \otimes H \rightarrow H \otimes C \otimes H$ is defined by

$$\tau(c \otimes h) = S(h_1) c_{1[-1]} \alpha_1(c_2) h_2 \otimes c_{1[0]} \otimes \alpha_2(c_2) h_3. \quad (2.44)$$

Conversely, if τ is a twisting of $C \otimes H$, then $(C \otimes H)^{\tau} = C \rtimes_{\alpha} H$, with weak coaction ρ and 2-cocycle α given by

$$\rho(c) = (\text{id} \otimes \text{id} \otimes \varepsilon_H) \tau(c \otimes 1), \quad (2.45)$$

$$\alpha(c) = (\text{id} \otimes \varepsilon_C \otimes \text{id}) \tau(c \otimes 1). \quad (2.46)$$

3. Twisted 2-cocycles. Let H be a Hopf algebra with bijective antipode S and let $\bar{S}$ be the composition inverse of S . Take an H -module coalgebra C , and let $B = C/CH^+$.

DEFINITION 3.1. A map $\alpha : C \rightarrow H \otimes H$, $\alpha(c) = \alpha_1(c) \otimes \alpha_2(c)$ is called a twisted 2-cocycle if the following conditions are satisfied for all $h \in H$ and $c \in C$:

$$(\text{id}_H \otimes \varepsilon_H) \alpha(c) = (\varepsilon_H \otimes \text{id}_H) \alpha(c) = \varepsilon_C(c) 1_H, \quad (3.1)$$

$$\alpha(c \cdot h) = S(h_1) \alpha_1(c) h_2 \otimes \bar{S}(h_4) \alpha_2(c) h_3, \quad (3.2)$$

$$\begin{aligned} \alpha_1(c_1) \alpha_1(c_3)_1 \otimes c_2 \cdot \alpha_2(c_1) \alpha_1(c_3)_2 \otimes \alpha_2(c_3) \\ = \alpha_1(c_1) \otimes c_2 \cdot \alpha_1(c_3) \alpha_2(c_1)_1 \otimes \alpha_2(c_3) \alpha_2(c_1)_2. \end{aligned} \quad (3.3)$$

Our first result is the fact that the twisted 2-cocycles can be used to define twistings on C .

PROPOSITION 3.2. *With notation as above, if $\alpha : C \rightarrow H \otimes H$ is a twisted 2-cocycle, then the map*

$$\tau_\alpha : C \longrightarrow H \otimes C, \quad \tau_\alpha(c) = \alpha_1(c_1) \otimes c_2 \cdot \alpha_2(c_1), \quad (3.4)$$

is a twisting of C .

PROOF. It follows easily from (3.1) that τ_α satisfies the normalizing condition (2.9). Next, we compute

$$\begin{aligned} (c \cdot h)_{-1} \otimes (c \cdot h)_0 &= \alpha_1((c \cdot h)_1) \otimes (c \cdot h)_2 \cdot \alpha_2((c \cdot h)_1) \\ &= \alpha_1(c_1 \cdot h_1) \otimes c_2 \cdot h_2 \alpha_2(c_1 \cdot h_1) \\ &= S(h_1) \alpha_1(c_1) h_2 \otimes c_2 \cdot h_5 \bar{S}(h_4) \alpha_2(c_1) h_3 \quad (3.5) \\ &= S(h_1) \alpha_1(c_1) h_2 \otimes c_2 \cdot \alpha_2(c_1) h_3 \\ &= S(h_1) c_{-1} h_2 \otimes c_0 \cdot h_3, \end{aligned}$$

and (2.13) follows easily. Finally, we compute the left- and right-hand sides of (2.14):

$$\begin{aligned} c_{-1} \otimes c_{0,1} \cdot c_{0,2,-1} \otimes c_{0,2,0} &= (1 \otimes \Delta_\tau) \tau_\alpha(c) = \alpha_1(c_1) \otimes (c_2 \cdot \alpha_2(c_1))_1 \cdot \alpha_1(((c_2 \cdot \alpha_2(c_1))_2)_1) \\ &\quad \otimes ((c_2 \cdot \alpha_2(c_1))_2)_2 \cdot \alpha_2(((c_2 \cdot \alpha_2(c_1))_2)_1) \\ &= \alpha_1(c_1) \otimes c_2 \cdot \alpha_2(c_1)_1 \alpha_1(c_3 \cdot \alpha_2(c_1)_2) \\ &\quad \otimes (c_4 \cdot \alpha_2(c_1)_3) \cdot \alpha_2(c_3 \cdot \alpha_2(c_1)_2) \\ &= \alpha_1(c_1) \otimes c_2 \cdot \alpha_2(c_1)_1 S(\alpha_2(c_1)_2) \alpha_1(c_3) \alpha_2(c_1)_3 \\ &\quad \otimes c_4 \cdot \alpha_2(c_1)_6 \bar{S}(\alpha_2(c_1)_5) \alpha_2(c_3) \alpha_2(c_1)_4 \quad (3.6) \\ &= \alpha_1(c_1) \otimes c_2 \cdot \alpha_1(c_3) \alpha_2(c_1)_1 \otimes c_4 \cdot \alpha_2(c_3) \alpha_2(c_1)_2, \\ c_{1,-1} c_{2,-1,1} \otimes c_{1,0} \cdot c_{2,-1,2} \otimes c_{2,0} &= \sum \alpha_1(c_{11}) \alpha_1(c_{21})_1 \otimes c_{12} \cdot \alpha_2(c_{11}) \cdot \alpha_1(c_{21})_2 \otimes c_{22} \cdot \alpha_2(c_{21}) \\ &= \alpha_1(c_1) \alpha_1(c_3)_1 \otimes c_2 \cdot \alpha_2(c_1) \alpha_1(c_3)_2 \otimes c_4 \cdot \alpha_2(c_3) \\ &= \alpha_1(c_1) \otimes c_2 \cdot \alpha_1(c_3) \alpha_2(c_1)_1 \otimes c_4 \cdot \alpha_2(c_3) \alpha_2(c_1)_2. \end{aligned}$$

Thus (2.14) follows, and τ_α is a twisting. $\square$

There is also a relation between twisted 2-cocycles and Harrison 2-cocycles. Let C be a right H -module coalgebra. Consider the trivial weak coaction $\rho(c) = 1 \otimes c$ and $\alpha : C \rightarrow H \otimes H$. Then the cocycle condition (2.36) and the twisted

comodule condition (2.37) of Definition 2.2 take the following form:

$$\begin{aligned} \alpha_1(c_2) \otimes \alpha_1(c_1) \alpha_2(c_2)_1 \otimes \alpha_2(c_1) \alpha_2(c_2)_2 \\ = \alpha_1(c_1) \alpha_1(c_2)_1 \otimes \alpha_2(c_1) \alpha_1(c_2)_2 \otimes \alpha_2(c_2), \end{aligned} \quad (3.7)$$

$$\alpha_1(c_2) \otimes \alpha_2(c_2) \otimes c_1 = \alpha_1(c_1) \otimes \alpha_2(c_1) \otimes c_2. \quad (3.8)$$

The set of Harrison 2-cocycles corresponding to the trivial weak coaction is denoted by $Z_{\text{Harr}}^2(H, C)$. Thus, $Z_{\text{Harr}}^2(H, C)$ consists of maps satisfying (2.35), (3.7), and (3.8). The set of twisted 2-cocycles $\alpha^t : C \otimes H \rightarrow H \otimes H$ in the sense of Definition 3.1 will be denoted by $Z_{\text{tw}}^2(H, C \otimes H)$.

