ON THE FRESNEL INTEGRALS AND THE CONVOLUTION

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The Fresnel cosine integral $C(x)$, the Fresnel sine integral $S(x)$, and the associated functions $C_+(x)$, $C_-(x)$, $S_+(x)$, and $S_-(x)$ are defined as locally summable functions on the real line. Some convolutions and neutrix convolutions of the Fresnel cosine integral and its associated functions with x_+^r and x^r are evaluated.

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The *Fresnel cosine integral* $C(x)$ is defined by

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos u^2 du, \quad (1)$$

(see [3]) and the associated functions $C_+(x)$ and $C_-(x)$ are defined by

$$C_+(x) = H(x)C(x), \quad C_-(x) = H(-x)C(x). \quad (2)$$

The *Fresnel sine integral* $S(x)$ is defined by

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin u^2 du, \quad (3)$$

(see [3]) and the associated functions $S_+(x)$ and $S_-(x)$ are defined by

$$S_+(x) = H(x)S(x), \quad S_-(x) = H(-x)S(x), \quad (4)$$

where H denotes Heaviside's function.

We define the function $I_r(x)$ by

$$I_r(x) = \int_0^x u^r \cos u^2 du \quad (5)$$

for $r = 0, 1, 2, \dots$. In particular,

$$I_0(x) = \sqrt{\frac{\pi}{2}} C(x), \quad I_1(x) = \frac{1}{2} \sin x^2, \quad I_2(x) = \frac{1}{2} x \sin x^2 - \frac{\sqrt{\pi}}{2\sqrt{2}} S(x). \quad (6)$$

We define the functions $\cos_+ x$, $\cos_- x$, $\sin_+ x$, and $\sin_- x$ by

$$\begin{aligned} \cos_+ x &= H(x) \cos x, & \cos_- x &= H(-x) \cos x, \\ \sin_+ x &= H(x) \sin x, & \sin_- x &= H(-x) \sin x. \end{aligned} \quad (7)$$

If the classical convolution $f * g$ of two functions f and g exists, then $g * f$ exists and

$$f * g = g * f. \quad (8)$$

Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \quad (9)$$

The classical definition of the convolution can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see [2].

DEFINITION 1. Let f and g be distributions in $\mathcal{D}'$. Then the *convolution* $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x + y) \rangle \rangle \quad (10)$$

for arbitrary φ in $\mathcal{D}'$, provided that f and g satisfy either of the conditions

- (a) either f or g has bounded support,
- (b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition, then (6) and (8) are satisfied.

THEOREM 2. *The convolution $(\cos_+ x^2) * x_+^r$ exists and*

$$(\cos_+ x^2) * x_+^r = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i}(x) x_+^i \quad (11)$$

for $r = 0, 1, 2, \dots$. In particular,

$$\begin{aligned} (\cos_+ x^2) * H(x) &= \sqrt{\frac{\pi}{2}} C_+(x), \\ (\cos_+ x^2) * x_+ &= -\frac{1}{2} \sin_+ x^2 + \sqrt{\frac{\pi}{2}} C(x) x_+. \end{aligned} \quad (12)$$

PROOF. It is obvious that $(\cos_+ x^2) * x_+^r = 0$ if $x < 0$. When $x > 0$, we have

$$\begin{aligned} (\cos_+ x^2) * x_+^r &= \int_0^x \cos t^2 (x - t)^r dt \\ &= \sum_{i=0}^r \binom{r}{i} \int_0^x x^i (-t)^{r-i} \cos t^2 dt \\ &= \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i}(x) x^i, \end{aligned} \quad (13)$$

proving (11). Equations (12) follow on using (6). □

COROLLARY 3. *The convolution $(\cos_- x^2) * x_-^r$ exists and*

$$(\cos_- x^2) * x_-^r = - \sum_{i=0}^r \binom{r}{i} I_{r-i}(x) x_-^i \quad (14)$$

for $r = 0, 1, 2, \dots$. In particular,

$$\begin{aligned} (\cos_- x^2) * H(-x) &= -\sqrt{\frac{\pi}{2}} C_-(x), \\ (\cos_- x^2) * x_- &= -\frac{1}{2} \sin_- x^2 - \sqrt{\frac{\pi}{2}} S(x) x_-. \end{aligned} \quad (15)$$

