COEFFICIENTS ESTIMATES FOR FUNCTIONS IN $B_n(\alpha)$

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Received 12 February 2003

We consider functions $f$, analytic in the unit disc and of the normalised form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For functions $f \in B_n(\alpha)$, the class of functions involving the Sălăgean differential operator, we give some coefficient estimates, namely, $|a_2|$, $|a_3|$, and $|a_4|$.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let $A$ be the class of functions $f$ which are analytic in the unit disc $D = \{z : |z| < 1\}$ and are of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j.$$  \hfill (1.1)

For functions $f \in A$, we introduce the subclass $B_n(\alpha)$ given by the following definition.

**Definition 1.1.** For $\alpha > 0$ and $n = 0, 1, 2, \ldots$, a function $f$ normalised by (1.1) belongs to $B_n(\alpha)$ if and only if, for $z \in D$,

$$\text{Re} \frac{D^n[f(z)]^\alpha}{z^\alpha} > 0,$$  \hfill (1.2)

where $D^n$ denotes the differential operator with $D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$ and $D^0 f(z) = f(z)$.

**Remark 1.2.** The differential operator $D^n$ was introduced by Sălăgean [5].

For $n = 1$, $B_1(\alpha)$ denotes the class of Bazilevič functions with logarithmic growth studied [4, 6, 7], amongst others. In [2], the author established some properties of the class $B_n(\alpha)$ including showing that $B_n(\alpha)$ forms a subclass of $S$, the class of all analytic, normalized, and univalent functions in $D$. The class $B_0(\alpha)$ was initiated by Yamaguchi [8].

2. Preliminary results. In proving our results, we need the following lemmas. However, we first denote $P$ to be the class of analytic functions with a positive real part in $D$. 
Lemma 2.1. Let \( p \in P \) and let it be of the form \( p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i \). Then
(i) \(|c_i| \leq 2\) for \( i \geq 1 \),
(ii) \(|c_2 - \mu c_1^2| \leq 2 \max\{1, |1 - 2\mu|\}\) for any \( \mu \in \mathbb{C} \).

Lemma 2.2 (see [3]). If the functions \( 1 + \sum_{v=1}^{\infty} b_v z^v \) and \( 1 + \sum_{v=1}^{\infty} c_v z^v \) belong to \( P \), then the same is true for the function \( 1 + (1/2) \sum_{v=1}^{\infty} b_v c_v z^v \).

Lemma 2.3 (see [3]). Let \( h(z) = 1 + h_1 z + h_2 z^2 + \cdots \) and let \( 1 + g(z) = 1 + g_1 z + g_2 z^2 + \cdots \) be functions in \( P \). Set \( \gamma_0 = 1 \) and for \( v \geq 1 \),
\[
\gamma_v = 2 - v \left[ 1 + \frac{1}{2} \sum_{\mu=1}^{v} \left( \frac{v}{\mu} \right) h_\mu \right].
\]
(2.1)

If \( A_k \) is defined by
\[
\sum_{v=1}^{\infty} (-1)^{v+1} y_{v-1} (g(z))^v = \sum_{k=1}^{\infty} A_k z^k,
\]
then
\[
|A_k| \leq 2.
\]
(2.3)

3. Results

Theorem 3.1. If \( \alpha > 0, n = 0, 1, 2, \ldots, \) and \( f \in B_n(\alpha) \) (\( n \) is fixed) with \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then the following inequalities hold:
\[
|a_2| \leq \frac{2\alpha^{n-1}}{(1 + \alpha)^n},
\]
(3.1)
\[
|a_3| \leq \begin{cases} 
\frac{2\alpha^{n-1}}{(2 + \alpha)^n} \left( 1 - \frac{\alpha - 1}{\alpha} \right) \left( \frac{\alpha^2 + 2\alpha}{\alpha^2 + 2\alpha + 1} \right)^n, & \text{for } 0 < \alpha < 1, \\
\frac{2\alpha^{n-1}}{(2 + \alpha)^n}, & \text{for } \alpha \geq 1,
\end{cases}
\]
(3.2)
\[
|a_4| \leq \begin{cases} 
\frac{2\alpha^{n-1}}{(3 + \alpha)^n} \left( 1 - (1 - \alpha)\alpha^{2n-2} \right) \left( 1 + \frac{(1 - 2\alpha)(2 + \alpha)^n \alpha^{-n-1}}{3(1 + \alpha)^{2n}} \right), & \text{for } 0 < \alpha < 1, \\
\frac{2\alpha^{n-1}}{(3 + \alpha)^n}, & \text{for } \alpha \geq 1,
\end{cases}
\]
(3.3)

