ON UNIFORMLY CLOSE-TO-CONVEX FUNCTIONS
AND UNIFORMLY QUASICONVEX FUNCTIONS

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Two new subclasses of uniformly convex and uniformly close-to-convex functions are introduced. We obtain inclusion relationships and coefficient bounds for these classes.

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1. The class \( UCC(\alpha) \). Denote by \( S \) the family consisting of functions

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

that are analytic and univalent in \( \Delta = \{ z : |z| < 1 \} \) and by \( C, S^*, \) and \( K \) the subfamilies of functions that are, respectively, convex, starlike, and close to convex in \( \Delta \). Noor and Thomas [7] introduced the class of functions known as quasiconvex functions. A normalized function of the form (1.1) is said to be quasiconvex in \( \Delta \) if there exists a convex function \( g \) with \( g(0) = 0, \ g'(0) = 1 \) such that for \( z \in \Delta \),

\[
\text{Re} \left( \frac{zf'(z)}{g'(z)} \right) > 0.
\]

Let \( Q \) denote the class of quasiconvex functions defined in \( \Delta \). It was shown that \( Q < K \), where \( < \) denotes subordination, so that every quasiconvex function is close to convex. Goodman [2, 3] introduced the classes UCV and UST of uniformly convex and uniformly starlike functions. In [10], Rønning defined the class \( UCV(\alpha) \), \( -1 \leq \alpha < 1 \), consisting of functions of the form (1.1) satisfying

\[
\text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] - \alpha \geq \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in \Delta.
\]

Geometrically, \( UCV(\alpha) \) is the family of functions \( f \) for which \( 1 + zf''(z)/f'(z) \) takes values that lie inside the parabola \( \Omega = \{ \omega : \text{Re}(\omega - \alpha) > |\omega - 1| \} \), which is symmetric about the real axis and whose vertex is \( w = (1 + \alpha)/2 \).
Since the function

\[ q_\alpha(z) = 1 + \frac{2(1 - \alpha)}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 \]  

(1.4)

maps \( \Delta \) onto this parabolic region, \( f \in \text{UCV}(\alpha) \) if and only if

\[ 1 + \frac{zf''(z)}{f'(z)} < q_\alpha(z). \]  

(1.5)

Rønning [10] also defined the family \( S_p(\alpha) \) consisting of functions \( zf'(z) \) when \( f \) is in \( \text{UCV}(\alpha) \). In particular, \( f \) is in \( S_p(\alpha) \) if and only if \( zf'(z)/f(z) \preceq q_\alpha(z) \).

Note for \( g(z) = zf'(z)/f(z) \) that

\[ g(z) + zg'(z)/g(z) = 1 + zf''(z)/f'(z), \]

and hence a result of Miller and Mocanu [6] shows that \( \text{UCV}(\alpha) \subset S_p(\alpha) \).

Kumar and Ramesha [4] investigated the class \( \text{UCC} \) of uniformly close-to-convex functions consisting of normalized functions of the form (1.1) satisfying

\[ f'(z)/g'(z) \preceq q_0(z), \]

where \( g(z) \in C \) and \( q_0(z) \) is given by (1.4) for \( \alpha = 0 \).

More generally, we give the following definition.

**Definition 1.1.** A function \( f \) is said to be uniformly close to convex of order \( \alpha \), \(-1 \leq \alpha < 1\), denoted by \( \text{UCC}(\alpha) \), if \( f'(z)/g'(z) \preceq q_\alpha(z) \), where \( q_\alpha(z) \) is as defined by (1.4) and \( g(z) \) is convex.

Since \( \Re q_\alpha(z) > 0 \), we see that \( \text{UCC}(\alpha) \) is a subclass of \( K \). To see that \( \text{UCC}(\alpha) \) also contains the family \( S_p(\alpha) \), we note for \( f \in S_p(\alpha) \subset S^* \) that \( f(z) = zg'(z) \) for some \( g \in C \). Hence, \( zf'(z)/f(z) = f'(z)/g'(z) \preceq q_\alpha(z) \).

We have thus proved the following inclusion chain.

**Theorem 1.2.** For \(-1 \leq \alpha < 1\), \( \text{UCV}(\alpha) \subset S_p(\alpha) \subset \text{UCC}(\alpha) \subset K \).

We next give a sufficient condition for a function to be in \( \text{UCC}(\alpha) \).