PROPOSITION 3.3. *Let C be a right H -module coalgebra. There exists a bijection between $Z_{\text{Harr}}^2(H, C)$ and $Z_{\text{tw}}^2(H, C \otimes H)$.*

PROOF. Take $\alpha^t \in Z_{\text{tw}}^2(H, C \otimes H)$ and write

$$\alpha^t(c \otimes h) = \sum \alpha_1^t(c \otimes h) \otimes \alpha_2^t(c \otimes h). \quad (3.9)$$

For all $c \in C$ and $h \in H$, we have

$$\begin{aligned} \alpha_1^t(c_1 \otimes h_1) \alpha_1^t(c_2 \otimes h_2)_1 \otimes \alpha_2^t(c_1 \otimes h_1) \alpha_1^t(c_2 \otimes h_2)_2 \otimes \alpha_2^t(c_2 \otimes h_2) \\ = \alpha_1^t(c_1 \otimes h_1) \otimes \alpha_1^t(c_2 \otimes h_2) \alpha_2^t(c_1 \otimes h_1)_1 \otimes \alpha_2^t(c_2 \otimes h_2) \alpha_2^t(c_1 \otimes h_1)_2. \end{aligned} \quad (3.10)$$

Now, define $\alpha : C \rightarrow H \otimes H$ by $\alpha(c) = \alpha^t(c \otimes 1)$. It is easy to see that α satisfies (2.35) and (3.8). Using (3.10), we compute

$$\begin{aligned} \alpha_1(c_2) \otimes \alpha_1(c_1) \alpha_2(c_2)_1 \otimes \alpha_2(c_1) \alpha_2(c_2)_2 \\ = \alpha_1(c_1) \otimes \alpha_1(c_2) \alpha_2(c_1)_1 \otimes \alpha_2(c_2) \alpha_2(c_1)_2 \\ = \alpha_1^t(c_1 \otimes 1) \otimes \alpha_1^t(c_2 \otimes 1) \alpha_2^t(c_1 \otimes 1)_1 \otimes \alpha_2^t(c_2 \otimes 1) \alpha_2^t(c_1 \otimes 1)_2 \\ = \alpha_1^t(c_1 \otimes 1) \alpha_1^t(c_2 \otimes 1)_1 \otimes \alpha_2^t(c_1 \otimes 1) \alpha_1^t(c_2 \otimes 1)_2 \otimes \alpha_2^t(c_2 \otimes 1) \\ = \alpha_1(c_1) \alpha_1(c_2)_1 \otimes \alpha_2(c_1) \alpha_1(c_2)_2 \otimes \alpha_2(c_2), \end{aligned} \quad (3.11)$$

and it follows that α also satisfies (3.7).

Conversely, let $\alpha \in Z_{\text{Harr}}^2(H, C)$, and define $\alpha^t : C \otimes H \rightarrow H \otimes H$ by

$$\alpha^t(c \otimes h) = S(h_1) \alpha_1(c) h_2 \otimes \bar{S}(h_4) \alpha_2(c) h_3. \quad (3.12)$$

We can easily show that α^t satisfies conditions (3.1) and (3.2) of Definition 3.1. A straightforward computation shows that (3.3) is also satisfied:

$$\begin{aligned}
 & \alpha_1^t(c_1 \otimes h_1) \alpha_1^t(c_3 \otimes h_3)_1 \otimes c_2 \otimes h_2 \alpha_2^t(c_1 \otimes h_1) \alpha_1^t(c_3 \otimes h_3)_2 \otimes \alpha_2^t(c_3 \otimes h_3) \\
 &= S h_1 \alpha_1(c_1) h_2 S(h_7) \alpha_1(c_3)_1 h_8 \\
 &\quad \otimes c_2 \otimes h_5 \bar{S}(h_4) \alpha_2(c_1) h_3 S(h_6) \alpha_1(c_3)_2 h_9 \otimes \bar{S}(h_{11}) \alpha_2(c_3) h_{10} \\
 &= S(h_1) \alpha_1(c_1) \alpha_1(c_3)_1 h_2 \otimes c_2 \otimes \alpha_2(c_1) \alpha_1(c_3)_2 h_3 \otimes \bar{S}(h_5) \alpha_2(c_3) h_4 \\
 &= S(h_1) \alpha_1(c_3) h_2 \otimes c_2 \otimes \alpha_1(c_1) \alpha_2(c_3)_1 h_3 \otimes \bar{S}(h_5) \alpha_2(c_1) \alpha_2(c_3)_2 h_4 \\
 &= S(h_1) \alpha_1(c_1) h_2 \otimes c_2 \otimes \alpha_1(c_3) \alpha_2(c_1)_1 h_3 \otimes \bar{S}(h_5) \alpha_2(c_3) \alpha_2(c_1)_2 h_4 \\
 &= S(h_1) \alpha_1(c_1) h_2 \otimes c_2 \otimes h_7 S(h_8) \alpha_1(c_3) h_9 \bar{S}(h_6) \alpha_2(c_1)_1 h_3 \\
 &\quad \otimes \bar{S}(h_{11}) \alpha_2(c_3) h_{10} \bar{S}(h_5) \alpha_2(c_1)_2 h_4 \\
 &= \alpha_1^t(c_1 \otimes h_1) \otimes c_2 \otimes h_2 \alpha_1^t(c_3 \otimes h_3) \alpha_2^t(c_1 \otimes h_1)_1 \otimes \alpha_2^t(c_3 \otimes h_3) \alpha_2^t(c_1 \otimes h_1)_2.
 \end{aligned} \tag{3.13}$$

So it follows that α^t is a twisted 2-cocycle. We leave it to the reader to show that the maps between $Z_{\text{Harr}}^2(H, C)$ and $Z_{\text{tw}}^2(H, C \otimes H)$ defined above are the inverses of each other. $\square$

4. Equivalence of twistings. In this section, we will define an equivalence relation on the set of twistings of an H -module coalgebra C . If a twisting is invertible, then all other twistings in the same equivalence class are also invertible.