Remark 3.2. When \( n = 1 \), the above results reduce to those obtained by Singh [6].
**Proof.** For \( f \in B_n(\alpha) \), **Definition 1.1** gives

\[
\text{Re} \frac{D^n f(z)^\alpha}{z^\alpha} > 0. \tag{3.4}
\]

Inequality (3.4) suggests that there exists \( p \in P \) such that for \( z \in D \),

\[
\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n p(z). \tag{3.5}
\]

Next, writing \( D^n f(z)^\alpha \) as \( z[D^{n-1} f(z)^\alpha]' \) and \( p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i \) in (3.5), it follows that

\[
[D^{n-1} f(z)^\alpha]' = \alpha^n \left( z^{\alpha-1} + \sum_{i=1}^{\infty} c_i z^{i+\alpha-1} \right) \tag{3.6}
\]

and integration gives

\[
\frac{D^{n-1} f(z)^\alpha}{z^\alpha} = \alpha^{n-1} \left[ 1 + \sum_{i=1}^{\infty} \frac{c_i z^i}{(i+\alpha)} \right]. \tag{3.7}
\]

Now, repeating the process, we are able to establish the following relation which holds in general for any \( k = 0, 1, 2, \ldots, n \)

\[
\frac{D^{n-k} f(z)^\alpha}{z^\alpha} = \alpha^{n-k} \left[ 1 + \sum_{i=1}^{\infty} \frac{c_i z^i}{(i+\alpha)^k} \right]. \tag{3.8}
\]

In particular, when \( n = k \), we have

\[
\frac{D^0 f(z)^\alpha}{z^\alpha} = \left( \frac{f(z)}{z} \right)^\alpha = 1 + \sum_{i=1}^{\infty} \frac{\alpha^n c_i z^i}{(i+\alpha)^n}. \tag{3.9}
\]

On comparing coefficients in (3.9) with \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), we obtain

\[
\alpha a_2 = \frac{\alpha^n c_1}{(1+\alpha)^n}, \tag{3.10}
\]

\[
\alpha a_3 = \frac{\alpha^n c_2}{(2+\alpha)^n} + \frac{\alpha(1-\alpha) a_2^2}{2}, \tag{3.11}
\]

\[
\alpha a_4 = \frac{\alpha^n c_3}{(3+\alpha)^n} + \frac{\alpha(1-\alpha)(\alpha-2) a_2^3}{6} + \alpha(1-\alpha) a_3 a_2. \tag{3.12}
\]

Inequality (3.1) follows easily from (3.10) for all \( \alpha > 0 \) since \( |c_1| \leq 2 \).
Eliminating $a_2$ in (3.11), we have

$$a_3 = \frac{\alpha^{n-1} c_2}{(2 + \alpha)^n} + \frac{(1 - \alpha)\left(\alpha^{n-1} c_1\right)^2}{2}$$

$$= \frac{\alpha^{n-1}}{(2 + \alpha)^n} \left[c_2 - (\alpha - 1)\frac{(2 + \alpha)^n}{2(1 + \alpha)^n} \alpha^{n-1} c_1^2\right]$$

$$= \frac{\alpha^{n-1}}{(2 + \alpha)^n} (c_2 - \mu c_1^2)$$

$$\leq \frac{2\alpha^{n-1}}{(2 + \alpha)^n} \max\{1, |1 - 2\mu|\},$$

where we used Lemma 2.1(ii) with

$$2\mu = \frac{(\alpha - 1)\alpha^{n-1}}{(1 + \alpha)^n} \left(\frac{2 + \alpha}{1 + \alpha}\right)^n.$$ 

(3.13)

Since $\mu \geq 0$ for $\alpha \geq 1$, both inequalities in (3.2) are easily obtained.