PROOF. Equations (14) and (15) follow on replacing x by $-x$ in (11) and (12), respectively, and noting that

$$I_r(-x) = (-1)^{r+1} I_r(x). \quad (16)$$

□

THEOREM 4. *The convolution $C_+(x) * x_+^r$ exists and*

$$C_+(x) * x_+^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1}(x) x_+^i \quad (17)$$

for $r = 0, 1, 2, \dots$. In particular,

$$\begin{aligned} C_+(x) * H(x) &= -\frac{1}{\sqrt{2\pi}} \sin_+ x^2 + C(x) x_+, \\ C_+(x) * x_+ &= \frac{1}{2\sqrt{2\pi}} \sin x^2 x_+ - \frac{1}{\sqrt{2\pi}} \sin_+ x^2 - \frac{1}{4} S_+(x) + \frac{1}{2} C(x) x_+^2. \end{aligned} \quad (18)$$

PROOF. It is obvious that $C_+(x) * x_+^r = 0$ if $x < 0$. When $x > 0$, we have

$$\begin{aligned} \sqrt{\frac{\pi}{2}} C_+(x) * x_+^r &= \int_0^x (x-t)^r \int_0^t \cos u^2 du dt \\ &= \int_0^x \cos u^2 \int_u^x (x-t)^r dt du \\ &= \frac{1}{r+1} \int_0^x \cos u^2 (x-u)^{r+1} du \\ &= \frac{1}{r+1} \int_0^x \cos u^2 \sum_{i=0}^{r+1} \binom{r+1}{i} x^i (-u)^{r-i+1} du \\ &= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1}(x) x_+^i. \end{aligned} \quad (19)$$

Equation (17) follows. Equations (18) follow on using (6). □

COROLLARY 5. *The convolution $C_{-}(x) * x_{-}^r$ exists and*

$$C_{-}(x) * x_{-}^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} I_{r-i+1}(x) x_{-}^i \quad (20)$$

for $r = 0, 1, 2, \dots$. In particular,

$$\begin{aligned} C_{-}(x) * H(-x) &= \frac{1}{\sqrt{2\pi}} \sin_{-} x^2 + C(x) x_{-}, \\ C_{-}(x) * x_{-} &= -\frac{1}{2\sqrt{2\pi}} \sin x^2 x_{-} + \frac{1}{\sqrt{2\pi}} \sin_{-} x^2 - \frac{1}{4} S_{-}(x) + \frac{1}{2} C(x) x_{-}^2. \end{aligned} \quad (21)$$

PROOF. Equations (20) and (21) follow on replacing x by $-x$ in (17) and (18), respectively, and using (16). $\square$

Definition 1 was extended in [1] with the next definition but first of all we let τ be a function in $\mathcal{D}$ having the following properties:

- (i) $\tau(x) = \tau(-x)$,
- (ii) $0 \leq \tau(x) \leq 1$,
- (iii) $\tau(x) = 1$, for $|x| \leq 1/2$,
- (iv) $\tau(x) = 0$, for $|x| \geq 1$.

The function τ_{ν} is now defined for $\nu > 0$ by

$$\tau_{\nu}(x) = \begin{cases} 1, & |x| \leq \nu, \\ \tau(\nu^{\nu} x - \nu^{\nu+1}), & x > \nu, \\ \tau(\nu^{\nu} x + \nu^{\nu+1}), & x < -\nu. \end{cases} \quad (22)$$

DEFINITION 6. Let f and g be distributions in $\mathcal{D}'$ and let $f_{\nu} = f\tau_{\nu}$ for $\nu > 0$. The *neutrix convolution product* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_{\nu} * g\}$, provided that the limit h exists in the sense that

$$N\text{-}\lim_{\nu \rightarrow \infty} \langle f_{\nu} * g, \varphi \rangle = \langle h, \varphi \rangle, \quad (23)$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [5], with its domain N' the positive real numbers, with negligible functions finite linear sums of the functions

$$\nu^{\lambda} \ln^{r-1} \nu, \quad \ln^r \nu, \quad \nu^r \sin \nu^2, \quad \nu^r \cos \nu^2 \quad (\lambda \neq 0, r = 1, 2, \dots) \quad (24)$$

and all functions which converge to zero in the normal sense as ν tends to infinity.