PROPOSITION 4.1. *Take $\tau, \lambda \in \mathcal{T}(C)$ and use notation (2.24). Consider $v \in \text{Hom}(C, H)$ satisfying the following identities, for all $h \in H, c \in C$:*

$$\varepsilon_H \circ v = \varepsilon_C, \quad v(c \cdot h) = S(h_1) v(c) h_2, \tag{4.1}$$

$$c_{1,(-1)} v(c_2)_1 \otimes c_{1,(0)} \cdot v(c_2)_2 = v(c_1) c_{2,-1} \otimes c_{2,0,1} \cdot v(c_{2,0,2}). \tag{4.2}$$

Then $\psi : C^\tau \rightarrow C^\lambda$, $\psi(c) = c_1 \cdot v(c_2)$, is a left B -colinear and right H -linear coalgebra map inducing the identity map on B . If $v \in \text{Reg}(C, H)$, then ψ is an isomorphism.

PROOF. Using the second identity in (4.1) and $B = C/CH^+$, we can easily prove that ψ is left B -colinear and right H -linear. Using the first identity in (4.1), we obtain that ψ induces a well-defined map $B \rightarrow B$, which is the identity. In order to prove that ψ is a coalgebra map, we need to check that

$$\psi(c_1 \cdot c_{2,-1}) \otimes \psi(c_{2,0}) = \psi(c)_1 \psi(c)_{2,(-1)} \otimes \psi(c)_{2,(0)}. \tag{4.3}$$

Again, we compute the left- and right-hand sides and see that they are equal:

$$\begin{aligned}
 & \psi(c)_1 \psi(c)_{2,(-1)} \otimes \psi(c)_{2,(0)} \\
 &= (c_1 \cdot v(c_2))_1 (c_1 \cdot v(c_2))_{2,(-1)} \otimes (c_1 \cdot v(c_2))_{2,(0)} \\
 &= c_1 \cdot v(c_3)_1 (c_2 \cdot v(c_3))_{2,(-1)} \otimes (c_2 \cdot v(c_3))_{2,(0)} \\
 &= c_1 \cdot v(c_3)_1 S(v(c_3)_2) c_{2,(-1)} v(c_3)_3 \otimes c_{2,(0)} \cdot v(c_3)_4 \\
 &= c_1 \cdot c_{2,(-1)} v(c_3)_1 \otimes c_{2,(0)} \cdot v(c_3)_2 \\
 &= c_1 \cdot v(c_2) c_{3,-1} \otimes c_{3,0,1} \cdot v(c_{3,0,2}), \tag{4.4} \\
 & \psi(c_1 \cdot c_{2,-1}) \otimes \psi(c_{2,0}) \\
 &= (c_1 \cdot c_{2,-1})_1 \cdot v((c_1 \cdot c_{2,-1})_2) \otimes (c_{2,0})_1 \cdot v((c_{2,0})_2) \\
 &= c_1 \cdot (c_{3,-1})_1 v(c_2 \cdot (c_{3,-1})_2) \otimes (c_{3,0})_1 \cdot v((c_{3,0})_2) \\
 &= c_1 \cdot c_{3,-1,1} S(c_{3,-1,2}) v(c_2) c_{3,-1,3} \otimes c_{3,0,1} \cdot v(c_{3,0,2}) \\
 &= c_1 \cdot v(c_2) c_{3,-1} \otimes c_{3,0,1} \cdot v(c_{3,0,2}).
 \end{aligned}$$

If $v \in \text{Reg}(C, H)$, then its inverse w also satisfies (4.1), and $\varphi : C^\lambda \rightarrow C^\tau$ defined by

$$\varphi(c) = c_1 \cdot w(c_2) \tag{4.5}$$

is the inverse of ψ . □

DEFINITION 4.2. It is said that $\tau, \lambda \in \mathcal{T}(C)$ are equivalent if there exists $v \in \text{Reg}(C, H)$ satisfying the conditions of Proposition 4.1. This is denoted by $\tau \sim \lambda$.

LEMMA 4.3. *The relation $\sim$ is an equivalence relation on $\mathcal{T}(C)$.*

PROOF. Clearly, $\tau \sim \tau$ through $v(c) = \varepsilon(c)1_H$.

Next, assume that $\tau \sim \lambda$, and take $v \in \text{Reg}(C, H)$ satisfying (4.1) and (4.2). Equation (4.2) is equivalent to

$$c_{(-1)} \otimes c_{(0)} = v(c_1) c_{2,-1} v^{-1}(c_3)_1 \otimes c_{2,0,1} \cdot v(c_{2,0,2}) v^{-1}(c_3)_2. \tag{4.6}$$

The inverse u of v satisfies (4.1). It also satisfies (4.2) since

$$\begin{aligned}
 & u(c_1) c_{2,(-1)} \otimes c_{2,(0)1} \cdot u(c_{2,(0),2}) \\
 &= u(c_1) v(c_2) c_{3,-1} v^{-1}(c_4)_1 \otimes c_{3,0,1} \cdot v(c_{3,0,3})_1 v^{-1}(c_4)_2 \\
 &\quad \cdot S(v^{-1}(c_4)_3) S(v(c_{3,0,3})_2) u(c_{3,0,2}) v(c_{3,0,3})_3 v^{-1}(c_4)_4 \tag{4.7} \\
 &= c_{1,-1} v^{-1}(c_2)_1 \otimes c_{1,0,1} \cdot u(c_{1,0,2}) v(c_{1,0,3}) v^{-1}(c_2)_2 \\
 &= c_{1,-1} u(c_2)_1 \otimes c_{1,0,1} \cdot u(c_2)_2,
 \end{aligned}$$

and it follows that $\lambda \sim \tau$.

