A NEW CHARACTERIZATION OF SOME ALTERNATING AND SYMMETRIC GROUPS

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We suppose that \( p = 2^\alpha 3^\beta + 1 \), where \( \alpha \geq 1 \), \( \beta \geq 0 \), and \( p \geq 7 \) is a prime number. Then we prove that the simple groups \( A_n \), where \( n = p, p + 1 \), or \( p + 2 \), and finite groups \( S_n \), where \( n = p, p + 1 \), are also uniquely determined by their order components. As corollaries of these results, the validity of a conjecture of J. G. Thompson and a conjecture of Shi and Bi (1990) both on \( A_n \), where \( n = p, p + 1 \), or \( p + 2 \), is obtained. Also we generalize these conjectures for the groups \( S_n \), where \( n = p, p + 1 \).

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1. Introduction. Let \( G \) be a finite group. We denote by \( \pi(G) \) the set of all prime divisors of \( |G| \). We construct the prime graph of \( G \) as follows. The prime graph \( \Gamma(G) \) of a group \( G \) is the graph whose vertex set is \( \pi(G) \), and two distinct primes \( p \) and \( q \) are joined by an edge (we write \( p \sim q \)) if and only if \( G \) contains an element of order \( pq \). Let \( t(G) \) be the number of connected components of \( \Gamma(G) \) and let \( \pi_1, \pi_2, \ldots, \pi_{t(G)} \) be the connected components of \( \Gamma(G) \). If \( 2 \in \pi(G) \), then we always suppose that \( 2 \in \pi_1 \).

Now \( |G| \) can be expressed as a product of coprime positive integers \( m_i \), \( i = 1, 2, \ldots, t(G) \), where \( \pi(m_i) = \pi_i \). These integers are called the order components of \( G \). The set of order components of \( G \) will be denoted by \( OC(G) \). Also we call \( m_2, \ldots, m_{t(G)} \) the odd-order components of \( G \). The order components of non-abelian simple groups having at least three prime graph components are obtained by Chen [7, Tables 1, 2, 3]. Similarly, the order components of non-abelian simple groups with two-order components can be obtained by using the tables in [18, 28].

The following groups are uniquely determined by their order components: Suzuki-Ree groups [6], Sporadic simple groups [4], \( PSL_2(q) \) [7], \( E_8(q) \) [2], \( G_2(q) \), where \( q \equiv 0 (\text{mod} \, 3) \) [3], \( F_4(q) \), where \( q \) is even [15], \( PSL_3(q) \), where \( q \) is an odd prime power [14], \( PSL_3(q) \), where \( q = 2^n \) [13], \( PSU_3(q) \), where \( q > 5 \) [16], and \( A_p \), where \( p \) and \( p - 2 \) are primes [12].

It was proved by Oyama [20] that a finite group which has the same table of characters as an alternating group \( A_n \) is isomorphic to \( A_n \). It was also proved by Koike [17] that a finite group which has the isomorphic subgroup-lattice as an alternating group \( A_n \) is isomorphic to \( A_n \).
Let $\pi_e(G)$ denote the set of orders of elements in $G$. Shi and Bi [27] proved that if $\pi_e(G) = \pi_e(A_n)$ and $|G| = |A_n|$, then $G \cong A_n$. Iranmanesh and Alavi [12] proved that if $p$ and $p - 2$ are primes and $\text{OC}(G) = \text{OC}(A_p)$, then $G \cong A_p$. Praeger and Shi [21] and Shi and Bi [26] proved that $A_8, A_9, A_{11}, A_{13}, S_7,$ and $S_8$ are characterizable by their element orders. Also recently, Kondrat’ev and Mazurov [19] and Zavarnitsin [29] proved that if $\pi_e(G) = \pi_e(A_n)$, where $n = s, s + 1, s + 2$ and $s$ is a prime number, then $G \cong A_n$.

Now we prove the following theorems.

**Theorem 1.1.** Let $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = A_n$, where $n = p, p + 1, p + 2$. Then $\text{OC}(G) = \text{OC}(M)$ if and only if $G \cong M$.