**Theorem 1.3.** If \( \sum_{n=2}^{\infty} n|a_n| \leq (1 - \alpha)/2 \), then \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is in \( \text{UCC}(\alpha) \), \(-1 \leq \alpha < 1\).

**Proof.** Setting \( g(z) = z \), we have

\[ f'(z)/g'(z) = f'(z)/f''(z) = 1 + \sum_{n=2}^{\infty} n a_n z^{n-1}, \]

so that for \( z \in \Delta \),

\[ \left| \frac{f'(z)}{g'(z)} - 1 \right| < \sum_{n=2}^{\infty} n|a_n| \leq 1 - \sum_{n=2}^{\infty} n|a_n| - \alpha \leq \Re f'(z) - \alpha. \]  

(1.6)

Thus \( f'(z)/g'(z) \) lies in the parabolic region \( \Omega = \{ \omega : |\omega - 1| < \Re(\omega - \alpha) \} \).

That is, \( f'(z)/g'(z) < q_\alpha(z) \), where \( q_\alpha(z) \) is as defined by (1.4). \( \Box \)
2. A convolution relation. We now prove a convolution result for the family UCC(\(\alpha\)). But first we need the following lemma.

**Lemma 2.1** (see [8]). Let \(\phi(z) \in C, \psi \in S^*\). If \(F(z)\) is analytic and \(\text{Re}\{F(z)\} > \alpha, -1 \leq \alpha < 1\), then

\[
\text{Re}\left\{ \frac{\phi \ast F \psi}{\phi \ast \psi} \right\} > \alpha, \quad z \in \Delta.
\] (2.1)

The above result was proved in [11] for the case \(\alpha = 0\).

**Theorem 2.2.** If \(f \in \text{UCC}(\alpha)\), then to each \(g \in S^*\), an \(h \in S^*\) may be associated for which \(\text{Re}(f \ast g)/h > (1 + \alpha)/2, z \in \Delta\).

**Proof.** If \(f \in \text{UCC}(\alpha)\), then \(f'/g'_1(z) \prec q_\alpha(z)\), where \(g_1(z) \in C\) and \(q_\alpha(z)\) is defined by (1.4). Hence, \(\text{Re}(f'(z)/g'_1(z)) > (1 + \alpha)/2\). Therefore, we can find an \(\psi \in S^*\) for which

\[
\text{Re} \left( \frac{zf'/\psi}{f} \right) > \frac{1 + \alpha}{2}.
\] (2.2)

Set \(F(z) = zf'/\psi(z)\). Then, for \(g \in S^*\), there corresponds a \(\phi \in C\) such that \(z\phi' = g\). Also \(f \ast g = zf' \ast \phi = \phi \ast F \psi \) and \(h = \phi \ast \psi \in S^*\). By Lemma 2.1,

\[
\text{Re} \left( \frac{\phi \ast F \psi}{\phi \ast \psi} \right) = \text{Re} \left( \frac{f \ast g}{h} \right) > \frac{1 + \alpha}{2},
\] (2.3)

and this proves the result. \(\square\)

3. Coefficient estimates. We need the following result by Rogosinski [9] to obtain coefficient bounds for the class UCC(\(\alpha\)).

**Lemma 3.1.** Let \(h(z) = 1 + \sum_{k=1}^\infty c_k z^k\) be subordinate to \(H(z) = 1 + \sum_{k=1}^\infty C_k z^k\). If \(H(z)\) is univalent in \(\Delta\) and \(H(\Delta)\) is convex, then \(|c_n| \leq |C_1|\).

**Theorem 3.2.** If \(f(z) = z + \sum_{n=2}^\infty a_n z^n \in \text{UCC}(\alpha)\), then

\[
|a_n| \leq (n - 1)c + 1, \quad n \geq 2,
\] (3.1)

where \(c = 4(1 - \alpha)/\pi^2\).

**Proof.** Set

\[
\Phi(z) = \frac{f'(z)}{g'(z)} = 1 + \sum_{k=1}^\infty c_k z^k
\] (3.2)

so that \(\Phi(z) \prec q_\alpha(z)\), where \(q_\alpha(z)\) is defined in (1.4).
Since \( q_\alpha(z) \) is univalent and maps \( \Delta \) onto a convex region, we may apply Lemma 3.1.