Now, assume that $\tau \sim \lambda$ and $\lambda \sim \gamma$, and take the corresponding maps $v, u \in \text{Reg}(C, H)$. Set $w = u * v$, and write

$$\tau(c) = c_{-1} \otimes c_0, \quad \lambda(c) = c_{(-1)} \otimes c_{(0)}, \quad \gamma(c) = c_{[-1]} \otimes c_{[0]}. \quad (4.8)$$

It is easily shown that w satisfies (4.1), v satisfies (4.6), and u satisfies

$$c_{[-1]} \otimes c_{[0]} = u(c_1)c_{2,(-1)}u^{-1}(c_3)_1 \otimes c_{2,(0),1} \cdot u(c_{2,(0),2})u^{-1}(c_3)_2. \quad (4.9)$$

We compute

$$\begin{aligned} c_{[-1]} \otimes c_{[0]} &= u(c_1)v(c_2)c_{3,-1}v^{-1}(c_4)_1u^{-1}(c_5)_1 \otimes c_{3,0,1} \\ &\quad \cdot v(c_{3,0,3})_1v^{-1}(c_4)_2S(v^{-1}(c_4)_3)S(v(c_{3,0,3})_2) \\ &\quad \cdot u(c_{3,0,2})v(c_{3,0,3})_3v^{-1}(c_4)_4u^{-1}(c_5)_2 \\ &= u(c_1)v(c_2)c_{3,-1}v^{-1}(c_4)_1u^{-1}(c_5)_1 \otimes c_{3,0,1} \cdot u(c_{3,0,2})v(c_{3,0,3})v^{-1}(c_4)_2u^{-1}(c_5)_2 \\ &= (u * v)(c_1)c_{2,-1}(u * v)^{-1}(c_3)_1 \otimes c_{2,0,1} \cdot (u * v)(c_{2,0,2})(u * v)^{-1}(c_3)_2, \end{aligned} \quad (4.10)$$

and this proves that $\tau \sim \gamma$. □

THEOREM 4.4. *Take $\tau \sim \lambda \in \mathcal{T}(C)$. If τ is invertible, then λ is also invertible.*

PROOF. Take $v \in \text{Reg}(C, H)$ satisfying the conditions in Proposition 4.1 and let $\psi : C^\tau \rightarrow C^\lambda$ be the coalgebra isomorphism given by

$$\psi(c) = c_1 \cdot v(c_2). \quad (4.11)$$

Let τ^{-1} be the inverse of τ , and write

$$\tau^{-1}(c) = c_{(-1)} \otimes c_{(0)}, \quad \tau(c) = c_{-1} \otimes c_0, \quad \lambda(c) = c_{(-1)} \otimes c_{(0)}. \quad (4.12)$$

Define $\mu : C \rightarrow H \otimes C$ by

$$\begin{aligned} \mu(c) &= c_{[-1]} \otimes c_{[0]} = \psi^{-1}(c)_{(-1)}v^{-1}(\psi^{-1}(c)_{(0)1})v(\psi^{-1}(c)_{(0)3})_1 \\ &\quad \otimes \psi^{-1}(c)_{(0)2} \cdot v(\psi^{-1}(c)_{(0)3})_2. \end{aligned} \quad (4.13)$$

Using the temporary notation $\psi^{-1}(c)_{(-1)} = a$ and $\psi^{-1}(c)_{(0)} = b$, it is not hard to prove that μ is a left inverse of λ . Indeed,

$$\begin{aligned}
(\mu * \lambda)(c) &= (m \otimes \text{id})(\text{id} \otimes \lambda)\mu(c) \\
&= av^{-1}(b_1)v(b_3)_1(b_2 \cdot v(b_3)_2)_{(-1)} \otimes (b_2 \cdot v(b_3))_{(0)} \\
&= av^{-1}(b_1)v(b_3)_1 S(v(b_3)_2)b_{2,(-1)}v(b_3)_3 \otimes b_{2,(0)} \cdot v(b_3)_4 \\
&= av^{-1}(b_1)b_{2,(-1)}v(b_3)_1 \otimes b_{2,(0)} \cdot v(b_3)_2 \quad (4.14) \\
&= av^{-1}(b_1)v(b_2)b_{3,-1} \otimes b_{3,0,1} \cdot v(b_{3,0,2}) \\
&= ab_{-1} \otimes b_{0,1} \cdot v(b_{0,2}) = 1 \otimes \psi^{-1}(c)_1 \cdot v(\psi^{-1}(c)_2) \\
&= 1 \otimes \psi(\psi^{-1}(c)) = 1 \otimes c = \sigma(c).
\end{aligned}$$

The proof of the fact that μ is also a right inverse of λ is much more technical. From the fact that v is invertible, and using (4.2), we obtain

$$\lambda(c) = c_{(-1)} \otimes c_{(0)} = v(c_1)c_{2,-1}v^{-1}(c_3)_1 \otimes c_{2,0,1} \cdot v(c_{2,0,2})v^{-1}(c_3)_2. \quad (4.15)$$

Now, set $\psi^{-1}(c) = c_1 \cdot v^{-1}(c_2)$. We compute

$$\begin{aligned}
(\lambda * \mu)(c) &= (m \otimes \text{id})(\text{id} \otimes \mu)\lambda(c) \\
&= v(c_1)c_{2,-1}v^{-1}(c_3)_1(c_{2,0,1} \cdot v(c_{2,0,2})v^{-1}(c_3)_2)_{[-1]} \\
&\quad \otimes (c_{2,0,1} \cdot v(c_{2,0,2})v^{-1}(c_3)_2)_{[0]} \\
&= v(c_1)c_{2,-1}v^{-1}(c_3)_1 S(v^{-1}(c_3)_2)S(v(c_{2,0,2})_1)(c_{2,0,1})_{[-1]} \\
&\quad \cdot v(c_{2,0,2})_2 v^{-1}(c_3)_3 \otimes (c_{2,0,1})_{[0]} \cdot v(c_{2,0,2})_3 v^{-1}(c_3)_4 \\
&= v(c_1)c_{2,-1}S(v(c_{2,0,2})_1)(c_{2,0,1})_{[-1]}v(c_{2,0,2})_2 v^{-1}(c_3)_1 \\
&\quad \otimes (c_{2,0,1})_{[0]} \cdot v(c_{2,0,2})_3 v^{-1}(c_3)_2 \\
&= v(c_1)c_{2,-1}S(v(c_{2,0,2})_1)\psi^{-1}(c_{2,0,1})_{(-1)}v^{-1}(\psi^{-1}(c_{2,0,1})_{(0),1}) \\
&\quad \cdot v(\psi^{-1}(c_{2,0,1})_{(0),3})_1 v(c_{2,0,2})_2 v^{-1}(c_3)_1 \otimes \psi^{-1}(c_{2,0,1})_{(0),2} \\
&\quad \cdot v(\psi^{-1}(c_{2,0,1})_{(0),3})_2 v(c_{2,0,2})_3 v^{-1}(c_3)_2 \\
&= v(c_1)c_{2,-1}S(v(c_{2,0,3})_1)(c_{2,0,1} \cdot v^{-1}(c_{2,0,2}))_{(-1)} \\
&\quad \cdot v^{-1}((c_{2,0,1} \cdot v^{-1}(c_{2,0,2}))_{(0),1})v((c_{2,0,1} \cdot v^{-1}(c_{2,0,2}))_{(0),3})_1 \\
&\quad \cdot v(c_{2,0,3})_2 v^{-1}(c_3)_1 \otimes (c_{2,0,1} \cdot v^{-1}(c_{2,0,2}))_{(0),2} \\
&\quad \cdot v((c_{2,0,1} \cdot v^{-1}(c_{2,0,2}))_{(0),3})_2 v(c_{2,0,3})_3 v^{-1}(c_3)_2 \\
&= v(c_1)c_{2,-1}S(v(c_{2,0,3})_1)S(v^{-1}(c_{2,0,2})_1)c_{2,0,1,(-1)}v^{-1}(c_{2,0,2})_2 \\
&\quad \cdot v^{-1}((c_{2,0,1,(-1)} \cdot v^{-1}(c_{2,0,2})_3)_1)v((c_{2,0,1,(-1)} \cdot v^{-1}(c_{2,0,2})_3)_3)_1 \\
&\quad \cdot v(c_{2,0,3})_2 v^{-1}(c_3)_1 \otimes (c_{2,0,1,(-1)} \cdot v^{-1}(c_{2,0,2})_3)_2 \\
&\quad \cdot v((c_{2,0,1,(-1)} \cdot v^{-1}(c_{2,0,2})_3)_3)_2 v(c_{2,0,3})_3 v^{-1}(c_3)_2
\end{aligned}$$