**Theorem 1.2.** Let $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = S_n$, where $n = p, p + 1$. Then $\text{OC}(G) = \text{OC}(M)$ if and only if $G \cong M$.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and we refer, for example, to [10]. Also frequently we use the results of Williams [28] and Kondrat’ev [18] about the prime graph of simple groups.

**2. Preliminary results**

**Remark 2.1.** Let $N$ be a normal subgroup of $G$ and $p \sim q$ in $\Gamma(G/N)$. Then $p \sim q$ in $\Gamma(G)$. In fact if $xN \in G/N$ has order $pq$, then there is a power of $x$ which has order $pq$.

**Definition 2.2** (see [11]). A finite group $G$ is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$, respectively.

**Lemma 2.3** (see [28, Theorem A]). If $G$ is a finite group with its prime graph having more than one component, then $G$ is one of the following groups:
(a) a Frobenius or 2-Frobenius group;
(b) a simple group;
(c) an extension of a $\pi_1$-group by a simple group;
(d) an extension of a simple group by a $\pi_1$-solvable group;
(e) an extension of a $\pi_1$-group by a simple group by a $\pi_1$-group.

**Lemma 2.4** (see [28, Lemma 3]). If $G$ is a finite group with more than one prime graph component and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $H$ and $G/K$ are $\pi_1$-groups and $K/H$ is a simple group, then $H$ is a nilpotent group.

The next lemma follows from [1, Theorem 2].

**Lemma 2.5.** Let $G$ be a Frobenius group of even order and let $H, K$ be Frobenius complement and Frobenius kernel of $G$, respectively. Then $t(\Gamma(G)) = 2,$
and the prime graph components of \( G \) are \( \pi(H) \), \( \pi(K) \) and \( G \) has one of the following structures:

(a) \( 2 \in \pi(K) \) and all Sylow subgroups of \( H \) are cyclic;
(b) \( 2 \in \pi(H) \), \( K \) is an abelian group, \( H \) is a solvable group, the Sylow subgroups of odd order of \( H \) are cyclic groups, and the \( 2 \)-Sylow subgroups of \( H \) are cyclic or generalized quaternion groups;
(c) \( 2 \in \pi(H) \), \( K \) is an abelian group, and there exists \( H_0 \leq H \) such that \( |H : H_0| \leq 2 \), \( H_0 = Z \times \text{SL}(2,5) \), \((|Z|,2.3.5) = 1\), and the Sylow subgroups of \( Z \) are cyclic.

The next lemma follows from [1, Theorem 2] and Lemma 2.4.

**Lemma 2.6.** Let \( G \) be a 2-Frobenius group of even order. Then \( t(\pi(G)) = 2 \) and \( G \) has a normal series \( 1 \leq H \leq K \leq G \) such that

(a) \( \pi_1 = \pi(G/K) \cup \pi(H) \) and \( \pi(K/H) = \pi_2 \);
(b) \( G/K \) and \( K/H \) are cyclic, \( |G/K| \) divides \( |\text{Aut}(K/H)|\), \((|G/K|,|K/H|) = 1 \), and \( |G/K| < |K/H| \);
(c) \( H \) is nilpotent and \( G \) is a solvable group.

**Lemma 2.7** (see [8, Lemma 8]). Let \( G \) be a finite group with \( t(\pi(G)) \geq 2 \) and let \( N \) be a normal subgroup of \( G \). If \( N \) is a \( \pi_1 \)-group for some prime graph component of \( G \) and \( m_1, m_2, \ldots, m_r \) are some order components of \( G \) but not a \( \pi_1 \)-number, then \( m_1 m_2 \cdots m_r \) is a divisor of \( |N| - 1 \).

The next lemma follows from [5, Lemma 1.4].

**Lemma 2.8.** Suppose that \( G \) and \( M \) are two finite groups satisfying \( t(\pi(M)) \geq 2 \), \( N(G) = N(M) \), where \( N(G) = \{ n \mid G \text{ has a conjugacy class of size } n \} \), and \( Z(G) = 1 \). Then \( |G| = |M| \).