Now
\[
q_\alpha(z) = 1 + \frac{8(1 - \alpha)}{\pi^2} z + \cdots,
\]
so that
\[
|c_n| \leq \frac{8(1 - \alpha)}{\pi^2}.
\] (3.3)

With \( g(z) = z + \sum_{k=2}^{\infty} b_k z^k \), we compare the coefficients of \( z^n \) for the expansion of \( \phi(z) \) to obtain
\[
(n + 1) |a_{n+1}| = c_n + \sum_{k=1}^{n-1} (k + 1)b_{k+1}c_{n-k} + (n + 1)b_{n+1}.
\] (3.4)

Since \( g(z) \) is convex, it is well known that \( |b_n| \leq 1, n = 1, 2, \ldots \). From (3.4), we get
\[
(n + 1) |a_{n+1}| \leq c n(n + 1) + (n + 1),
\] (3.5)
and the proof is complete.

4. The class \( UQC(\alpha) \). We now introduce a natural analogue to the class \( UCV(\alpha) \) in terms of Alexander’s result on convex functions [1, page 43].

**Definition 4.1.** A normalized function of the form (1.1) is said to be uniformly quasiconvex of order \( \alpha \), \(-1 \leq \alpha < 1\), in \( \Delta \), denoted by \( UQC(\alpha) \), if there exists a convex function \( g(z) \) with \( g(0) = 0, g'(0) = 1 \), such that
\[
(zf'(z))' \prec q_\alpha(z),
\] (4.1)
where \( q_\alpha(z) \) is as defined by (1.4).

**Remark 4.2.** (1) By setting \( f(z) = g(z) \), we see that \( UCV(\alpha) \subseteq UQC(\alpha) \).

(2) We see that \( f \in UQC(\alpha) \) if and only if \( zf' \in UCC(\alpha) \).

In view of the above remark, we obtain from Theorem 1.3 a sufficient coefficient bound for inclusion in the family \( UQC(\alpha) \).

**Theorem 4.3.** If \( \sum_{n=2}^{\infty} n^2 |a_n| \leq (1 - \alpha)/2 \), then \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in UQC(\alpha) \).

We next prove a theorem which shows that every function in \( UQC(\alpha) \) is close to convex and hence univalent. We need a result due to Miller and Mocanu [5].

**Lemma 4.4.** Let \( M(z) \) and \( N(z) \) be regular in \( \Delta \) with \( M(z) = N(z) = 0 \) and let \( \alpha \) be real. If \( N(z) \) maps \( \Delta \) onto a possibly many-sheeted region which is starlike with respect to the origin, then for \( z \in \Delta \),
\[
\text{Re} \frac{M'(z)}{N'(z)} > \alpha \Rightarrow \text{Re} \frac{M(z)}{N(z)} > \alpha.
\] (4.2)
**Theorem 4.5.** If $F(z) \in \text{UQC}(\alpha)$, then $F(z) \in K$ and hence it is univalent in $\Delta$.

**Proof.** Since

$$\frac{(zf'(z))'}{g'(z)} < q_\alpha(z) \Rightarrow \Re \left\{ \frac{(zf'(z))'}{g'(z)} \right\} > \frac{1 + \alpha}{2},$$

an application of Lemma 4.4, with $M(z) = zf'(z)$, $N(z) = g(z)$, proves the result.

**Theorem 4.6.** If $f(z) \in \text{UQC}(\alpha)$, then $H(z) = \int_0^zf'(t)dt$ is in $\text{UCC}(\alpha)$.

**Proof.** If $f(z) \in \text{UQC}(\alpha)$, then there exists a function $g(z) \in C$ such that $(zf'(z))'/g'(z) < q_\alpha(z)$, where $q_\alpha(z)$ is as given by (1.4). The result now follows on observing that $H'(z) = (zf'(z))'$.

We close with coefficient estimates for the class $\text{UQC}(\alpha)$.

**Theorem 4.7.** If $f(z) = z + \sum_{n=2}^{\infty}a_nz^n \in \text{UQC}(\alpha)$, then

$$|a_n| \leq \frac{(n-1)c + 1}{n}, \quad n \geq 2,$$

where $c = 4(1 - \alpha)/\pi^2$.

**Proof.** Proceeding on the same lines as in the proof of Theorem 3.2, we obtain the result.

**Remark 4.8.** When $\alpha = 0$, $\text{UQC}(0) = Q$ [6] and we see that the bounds are lower than the corresponding bounds for $Q$ in [6].

**References**


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