$$\begin{aligned}
&= v(c_1)c_{2,-1}S(v^{-1}(c_{2,0,2})_1v(c_{2,0,3})_1)c_{2,0,1,\langle -1 \rangle}v^{-1}(c_{2,0,2})_2 \\
&\quad \cdot v^{-1}(c_{2,0,1,\langle 0 \rangle,1} \cdot v^{-1}(c_{2,0,2})_3)v(c_{2,0,1,\langle 0 \rangle,3} \cdot v^{-1}(c_{2,0,2})_5)_1v(c_{2,0,3})_2 \\
&\quad \cdot v^{-1}(c_3)_1 \otimes c_{2,0,1,\langle 0 \rangle,2} \cdot v^{-1}(c_{2,0,2})_4 \\
&\quad \cdot v(c_{2,0,1,\langle 0 \rangle,3} \cdot v^{-1}(c_{2,0,2})_5)_2v(c_{2,0,3})_3v^{-1}(c_3)_2 \\
&= v(c_1)c_{2,-1}S(v^{-1}(c_{2,0,2})_1v(c_{2,0,3})_1)c_{2,0,1,\langle -1 \rangle}v^{-1}(c_{2,0,2})_2 \\
&\quad \cdot S(v^{-1}(c_{2,0,2})_3)v^{-1}(c_{2,0,1,\langle 0 \rangle,1})v^{-1}(c_{2,0,2})_4S(v^{-1}(c_{2,0,2})_6)_1 \\
&\quad \cdot v(c_{2,0,1,\langle 0 \rangle,3})_1(v^{-1}(c_{2,0,2})_7)_1v(c_{2,0,3})_2v^{-1}(c_3)_1 \\
&\quad \otimes c_{2,0,1,\langle 0 \rangle,2} \cdot v^{-1}(c_{2,0,2})_5S(v^{-1}(c_{2,0,2})_6)_2 \\
&\quad \cdot v(c_{2,0,1,\langle 0 \rangle,3})_2(v^{-1}(c_{2,0,2})_7)_2v(c_{2,0,3})_3v^{-1}(c_3)_2 \\
&= v(c_1)c_{2,-1}S(v^{-1}(c_{2,0,2})_1v(c_{2,0,3})_1)c_{2,0,1,\langle -1 \rangle}v^{-1}(c_{2,0,2})_2S(v^{-1}(c_{2,0,2})_3) \\
&\quad \cdot v^{-1}(c_{2,0,1,\langle 0 \rangle,1})v^{-1}(c_{2,0,2})_4S(v^{-1}(c_{2,0,2})_7)v(c_{2,0,1,\langle 0 \rangle,3})_1 \\
&\quad \cdot v^{-1}(c_{2,0,2})_8v(c_{2,0,3})_2v^{-1}(c_3)_1 \otimes c_{2,0,1,\langle 0 \rangle,2} \cdot v^{-1}(c_{2,0,2})_5 \\
&\quad \cdot S(v^{-1}(c_{2,0,2})_6)v(c_{2,0,1,\langle 0 \rangle,3})_2v^{-1}(c_{2,0,2})_9v(c_{2,0,3})_3v^{-1}(c_3)_2 \\
&= v(c_1)c_{2,-1}S(v^{-1}(c_{2,0,2})_1v(c_{2,0,3})_1)c_{2,0,1,\langle -1 \rangle}v^{-1}(c_{2,0,1,\langle 0 \rangle,1}) \\
&\quad \cdot v(c_{2,0,1,\langle 0 \rangle,3})_1v^{-1}(c_{2,0,2})_2v(c_{2,0,3})_2v^{-1}(c_3)_1 \\
&\quad \otimes c_{2,0,1,\langle 0 \rangle,2} \cdot v(c_{2,0,1,\langle 0 \rangle,3})_2v^{-1}(c_{2,0,2})_3v(c_{2,0,3})_3v^{-1}(c_3)_2 \\
&= v(c_1)c_{2,-1}c_{2,0,\langle -1 \rangle}v^{-1}(c_{2,0,\langle 0 \rangle,1})v(c_{2,0,\langle 0 \rangle,3})_1v^{-1}(c_3)_1 \\
&\quad \otimes c_{2,0,\langle 0 \rangle,2} \cdot v(c_{2,0,\langle 0 \rangle,3})_2v^{-1}(c_3)_2 \\
&= v(c_1)v^{-1}(c_{2,1})v(c_{2,3})_1v^{-1}(c_3)_1 \otimes c_{2,2} \cdot v(c_{2,3})_2v^{-1}(c_3)_2 \\
&= v(c_1)v^{-1}(c_2)v(c_4)_1v^{-1}(c_5)_1 \otimes c_3 \cdot v(c_4)_2v^{-1}(c_5)_2 \\
&= 1 \otimes c = \sigma(c),
\end{aligned}$$