**Lemma 2.9** (see [5, Lemma 1.5]). Let \( G_1 \) and \( G_2 \) be finite groups satisfying \( |G_1| = |G_2| \) and \( N(G_1) = N(G_2) \). Then \( t(\pi(G_1)) = t(\pi(G_2)) \) and \( OC(G_1) = OC(G_2) \).

**Lemma 2.10.** Let \( G \) be a finite group and let \( M \) be a non-abelian finite group with \( t(M) = 2 \) satisfying \( OC(G) = OC(M) \).

1. Let \( |M| = m_1 m_2 \), \( OC(M) = \{ m_1, m_2 \} \), and \( \pi(m_i) = \pi_i \) for \( i = 1, 2 \). Then \( |G| = m_1 m_2 \) and one of the following holds:
   (a) \( G \) is a Frobenius or 2-Frobenius group;
   (b) \( G \) has a normal series \( 1 \leq H \leq K \leq G \) such that \( G/K \) is a \( \pi_1 \)-group, \( H \) is a nilpotent \( \pi_1 \)-group, and \( K/H \) is a non-abelian simple group. Moreover, \( OC(K/H) = \{ m'_{1}, m'_{2}, \ldots, m'_{j}, m_2 \} \), \( |K/H| = m'_1 m'_2 \cdots m'_j m_2 \), and \( m'_1 m'_2 \cdots m'_j | m_1 \), where \( \pi(m'_j) = \pi'_j \), \( 1 \leq j \leq s \).
2. In case (b), \( |G/K| | |\text{Out}(K/H)| \).

**Proof.** The proof of (1) follows from the above lemmas. Since \( t(G) \geq 2 \), we have \( t(G/H) \geq 2 \). Otherwise \( t(G/H) = 1 \), so that \( t(G) = 1 \). Since \( H \) is a \( \pi_1 \)-group,
we arrive at a contradiction. Moreover, we have $Z(G/H) = 1$. For any $xH \in G/H$ and $xH \notin K/H$, $xH$ induces an automorphism of $K/H$ and this automorphism

### Table 2.1

<table>
<thead>
<tr>
<th>$p$</th>
<th>$A_5, A_6; L_2(q)$, where $q$ is a Fermat prime, a Mersenne prime, or $q = 2^n, n \geq 3, L_3(2^2), Sz(2^{2n+1}), n \geq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$A_5, A_6, A_7; M_{11}, M_{22}; L_2(q)$, where $q = 7^2, 5^n, or 2.3^n + 1 which is a prime, $n \geq 1, L_3(2^2), S_4(q), q = 3, 7, U_4(3); Sz(q), q = 2^3, 2^5$</td>
</tr>
<tr>
<td>13</td>
<td>$A_13, A_{14}, A_{15}; Suz, Fi_{22}; L_2(q), q = 3^2, 5^2, 13^n, or 2.13^n - 1 which is a prime, $n \geq 1, L_3(3), L_4(3), O_7(3), S_4(3), S_6(3), O^+_8(3), G_2(q), q = 3^2, 3; F_4(2), U_3(q), q = 2^2, 23, Sz(2^3), 3D_4(2), 2E_6(2), 2F_4(2)'$</td>
</tr>
<tr>
<td>17</td>
<td>$A_{17}, A_{18}, A_{19}; J_3, He, F_{23}, F_{24}'; L_2(q), q = 2^{17}, 17^n, 2.17^n + 1 which is a prime, $n \geq 1, S_4(4), S_6(2), F_4(2), O^+_8(2), O^+_7(2), 2E_6(2)$</td>
</tr>
<tr>
<td>37</td>
<td>$A_{19}, A_{20}, A_{21}; J_1, J_3, O'N, Th, HN; L_2(q), q = 19^n, 2.19^n - 1 which is a prime, $n \geq 1, L_3(7), U_3(2^3), R(3^3), 2F_4(2^3)$</td>
</tr>
<tr>
<td>73</td>
<td>$A_{37}, A_{38}, A_{39}; J_4, L_4'; L_2(q), q = 37^n, 2.37^n - 1 which is a prime, $n \geq 1, U_3(11), R(3^3), 2F_4(2^3)$</td>
</tr>
<tr>
<td>109</td>
<td>$A_{109}, A_{110}, A_{111}; L_2(q), q = 109^n, 2.109^n - 1 which is a prime, $n \geq 1, 2F_4(2^3)$</td>
</tr>
<tr>
<td>$p = 2^m + 1, m = 2^s$</td>
<td>$A_p, A_{p+1}, A_{p+2}; L_2(q), q = 2^m, p^k, 2 \cdot p^k + 1 which is a prime, s \geq k \geq 1, S_a(2^b), a = 2c+1 and b = 2d, c \geq 1, c + d = s, F_4(2^c), e \geq 1, 4e = 2s, O_2(m+1)(2), s \geq 2, O_a(2^b), c \geq 2, c + d = s$</td>
</tr>
<tr>
<td>Other</td>
<td>$A_p, A_{p+1}, A_{p+2}; L_2(q), q = p^k, 2 \cdot p^k - 1 which is a prime, k \geq 1$</td>
</tr>
</tbody>
</table>
is trivial if and only if $xH \in Z(G/H)$. Therefore, $G/K \leq \text{Out}(K/H)$ and since $Z(G/H) = 1$, (2) follows.