We now prove (3.3). Using (3.10) and (3.11) in (3.12) gives

$$a_4 = \frac{\alpha^{n-1}}{(3 + \alpha)^n} \left[c_3 + \frac{(1 - \alpha)(3 + \alpha)^n\alpha^{n-1}}{(1 + \alpha)^n} \left(\frac{c_1 c_2}{(2 + \alpha)^n} + \frac{(1 - 2\alpha)\alpha^{n-1} c_1^2}{6(1 + \alpha)^{2n}}\right)\right].$$

(3.15)

First, we consider the case $0 < \alpha < 1/2$. Applying the triangle inequality with Lemma 2.1(i) in (3.15) results in the inequality

$$|a_4| \leq \frac{2\alpha^{n-1}}{(3 + \alpha)^n} \left[1 + \frac{2(1 - \alpha)(3 + \alpha)^n\alpha^{n-1}}{(1 + \alpha)^n} \left(\frac{1}{(2 + \alpha)^n} + \frac{(1 - 2\alpha)\alpha^{n-1}}{3(1 + \alpha)^{2n}}\right)\right]$$

(3.16)

which is the first inequality in (3.3).

For the case $1/2 \leq \alpha < 1$, we use Carathéodory-Toeplitz result which states that for some $\varepsilon$ with $|\varepsilon| < 1$,

$$c_2 = \frac{c_1^2}{2} + \varepsilon \left(2 - \frac{|c_1|^2}{2}\right).$$

(3.17)

Thus, (3.15) becomes

$$a_4 = \frac{\alpha^{n-1}}{(3 + \alpha)^n} \left[c_3 + \frac{(1 - \alpha)(3 + \alpha)^n\alpha^{n-1} c_1}{(1 + \alpha)^n}\right.$$  

$$\times \left(\frac{c_1^2}{2(2 + \alpha)^n} + \frac{(1 - 2\alpha)\alpha^{n-1} c_1^2}{6(1 + \alpha)^{2n}} + \frac{\varepsilon}{(2 + \alpha)^n} \left(\frac{2 - |c_1|^2}{2}\right)\right)\right].$$

(3.18)
We then have

\[ |a_4| \leq \frac{\alpha^{n-1}}{(3 + \alpha)^n} \left( |c_3| + \frac{(1 - \alpha)(3 + \alpha)^n \alpha^{n-1}}{(1 + \alpha)^n (2 + \alpha)^n} |c_1| \left( \frac{c_1^2}{2} w - \frac{|c_1|^2}{2} \varepsilon + 2\varepsilon \right) \right), \]

(3.19)

where

\[ w = 1 + \frac{(1 - 2\alpha)\alpha^{n-1}(2 + \alpha)^n}{3(1 + \alpha)^{2n}}. \]

(3.20)

Since \( 0 < w \leq 1 \) and \( |\varepsilon| < 1 \), it is easily shown that

\[ |a_4| \leq \frac{\alpha^{n-1}}{(3 + \alpha)^n} \left( |c_3| + \frac{(1 - \alpha)(3 + \alpha)^n \alpha^{n-1}}{(1 + \alpha)^n (2 + \alpha)^n} |c_1| \left( \frac{|c_1|^2}{2} (w - 1) + 2 \right) \right) \]

(3.21)

and the result follows trivially when using \( |c_1| \leq 2 \) and \( |c_3| \leq 2 \).

Finally, we consider (3.3) for the case \( \alpha \geq 1 \). Here, we use a method introduced by Nehari and Netanyahu [3] which was also used by Singh [6] and the author in [1].

First, let \( h \) and \( g \) be defined as in Lemma 2.3, and since \( p \in P \), Lemma 2.2 indicates that

\[ 1 + G(z) = 1 + \frac{1}{2} \sum_{k=1}^{\infty} g_k c_k z^k \]

(3.22)

also belongs to \( P \).