(4.16)

and it follows that λ is convolution invertible. $\square$

THEOREM 4.5. *Let C be a right H -comodule algebra and consider $\tau, \lambda \in \mathcal{T}(C \otimes H)$. Then τ and λ are equivalent in the sense of [Definition 4.2](#) if and only if there is a left C -colinear and right H -linear coalgebra isomorphism between the crossed coproducts $C \rtimes_{\alpha} H$, ρ and $C \rtimes'_{\alpha'} H$, ρ' corresponding to τ and λ .*

PROOF. Consider (2.38). If $\tau \sim \lambda$, then there exists $v \in \text{Reg}(C \otimes H, H)$ satisfying (4.1) and (4.2). Define

$$u : C \longrightarrow H, \quad u(c) = v^{-1}(c \otimes 1). \quad (4.17)$$

If we can show that u satisfies (2.40) and (2.41), then one implication is proved by Lemma 2.3. It follows from (4.2) that

$$\begin{aligned} & (c_1 \otimes 1)_{(-1)} v(c_2 \otimes 1)_1 \otimes (c_1 \otimes 1)_{(0)} \cdot v(c_2 \otimes 1)_2 \\ &= v(c_1 \otimes 1)(c_2 \otimes 1)_{-1} \otimes (c_2 \otimes 1)_{0,1} \cdot v((c_2 \otimes 1)_{0,2}). \end{aligned} \quad (4.18)$$

Applying $1 \otimes 1 \otimes \varepsilon$ to both sides, we find

$$\begin{aligned} & (c_1 \otimes 1)_{(-1)} v(c_2 \otimes 1) \otimes (1 \otimes \varepsilon)(c_1 \otimes 1)_{(0)} \\ &= v(c_1 \otimes 1)(c_2 \otimes 1)_{-1} \otimes (1 \otimes \varepsilon)(c_2 \otimes 1)_0, \end{aligned} \quad (4.19)$$

and using (2.45), we obtain

$$c_{(-1)} \otimes c_{(0)} = u^{-1}(c_1)c_{2[-1]}u(c_3) \otimes c_{2[0]}. \quad (4.20)$$

So u satisfies (2.40).

Applying $1 \otimes \varepsilon \otimes 1$ to both sides of (4.18), we find

$$\begin{aligned} \alpha'(c) &= v(c_1 \otimes 1)(c_2 \otimes 1)_{-1} v^{-1}(c_3 \otimes 1)_1 \\ &\quad \otimes (\varepsilon \otimes 1)((c_2 \otimes 1)_{0,1} \cdot v((c_2 \otimes 1)_{0,2})v^{-1}(c_3 \otimes 1)_2). \end{aligned} \quad (4.21)$$

It follows from (2.44) that

$$\begin{aligned} & (c \otimes 1)_0 = c_{1[0]} \otimes \alpha_2(c_2), \\ & (\varepsilon \otimes 1)((c_2 \otimes 1)_{0,1} \cdot v((c_2 \otimes 1)_{0,2})v^{-1}(c_3 \otimes 1)_2) \\ &= (\varepsilon \otimes 1)(c_{2,[0],1} \otimes \alpha_2(c_3)_1 v(c_{2,[0],2} \otimes \alpha_2(c_3)_2)v^{-1}(c_4 \otimes 1)_2) \\ &= \alpha_2(c_3)_1 v(c_{2,[0]} \otimes \alpha_2(c_3)_2)v^{-1}(c_4 \otimes 1)_2 \\ &= v(c_{2,[0]} \otimes 1)\alpha_2(c_3)v^{-1}(c_4 \otimes 1)_2 \\ &= v((1 \otimes \varepsilon)(c_2 \otimes 1)_0 \otimes 1)(\varepsilon \otimes 1)(c_3 \otimes 1)_0 v^{-1}(c_4 \otimes 1)_2. \end{aligned} \quad (4.22)$$

So,

$$\begin{aligned} \alpha'(c) &= v(c_1 \otimes 1)(c_2 \otimes 1)_{-1}(c_3 \otimes 1)_{-1} v^{-1}(c_4 \otimes 1)_1 \\ &\quad \otimes v((1 \otimes \varepsilon)(c_2 \otimes 1)_0 \otimes 1)(\varepsilon \otimes 1)(c_3 \otimes 1)_0 v^{-1}(c_4 \otimes 1)_2 \\ &= u^{-1}(c_1)c_{2[-1]}\alpha_1(c_3)u(c_4)_1 \otimes u^{-1}(c_{2[0]})\alpha_2(c_3)u(c_4)_2, \end{aligned} \quad (4.23)$$

and (2.41) follows.

Conversely, assume that the two crossed coproducts are isomorphic. By [Lemma 2.3](#), there exists $u \in \text{Reg}(C, H)$ satisfying (2.40) and (2.41). Define

$$v : C \otimes H \longrightarrow H, \quad v(c \otimes h) = S(h_1)u^{-1}(c)h_2. \quad (4.24)$$

Then

$$\begin{aligned} \varepsilon_H v(c \otimes h) &= \varepsilon(S(h_1)u^{-1}(c)h_2) = \varepsilon(c)\varepsilon(h), \\ v((c \otimes h) \cdot g) &= v(c \otimes hg) = S(hg)_1 u^{-1}(c)(hg)_2 \\ &= S(g_1)S(h_1)u^{-1}(c)h_2 g_2 = S(g_1)v(c \otimes h)g_2. \end{aligned} \quad (4.25)$$

So,

$$\begin{aligned} \lambda(c \otimes h) &= (c \otimes h)_1 \otimes (c \otimes h)_0 \\ &= S(h_1)c_{1,(-1)}\alpha'_1(c_2)h_2 \otimes c_{1,(0)} \otimes \alpha'_2(c_2)h_3 \\ &= S(h_1)u^{-1}(c_1)c_{2,[-1]}u(c_3)u^{-1}(c_4)c_{5,[-1]}\alpha_1(c_6)u(c_7)_1 h_2 \\ &\quad \otimes c_{2,[0]} \otimes u^{-1}(c_{5,[0]})\alpha_2(c_6)u(c_7)_2 h_3 \\ &= S(h_1)u^{-1}(c_1)c_{2,[-1]}\alpha_1(c_3)u(c_4)_1 h_2 \otimes c_{2,[0],1} \\ &\quad \otimes u^{-1}(c_{2,[0],2})\alpha_2(c_3)u(c_4)_2 h_3 \\ &= S(h_1)u^{-1}(c_1)h_2 S(h_3)c_{2,[-1]}\alpha_1(c_3)h_4 S(h_7)u(c_4)_1 h_8 \\ &\quad \otimes c_{2,[0],1} \otimes u^{-1}(c_{2,[0],2})\alpha_2(c_3)h_5 S(h_6)u(c_4)_2 h_9 \\ &= S(h_1)u^{-1}(c_1)h_2 S(h_3)c_{2,[-1]}\alpha_1(c_3)h_4 S(h_9)u(c_4)_1 h_{10} \otimes c_{2,[0],1} \\ &\quad \otimes \alpha_2(c_3)_1 h_5 S(h_6)S(\alpha_2(c_3)_2)u^{-1}(c_{2,[0],2})\alpha_2(c_3)_3 h_7 S(h_8)u(c_4)_2 h_{11} \\ &= v(c_1 \otimes h_1)(c_2 \otimes h_2)_{-1} v^{-1}(c_3 \otimes h_3)_1 \\ &\quad \otimes (c_2 \otimes h_2)_{0,1} \cdot v((c_2 \otimes h_2)_{0,2})v^{-1}(c_3 \otimes h_3)_2. \end{aligned} \quad (4.26)$$