**Definition 2.11.** A group $G$ is called a $C_{pp}$ group if the centralizers of its elements of order $p$ in $G$ are $p$-groups.

**Lemma 2.12** (see [9]). Let $p$ be a prime and $p = 2^\alpha 3^\beta + 1$, $\alpha \geq 0$ and $\beta \geq 0$. Then any finite simple $C_{pp}$ group is given by Table 2.1.

3. Characterization of some alternating and symmetric groups. In the sequel, we suppose that $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number.

**Lemma 3.1.** Let $G$ be a finite group and let $M$ be $A_n$, where $n = p, p + 1$, or $p + 2$, or $S_n$, where $n = p, p + 1$. If $\text{OC}(G) = \text{OC}(M)$, then $G$ is neither a Frobenius group nor a 2-Frobenius group.

**Proof.** If $G$ is a Frobenius group, then by Lemma 2.5, $\text{OC}(G) = \{|H|, |K|\}$, where $K$ and $H$ are Frobenius kernel and Frobenius complement of $G$, respectively. Since $|H| \mid |K| - 1$, we have $|H| < |K|$. Therefore, $2 \nmid |H|$, and hence $2 \nmid |K|$. So, $|H| = p, |K| = |G|/p$. We claim that there exists a prime $p'$ such that $3n/4 < p'$. Note that $p \leq n$, and hence $p'^2 \nmid |A_n|$. Let $\beta(n)$ be the number of prime numbers less than or equal to $n$. In fact, by [22, Theorem 2] we have

$$\frac{n}{\log n - 1/2} < \beta(n) < \frac{n}{\log n - 3/2}, \quad (3.1)$$

where $n \geq 67$. Thus

$$\beta(n) - \beta\left(\frac{3n}{4}\right) > \frac{n}{\log n - 1/2} - \frac{3n/4}{\log(3n/4) - 3/2}. \quad (3.2)$$

When $n \geq 405$, we get $\beta(n) - \beta(3n/4) > 1$, and for $n < 405$, we can immediately obtain the result by checking the table of prime numbers. Now let $P'$ be the $p'$-Sylow subgroup of $K$. Since $K$ is nilpotent, $P' \trianglelefteq G$. Then $p \nmid p' - 1$, by Lemma 2.7, which is a contradiction since $p' < p$. Therefore, $G$ is not a Frobenius group.

Now let $G$ be a 2-Frobenius group. By Lemma 2.6, there is a normal series $1 \leq H \leq K \leq G$ such that $|K/H| = p$ and $|G/K| < p$. So, $|H| \neq 1$ since $|G| = |G/K| \cdot |K/H| \cdot |H|$. Since $2 \nmid |H|$, let $p'$ be as above and let $P'$ be the $p'$-Sylow subgroup of $H$. Now, $p \nmid p' - 1$, which is impossible. Hence, $G$ is not a 2-Frobenius group.