Note that in this definition the convolution product $f_{\nu} * g$ is defined in Gel'fand and Shilov's sense, since the distribution f_{ν} has bounded support.

It was proved in [1] that if $f * g$ exists in the classical sense or by Definition 1, then $f \circledast g$ exists and

$$f \circledast g = f * g. \quad (25)$$

The following theorem was also proved in [1].

THEOREM 7. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'. \quad (26)$$

We need the following lemma.

LEMMA 8. *If $I_r = N\text{-}\lim_{\nu \rightarrow \infty} I_r(\nu)$, then*

$$\begin{aligned} I_{4r} &= \frac{(-1)^r (4r)! \sqrt{\pi}}{2^{4r+1} (2r)! \sqrt{2}}, \\ I_{4r+1} &= 0, \\ I_{4r+2} &= \frac{(-1)^r (4r+1)! \sqrt{\pi}}{2^{4r+2} (2r)! \sqrt{2}}, \\ I_{4r+3} &= \frac{(-1)^{r+1} (2r)!}{2} \end{aligned} \quad (27)$$

for $r = 0, 1, 2, \dots$

PROOF. It is easily proved that

$$I_3(x) = \frac{1}{2} x^2 \sin x^2 - \frac{1}{2} + \frac{1}{2} \cos x^2 \quad (28)$$

and it follows from (6) and (28) that (27) hold when $r = 0$, since

$$S(\infty) = C(\infty) = \frac{1}{2}, \quad (29)$$

see Olver [4].

We also have

$$\begin{aligned} I_{2r}(x) &= \frac{1}{2} x^{2r-1} \sin x^2 + \frac{2r-1}{4} x^{2r-3} \cos x^2 - \frac{(2r-1)(2r-3)}{4} I_{2r-4}(x), \\ I_{2r+1}(x) &= \frac{1}{2} x^{2r} \sin x^2 + \frac{r}{2} x^{2r-2} \cos x^2 - r(r-1) I_{2r-3}(x) \end{aligned} \quad (30)$$

and it follows that

$$\begin{aligned} N\text{-}\lim_{\nu \rightarrow \infty} I_{2r}(\nu) &= -\frac{(2r)!(r-2)!}{2^4(2r-4)!r!} N\text{-}\lim_{\nu \rightarrow \infty} I_{2r-4}(\nu), \\ N\text{-}\lim_{\nu \rightarrow \infty} I_{2r+1}(\nu) &= -\frac{r!}{(r-2)!} N\text{-}\lim_{\nu \rightarrow \infty} I_{2r-3}(\nu). \end{aligned} \quad (31)$$

Equations (27) now follow by induction. $\square$

THEOREM 9. *The neutrix convolution $(\cos_+ x^2) * x^r$ exists and*

$$(\cos_+ x^2) \circledast x^r = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i} x^i \quad (32)$$

for $r = 0, 1, 2, \dots$. In particular,

$$\begin{aligned} (\cos_+ x^2) \circledast 1 &= \frac{\sqrt{\pi}}{2\sqrt{2}}, \\ (\cos_+ x^2) \circledast x &= \frac{\sqrt{\pi}}{2\sqrt{2}} x. \end{aligned} \quad (33)$$

PROOF. We put $(\cos_+ x^2)_\nu = (\cos_+ x^2) \tau_\nu(x)$. Then the convolution $(\cos_+ x^2)_\nu * x^r$ exists and

$$(\cos_+ x^2)_\nu * x^r = \int_0^\nu \cos t^2 (x-t)^r dt + \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t) \cos t^2 (x-t)^r dt. \quad (34)$$

□

Now,

$$\begin{aligned} \int_0^\nu \cos t^2 (x-t)^r dt &= \sum_{i=0}^r \binom{r}{i} \int_0^\nu x^i (-t)^{r-i} \cos t^2 dt \\ &= \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i}(\nu) x^i \end{aligned} \quad (35)$$

and it follows that

$$N\text{-}\lim_{\nu \rightarrow \infty} \int_0^\nu \cos t^2 (x-t)^r dt = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i} x^i. \quad (36)$$

Further, it is easily seen that, for each fixed x ,

$$\lim_{\nu \rightarrow \infty} \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t) \cos t^2 (x-t)^r dt = 0 \quad (37)$$

and (32) follows from (34), (36), and (37). Equations (33) follow immediately.