Next, it follows from (2.2) that, with \( g \) replaced by \( G \),

\[ |A_3| = \left| \frac{1}{2} g_3 c_3 - \frac{1}{2} y_1 g_1 g_2 c_1 c_2 + \frac{1}{8} y_2 g_1^3 c_1^3 \right|. \]

(3.23)

Rewriting (3.15) as

\[ \alpha^{1-n}(3 + \alpha)^n a_4 = c_3 + \frac{(1 - \alpha)(3 + \alpha)^n \alpha^{n-1}}{(1 + \alpha)^n (2 + \alpha)^n} c_1 c_2 \]

\[ + \frac{(1 - \alpha)(1 - 2\alpha)(3 + \alpha)^n \alpha^{2n-2}}{6(1 + \alpha)^{3n}} c_1^3 \]

(3.24)

and comparing it with (3.23), the required result is easily obtained since, by Lemma 2.3, \( |A_3| = ((3 + \alpha)^n/\alpha^{n-1})|a_4| \leq 2 \). This however is only true if we can show the existence of functions \( h \) and \( \psi \) in \( P \) where \( \psi(z) = 1 + g(z) \). To be simple, we choose \( \psi(z) = (1 + z)/(1 - z) \). Thus, now it remains to construct and show that an \( h \in P \).
Now since $g_1 = g_2 = g_3 = 2$, it follows from (3.23) and (3.24) that

$$2y_1 = \frac{(\alpha - 1)(3 + \alpha)^n \alpha^{n-1}}{(1 + \alpha)^n (2 + \alpha)^n},$$  \hspace{1cm} (3.25)

$$y_2 = \frac{(1 - \alpha)(1 - 2\alpha)(3 + \alpha)^n \alpha^{2n-2}}{6(1 + \alpha)^{3n}}.$$  \hspace{1cm} (3.26)

However, from (2.1), we have

$$y_1 = \frac{1}{2} \left( 1 + \frac{1}{2}h_1 \right),$$  \hspace{1cm} (3.27)

$$y_2 = \frac{1}{4} \left( 1 + h_1 + \frac{1}{2}h_2 \right).$$  \hspace{1cm} (3.28)

Solving for $h_1$ by eliminating $y_1$ from (3.25) and (3.27), we obtain

$$|h_1| = 2 \left| \frac{(\alpha - 1)(3 + \alpha)^n \alpha^{n-1}}{(1 + \alpha)^n (2 + \alpha)^n} - 1 \right|.$$  \hspace{1cm} (3.29)

Quite trivially, it can be seen that $|h_1| \leq 2$ for $\alpha \geq 1$.

In a similar manner, eliminating $y_2$ from (3.26) and (3.28) and using $h_1$ given by (3.29), we have

$$h_2 = 2 \left\{ 1 - \frac{2}{3} \left( 1 - \frac{1}{\alpha} \right) \left( \frac{\alpha^2 + 3\alpha}{\alpha^2 + 3\alpha + 2} \right)^n \left[ \frac{1 - 2\alpha}{\alpha} \left( \frac{\alpha^2 + 2\alpha}{\alpha^2 + 2\alpha + 1} + 3 \right) \right] \right\}.$$  \hspace{1cm} (3.30)

For $\alpha \geq 1$, elementary calculations show that $|h_2| \leq 2$.

Next, we construct $h$ by first setting it to be of the form

$$h(z) = \frac{\mu_1 (1 - z)}{1 + z} + \frac{\mu_2 (1 + \lambda z^2)}{1 - \lambda z^2},$$  \hspace{1cm} (3.31)

with

$$\mu_1 = 1 - \frac{(\alpha - 1)(3 + \alpha)^n \alpha^{n-1}}{(1 + \alpha)^n (2 + \alpha)^n},$$

$$\mu_2 = \frac{(\alpha - 1)(3 + \alpha)^n \alpha^{n-1}}{(1 + \alpha)^n (2 + \alpha)^n},$$

$$\lambda = 1 - \frac{2}{3} \left[ \frac{1 - 2\alpha}{\alpha} \left( \frac{\alpha^2 + 2\alpha}{\alpha^2 + 2\alpha + 1} \right)^n + 3 \right].$$  \hspace{1cm} (3.32)

It is readily seen that for $\alpha \geq 1$, both $\mu_1$ and $\mu_2$ are nonnegative and $\mu_1 + \mu_2 = 1$. Further, with a little bit of manipulation, it can be shown that $|\lambda| \leq 1$ and the coefficients of $z$ and $z^2$ in the expansion of $h$ are respectively those given by (3.29) and (3.30). Hence $h \in P$ and thus $|a_4| \leq 2\alpha^{n-1}/(3 + \alpha)^n$, the second inequality in (3.3). This completes the proof of Theorem 3.1.

\[\square\]
REFERENCES


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