**Lemma 3.2.** Let $G$ be a finite group and $M = A_n$, where $n = p, p + 1$, or $p + 2$, or $S_n$, where $n = p, p + 1$. If $\text{OC}(G) = \text{OC}(M)$, then $G$ has a normal series $1 \leq H \leq K \leq G$ such that $H$ and $G/K$ are $\pi_1$-groups and $K/H$ is a simple group. Moreover, the odd-order component of $M$ is equal to an odd-order component of $K/H$. In particular, $t(\Gamma(K/H)) \geq 2$. Also $|G/H|$ divides $|\text{Aut}(K/H)|$, and in fact $G/H \leq \text{Aut}(K/H)$. 


PROOF. The first part of the lemma follows from the above lemmas since the prime graph of $M$ has two prime graph components. For primes $p$ and $q$, if $K/H$ has an element of order $pq$, then $G$ has one. Hence, by the definition of prime graph component, the odd-order component of $M$ is an odd-order component of $K/H$. Since $K/H \triangleleft G/H$ and $C_{G/H}(K/H) = 1$, we have

$$G/H = \frac{N_{G/H}(K/H)}{C_{G/H}(K/H)} \cong T, \quad T \leq \text{Aut}(K/H).$$  \hspace{1cm} (3.3)

**Theorem 3.3.** Let $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = A_n$, where $n = p, p + 1, p + 2$. Then $OC(G) = OC(M)$ if and only if $G \cong M$.

**Proof.** By Lemma 3.2, $G$ has a normal series $1 \leq H \leq K \leq G$ such that $\pi(H) \cup \pi(G/K) \subset \pi_1$, $K/H$ is a non-abelian simple group, $t(\Gamma(K/H)) \geq 2$, and the odd-order component of $M$ is an odd-order component of $K/H$. Therefore, $K/H$ is a finite simple $C_{pp}$ group. Now using Table 2.1, we consider each possibility of $K/H$ separately.

In the sequel, we frequently use the results of [28, Table I] and [18, Tables 2, 3].

**Step 1.** Let $p = 7, 13, 17, 19, 37, 73$, or 109.

Since the proofs of these cases are similar, we state only one of them, say $p = 13$. Using Table 2.1, we have

1. $K/H \cong S_{uz}$ or $F_{22}$. It is a contradiction since $3^7 \nmid |S_{uz}|$ and $3^9 \nmid |F_{22}|$ but $3^7 \nmid |A_n|$, where $n = 13, 14, 15$;
2. $K/H \cong L_2(27)$, $L_2(25)$, $L_3(3)$, $L_4(3)$, $S_2(8)$, $(2F_4(2))'$, or $U_3(4)$. If $K/H \cong L_2(27)$, then $|G/K| = |H| \cdot |G/K| \neq 1$. By Lemma 2.6, $(|G/K| \mid \text{Out}(K/H)) = 6$. So, $|H| \neq 1$. Let $P$ be the 5-Sylow subgroup of $H$. But since $H$ is nilpotent, $P \triangleleft G$. Hence, $13 \mid (|P| - 1)$, which is a contradiction. Other cases are similar;
3. $K/H \cong L_2(13^r)$ or $L_2(2.13^r - 1)$, where $2.13^r - 1$ is a prime, $r \geq 1$. Note that $13^2 \nmid |G|$, hence $r = 1$. So, $K/H \cong L_2(13)$ or $L_2(25)$, and we can proceed similar to (2);
4. $K/H \cong O_7(3)$. It is a contradiction since $3^9 \mid |O_7(3)|$ but $3^9 \nmid |A_n|$;
5. $K/H \cong S_4(5)$ or $S_6(3)$. It is a contradiction since $5^4 \mid |S_4(5)|$ but $5^4 \nmid |A_n|$. Also $3^9 \mid |S_6(3)|$ but $3^9 \nmid |A_n|$;
6. $K/H \cong O_8^+(3)$. It is a contradiction since $3^{12} \mid |O_8^+(3)|$ but $3^{12} \nmid |A_n|$;
7. $K/H \cong G_2(3)$ or $G_2(8)$. If $K/H \cong G_2(3)$, then we get a contradiction since for $n = 13, 14$ we have $3^6 \mid |G_2(3)|$ but $3^6 \nmid |A_n|$. For $n = 15$, since $\mid \text{Out}(G_2(3))\mid = 2$, we have $|H| \neq 1$. Now we proceed similar to (2). If $K/H \cong G_2(8)$, then we get a contradiction since $2^{18} \mid |G_2(8)|$ but $2^{18} \nmid |A_n|$;
8. $K/H \cong F_4(2)$. It is a contradiction since $17 \mid |F_4(2)|$ but $17 \nmid |A_n|$;
9. $K/H \cong U_3(23)$. It is a contradiction since $23 \mid |U_3(23)|$ but $23 \nmid |A_n|$;
(10) \( K/H \cong 3^3 D_4(2) \) or \( 2 E_6(2) \). It is a contradiction since \( 2^{12} \nmid |A_n| \). Also \( 19 \nmid |A_n| \).