COROLLARY 10. *The neutrix convolution $\cos_- x^2 \circledast x^r$ exists and*

$$(\cos_- x^2) \circledast x^r = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i+1} I_{r-i} x^i \quad (38)$$

for $r = 0, 1, 2, \dots$. In particular,

$$\begin{aligned} (\cos_- x^2) \circledast 1 &= -\frac{\sqrt{\pi}}{2\sqrt{2}}, \\ (\cos_- x^2) \circledast x &= -\frac{\sqrt{\pi}}{2\sqrt{2}} x. \end{aligned} \quad (39)$$

PROOF. Equation (38) follows on replacing x by $-x$ in (32) and noting that I_r must be replaced by

$$N\text{-}\lim_{\nu \rightarrow \infty} I_r(-\nu) = (-1)^{r-1} N\text{-}\lim_{\nu \rightarrow \infty} I_r(\nu) = (-1)^{r-1} I_r. \quad (40)$$

Equations (33) follow. $\square$

COROLLARY 11. *The convolution $(\cos x^2) \circledast x^r$ exists and*

$$(\cos x^2) \circledast x^r = 0 \quad (41)$$

for $r = 0, 1, 2, \dots$

PROOF. Equation (41) follows from (32) and (38) on noting that $\cos x^2 = \cos_+ x^2 + \cos_- x^2$. $\square$

THEOREM 12. *The neutrix convolution $C_+(x) \circledast x^r$ exists and*

$$C_+(x) \circledast x^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1} x^i \quad (42)$$

for $r = 0, 1, 2, \dots$. In particular

$$C_+(x) \circledast 1 = 0, \quad (43)$$

$$C_+(x) \circledast x = \frac{1}{8}. \quad (44)$$

PROOF. We put $[C_+(x)]_\nu = C_+(x)\tau_\nu(x)$. Then the convolution product $[C_+(x)]_\nu * x^r$ exists and

$$[C_+(x)]_\nu * x^r = \int_0^\nu C(t)(x-t)^r dt + \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t)C(t)(x-t)^r dt. \quad (45)$$

We have

$$\begin{aligned} & \sqrt{\frac{\pi}{2}} \int_0^\nu C(t)(x-t)^r dt \\ &= \int_0^\nu (x-t)^r \int_0^t \cos u^2 du dt \\ &= \int_0^\nu \cos u^2 \int_u^\nu (x-t)^r dt du \\ &= -\frac{1}{r+1} \int_0^\nu \cos u^2 [(x-\nu)^{r+1} - (x-u)^{r+1}] du \\ &= -\frac{1}{r+1} \int_0^\nu \sum_{i=0}^r \binom{r+1}{i} x^i [(-\nu)^{r-i+1} - (-u)^{r-i+1}] \cos u^2 du \end{aligned} \quad (46)$$

and it follows that

$$N\text{-}\lim_{v \rightarrow \infty} \int_0^v C(t)(x-t)^r dt = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1} x^i. \quad (47)$$

Further, it is easily seen that, for each fixed x ,

$$\lim_{v \rightarrow \infty} \int_v^{v+v^{-v}} \tau_v(t) C(t)(x-t)^r dt = 0 \quad (48)$$

and (42) now follows immediately from (45), (47), and (48). $\square$

COROLLARY 13. *The neutrix convolution $C_-(x) \circledast x^r$ exists and*

$$C_-(x) \circledast x^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i} I_{r-i+1} x^i \quad (49)$$

for $r = 0, 1, 2, \dots$. In particular,

$$C_-(x) \circledast 1 = 0, \quad (50)$$

$$C_-(x) \circledast x = -\frac{1}{8}. \quad (51)$$

PROOF. Equation (49) follows on replacing x by $-x$ and I_r by $(-1)^{r-1} I_r$ in (42). Equations (50) and (51) follow. $\square$

COROLLARY 14. *The neutrix convolution $C(x) \circledast x^r$ exists and*

$$C(x) \circledast x^r = 0 \quad (52)$$

for $r = 0, 1, 2, \dots$.

PROOF. Equation (52) follows from (43) and (50) on noting that $C(x) = C_+(x) + C_-(x)$. $\square$

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