This shows that $\tau \sim \lambda$. □

5. Twisting Hopf-Galois coextensions. Let H be a Hopf algebra with bijective antipode S , and C a right H -module coalgebra. As before, we use the following notation:

$$B = C/I, \quad I = \{c(h - \varepsilon(h)) \mid h \in H, c \in C\}. \quad (5.1)$$

For $\tau \in \mathcal{T}(C)$, we have that $C^\tau/I^\tau = C/I = B$.

Now, assume that C/B is an H -Galois coextension (see [5]). This means that the canonical map

$$\beta : C \otimes H \longrightarrow C \square_B C, \quad \beta(c \otimes h) = c_1 \otimes c_2 \cdot h, \quad (5.2)$$

is a bijection. Recall that, in this situation, $C \square_B C$ is the cotensor product

$$C \square_B C = \left\{ \sum_i C_i \otimes d_i \in C \otimes C \mid \sum_i C_{i_1} \otimes \pi(c_{i_2}) \otimes d_i = \sum_i C_i \otimes \pi(d_{i_1}) \otimes d_{i_2} \right\}, \quad (5.3)$$

where $\pi : C \rightarrow B$ is the natural epimorphism.

LEMMA 5.1. *With notation as above, consider the map*

$$\beta' : C \otimes H \longrightarrow C \square_B C, \quad \beta'(c \otimes h) = c_1 \cdot h \otimes c_2. \quad (5.4)$$

If the antipode S is bijective, then β is bijective (resp., injective, resp., surjective) if and only if β' is bijective (resp., injective, resp., surjective).

PROOF. The map

$$\phi : C \otimes H \longrightarrow C \otimes H, \quad \phi(c \otimes h) = c \cdot h_1 \otimes S(h_2) \quad (5.5)$$

is a bijection with inverse

$$\phi^{-1}(c \otimes h) = c \cdot h_2 \otimes \bar{S}h_1. \quad (5.6)$$

Then the statement follows from the fact that $\beta' = \beta \circ \phi$. $\square$

THEOREM 5.2. *Take $\tau \in U(\mathcal{T}(C))$. Then C^τ/B is an H -Galois coextension if and only if C/B is an H -Galois coextension.*

PROOF. Let λ be the inverse of τ . As before, we use the notation (2.24). Let β^τ be the canonical map corresponding to the coextension C^τ/B , that is,

$$\beta^\tau(c \otimes h) = c_1 \cdot c_{2,-1} h \otimes c_{2,0}. \quad (5.7)$$

Consider the following diagram:

$$\begin{array}{ccc} C \otimes H & \xrightarrow{\beta} & C \square_B C \\ f \downarrow & & \downarrow g \\ C \otimes H & \xrightarrow{\beta^\tau} & C \square_B C, \end{array} \quad (5.8)$$

where

$$f(c \otimes h) = c_0 \otimes \bar{S}(c_{-1})h, \quad g(c \otimes d) = c_0 \cdot \bar{S}(c_{-1}) \otimes d, \quad (5.9)$$

f and g are bijections, with inverses given by

$$f^{-1}(c \otimes h) = c_{(0)} \otimes \bar{S}(c_{(-1)})h, \quad g^{-1}(c \otimes d) = c_{(0)} \cdot \bar{S}(c_{(-1)}) \otimes d. \quad (5.10)$$

We can also compute

$$\begin{aligned}
 \beta^\tau f(c \otimes h) &= \beta^\tau (c_0 \otimes \bar{S}(c_{-1})h) \\
 &= c_{0,1} \cdot c_{0,2,-1} \bar{S}(c_{-1})h \otimes c_{0,2,0} \\
 &= c_{1,0} \cdot c_{2,-1,2} \bar{S}(c_{1,-1}c_{2,-1,1})h \otimes c_{2,0} \\
 &= c_{1,0} \cdot c_{2,-1,2} \bar{S}(c_{2,-1,1}) \bar{S}(c_{1,-1})h \otimes c_{2,0} \\
 &= c_{1,0} \cdot \bar{S}(c_{1,-1})h \otimes c_2 \\
 &= c_{1,0} \cdot h_3 \bar{S}(h_2) \bar{S}(c_{1,-1})h_1 \otimes c_2 \\
 &= c_{1,0} \cdot h_3 \bar{S}(S(h_1)c_{1,-1}h_2) \otimes c_2 \\
 &= (c_1 \cdot h)_0 \cdot \bar{S}((c_1 \cdot h)_{-1}) \otimes c_2 \\
 &= g(c_1 \cdot h \otimes c_2) = g\beta(c \otimes h).
 \end{aligned} \tag{5.11}$$

This shows that (5.8) is commutative and it follows that β is bijective if and only if β^τ is bijective. $\square$

THEOREM 5.3. *Let C/B be an H -Galois coextension and take $\tau, \lambda \in \mathcal{T}(C)$. Every left B -colinear and right H -linear coalgebra map*

$$\psi : C^\tau \longrightarrow C^\lambda \tag{5.12}$$

is of the form $\psi(c) = c_1 \cdot v(c_2)$, where $v \in \text{Hom}(C, H)$ satisfies conditions (4.1) and (4.2) of Proposition 4.1. If ψ is an isomorphism, then $v \in \text{Reg}(C, H)$.