(11) \( K/H \) is an alternating group, namely \( A_{13}, A_{14}, \) or \( A_{15} \).

First suppose that \( n = 13 \). Since \( |K/H| \leq |A_{13}| \), \( K/H \cong A_{13} \). But \( |G| = |A_{13}| \), and hence \( H = 1 \) and \( K = G \cong A_{13} \). If \( n = 14 \), then \( K/H \cong A_{13} \) or \( A_{14} \). But if \( r \neq 6 \), then \( \text{Aut}(A_r) = S_r \), and hence \( |\text{Out}(A_r)| = 2 \). If \( K/H \cong A_{13} \), then \( |G/K| \mid 2 \), and hence \( |H| \neq 1 \). Now we get a contradiction similar to (2). Therefore, \( K/H \cong A_{14} \), and hence \( G \cong A_{14} \). If \( n = 15 \), we do similarly.

**Step 2.** Let \( p = 2^m + 1 \), where \( m = 2^s \).

Using Table 2.1, we have

(i) \( K/H \cong L_2(2^m) \). Note that for every \( m \) we have \( |L_2(2^m)| \mid |G| \). Using Lemma 2.6, \( |G/K| \mid |\text{Out}(K/H)| \). Also \( |\text{Out}(L_2(2^m))| = m \). Hence, \( |H| \neq 1 \). Now let \( p' \) be a prime number less than \( p \) such that

\[
p' \mid \frac{|A_n|}{m|K/H|}.
\]

Let \( P' \) be the \( p' \)-Sylow subgroup of \( H \). Since \( H \) is nilpotent, \( P' \trianglelefteq G \). Hence, \( p \mid (|P| - 1) \), which is a contradiction;

(ii) \( K/H \cong L_2(p^k) \) or \( L_2(2p^k) \), where \( 2p^k + 1 \) is a prime and \( 1 \leq k \leq s \).

We know that \( p \mid |A_n| \), hence \( k = 1 \). Now we proceed similar to (i);

(iii) \( K/H \cong S_d(2^b) \), where \( a = 2^{c+1} \) and \( b = 2^d \), \( c \geq 1 \), \( c + d = s \). Let \( q = 2^b \) and \( f = 2^c \). Then \( p = q^f + 1 \) and we have

\[
|S_d(2^b)| = q^{f^2}(q^f - 1)(q^f + 1)\Pi_{i=1}^{f-1}(q^i - 1)(q^i + 1).
\]

Each factor of the form \( (q^j + 1) \) is less than or equal to \( p \) and therefore divides \( |A_n| \). Also \( q^{f^2} = (2^m)^j \leq 2^{m^2} \leq 2^{m^2} \). \( |S_d(2^b)| \mid |A_n| \). But \( |S_d(2^b)| = b \). Then \( |H| \neq 1 \) and we can proceed similar to (i);

(iv) \( K/H \cong F_4(2^e) \), where \( e \geq 1 \), \( 4e = 2^s \), or \( O_{2(m+1)}(2) \), where \( s \geq 2 \), or \( O_n^2(2^b) \), where \( c \geq 2 \), \( c + d = s \). Again this part is similar to (iii);

(v) \( K/H \cong A_p, A_{p+1}, A_{p+2} \).