PROOF. We use the notation (2.24). As in [5], we consider the map

$$\bar{\tau} = (\varepsilon \otimes 1)\beta^{-1} : C \square_B C \longrightarrow H. \tag{5.13}$$

We introduce the notation $c \diamond d = \bar{\tau}(c \otimes d)$. Then we have the following properties:

$$\varepsilon_H(c \diamond d) = \varepsilon_C(c)\varepsilon_C(d), \tag{5.14}$$

$$(c \diamond d)h = c \diamond (d \cdot h), \tag{5.15}$$

$$(c \cdot h) \diamond d = S(h)(c \diamond d), \tag{5.16}$$

$$c_1 \cdot (c_2 \diamond d) = \varepsilon(c)d. \tag{5.17}$$

The map

$$v : C \longrightarrow H, \quad v(c) = c_1 \diamond \psi(c_2), \tag{5.18}$$

satisfies the property

$$c_1 \cdot v(c_2) = c_1 \cdot (c_2 \diamond \psi(c_3)) = \psi(c). \tag{5.19}$$

Since ψ is a coalgebra,

$$\varepsilon_H v(c) = \varepsilon_H(c_1 \diamond \psi(c_2)) = \varepsilon(\psi(c)) = \varepsilon(c), \quad (5.20)$$

and it follows that $\varepsilon_H \circ v = \varepsilon_C$.

It follows from (5.16) and (5.17) that

$$\begin{aligned} v(c \cdot h) &= (c \cdot h)_1 \diamond \psi((c \cdot h)_2) = c_1 \cdot h_1 \diamond \psi(c_2 \cdot h_2) \\ &= S(h_1)(c_1 \diamond \psi(c_2))h_2 = S(h_1)v(c)h_2. \end{aligned} \quad (5.21)$$

Since ψ is a coalgebra map, we have that

$$\begin{aligned} \psi(c_1 \cdot c_{2,-1}) \otimes \psi(c_{2,0}) &= \psi(c)_1 \psi(c)_{2,(-1)} \otimes \psi(c)_{2,(0)}, \\ c_1 \cdot v(c_2)c_{3,-1} \otimes c_{3,0,1} \cdot v(c_{3,0,2}) \\ &= (c_1 \cdot v(c_2))_1 (c_1 \cdot v(c_2))_{2,(-1)} \otimes (c_1 \cdot v(c_2))_{2,(0)} \\ &= c_1 \cdot v(c_3)_1 (c_2 \cdot v(c_3))_{2,(-1)} \otimes (c_2 \cdot v(c_3))_{2,(0)} \\ &= c_1 \cdot c_{2,(-1)} v(c_3)_1 \otimes c_{2,(0)} \cdot v(c_3)_2, \end{aligned} \quad (5.22)$$

which is equivalent to

$$\begin{aligned} c_1 \otimes c_2 \cdot v(c_3)c_{4,-1} \otimes c_{4,0,1} \cdot v(c_{4,0,2}) \\ = c_1 \otimes c_2 \cdot c_{3,(-1)} v(c_4)_1 \otimes c_{3,(0)} \cdot v(c_4)_2. \end{aligned} \quad (5.23)$$

After we apply β^{-1} to both sides, we obtain

$$\begin{aligned} c_1 \otimes v(c_2)c_{3,-1} \otimes c_{3,0,1} \cdot v(c_{3,0,2}) &= c_1 \otimes c_{2,(-1)} v(c_3)_1 \otimes c_{2,(0)} \cdot v(c_3)_2, \\ v(c_1)c_{2,-1} \otimes c_{2,0,1} \cdot v(c_{2,0,2}) &= c_{1,(-1)} v(c_2)_1 \otimes c_{1,(0)} \cdot v(c_2)_2. \end{aligned} \quad (5.24)$$

If ψ is an isomorphism, then $\psi^{-1} : C^\lambda \rightarrow C^\tau$ is a left B -colinear and right H -linear coalgebra map. Then we have a map $w : C \rightarrow H$ satisfying (4.1) and (4.2) such that

$$\psi^{-1}(c) = c_1 \cdot w(c_2). \quad (5.25)$$

For all $c \in C$, we have that

$$c = c_1 \cdot v(c_2)w(c_3) = c_1 \cdot w(c_2)v(c_3). \quad (5.26)$$

Proceeding as in the proof of (5.24), we find that v is convolution invertible. $\square$

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REFERENCES

- [1] M. Beattie, C.-Y. Chen, and J. J. Zhang, *Twisted Hopf comodule algebras*, Comm. Algebra **24** (1996), no. 5, 1759–1775.
- [2] M. Beattie and B. Torrecillas, *Twistings and Hopf Galois extensions*, J. Algebra **232** (2000), no. 2, 673–696.
- [3] R. J. Blattner, M. Cohen, and S. Montgomery, *Crossed products and inner actions of Hopf algebras*, Trans. Amer. Math. Soc. **298** (1986), no. 2, 671–711.
- [4] R. J. Blattner and S. Montgomery, *Crossed products and Galois extensions of Hopf algebras*, Pacific J. Math. **137** (1989), no. 1, 37–54.
- [5] T. Brzeziński and P. M. Hajac, *Coalgebra extensions and algebra coextensions of Galois type*, Comm. Algebra **27** (1999), no. 3, 1347–1367.
- [6] S. Caenepeel, *Harrison cocycles and the group of Galois coobjects*, Algèbre Non Commutative, Groupes Quantiques et Invariants (Reims, 1995), Sémin. Congr., vol. 2, Société Mathématique de France, Paris, 1997, pp. 83–101.
- [7] S. Caenepeel, S. Dăscălescu, G. Militaru, and F. Panaite, *Coalgebra deformations of bialgebras by Harrison cocycles, copairings of Hopf algebras and double crosscoproducts*, Bull. Belg. Math. Soc. Simon Stevin **4** (1997), no. 5, 647–671.
- [8] S. Dăscălescu, G. Militaru, and Ş. Raianu, *Crossed coproducts and cleft coextensions*, Comm. Algebra **24** (1996), no. 4, 1229–1243.
- [9] S. Dăscălescu, C. Năstăsescu, and Ş. Raianu, *Hopf Algebras. An Introduction*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 235, Marcel Dekker, New York, 2001.
- [10] S. Dăscălescu, Ş. Raianu, and Y. H. Zhang, *Finite Hopf-Galois coextensions, crossed coproducts, and duality*, J. Algebra **178** (1995), no. 2, 400–413.
- [11] Y. Doi and M. Takeuchi, *Cleft comodule algebras for a bialgebra*, Comm. Algebra **14** (1986), no. 5, 801–817.
- [12] T. X. Ju and C. R. Cai, *Twisted Hopf module coalgebras*, Comm. Algebra **28** (2000), no. 1, 307–320.
- [13] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conference Series in Mathematics, vol. 82, American Mathematical Society, Rhode Island, 1993.
- [14] M. E. Sweedler, *Cohomology of algebras over Hopf algebras*, Trans. Amer. Math. Soc. **133** (1968), 205–239.
- [15] ———, *Hopf Algebras*, Mathematics Lecture Note Series, W. A. Benjamin, New York, 1969.

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