First suppose that \( n = p \). Since \( |K/H| \leq |A_p| \), \( K/H \cong A_p \). But \( |G| = |A_p| \), and hence \( H = 1 \) and \( K = G \cong A_p \). If \( n = p + 1 \), then \( K/H \cong A_p \) or \( A_{p+1} \). But if \( r \neq 6 \), then \( \text{Aut}(A_r) = S_r \), and hence \( |\text{Out}(A_r)| = 2 \). If \( K/H \cong A_p \), then \( |G/K| \mid 2 \), and hence \( |H| \neq 1 \). Now we get a contradiction similar to (i). Therefore, \( K/H \cong A_{p+1} \), and hence \( G \cong A_{p+1} \). If \( n = p + 2 \), we do similarly.

**Step 3.** For other primes \( p \), we have \( K/H \cong A_p, A_{p+1}, A_{p+2} \); \( L_2(q) \), where \( q = p^k \), \( 2p^k - 1 \) which is a prime, \( k \geq 1 \).

In fact the proof of this step is exactly similar to that of **Step 2** and we omit it for convenience.
Theorem 3.4. If $G$ is a non-abelian finite group with connected prime graph, then $G$ is not characterizable with its order component.

Proof. Clearly, $\text{OC}(G) = \text{OC}(\mathbb{Z}|G|)$, but $G \not\cong \mathbb{Z}|G|$. □

Corollary 3.5. Every simple group with one component (see [28, Table I]) is not characterizable with this method.

Theorem 3.6. Let $n$ be a positive integer. If there exist at least two non-isomorphic abelian groups of order $n$, then abelian groups of order $n$ are not characterizable with their order component.

Proof. The proof is obvious. □

Remark 3.7. It was a conjecture that every finite simple group $M$, where $\Gamma(M)$ is not connected, is characterizable with its order components. But the following example is a counterexample.

Example 3.8. If $q$ is an odd-prime power and $n = 2^k \geq 4$, then $\text{OC}(S_{2n}(q)) = \text{OC}(O_{2n+1}(q))$, but obviously $S_{2n}(q) \not\cong O_{2n+1}(q)$.

Theorem 3.9. Let $p = 2^{\alpha}3^{\beta} + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = S_n$, where $n = p, p + 1$. Then $\text{OC}(G) = \text{OC}(M)$ if and only if $G \cong M$.

Proof. Similar to the proof of Theorem 3.3, since $G$ is a $C_{pp}$ group, we have $K/H \cong A_n$. Now using Lemma 3.2, we have

$$A_n \leq \frac{G}{H} \leq \text{Aut}(A_n) = S_n.$$  \hspace{1cm} (3.6)

Therefore, $G/H \cong A_n$ or $\text{Aut}(A_n) = S_n$. If $G/H \cong A_n$, then $|H| = 2$ and $H < G$. Hence, $H \subseteq Z(G) = 1$, which is a contradiction. Therefore, $G/H \cong S_n$, and since $|G| = |S_n|$, we have $G \cong S_n$. □

4. Some related results

Remark 4.1. It is a well known conjecture of J. G. Thompson that if $G$ is a finite group with $Z(G) = 1$ and $M$ is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

We can generalize this conjecture for the groups under discussion by our characterization of these groups.

Corollary 4.2. Let $G$ be a finite group with $Z(G) = 1$ and let $M$ be $A_p, A_{p+1}, A_{p+2}, S_p$, or $S_{p+1}$. If $N(G) = N(M)$, then $G \cong M$.

Proof. By Lemmas 2.8 and 2.9, if $G$ and $M$ are two finite groups satisfying the conditions of Corollary 4.2, then $\text{OC}(G) = \text{OC}(M)$. So, Theorems 3.3 and 3.9 imply this corollary. □
**Remark 4.3.** Shi and Bi in [26] put forward the following conjecture.

**Shi’s Conjecture.** Let $G$ be a group and $M$ a finite simple group. Then $G \cong M$ if and only if

(i) $|G| = |M|$,  
(ii) $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the set of orders of elements in $G$.

This conjecture is valid for sporadic simple groups [24], groups of alternating type [27], and some simple groups of Lie type [23, 25, 26]. As a consequence of Theorems 3.3 and 3.9, we prove a generalization of this conjecture for the groups under discussion.

**Corollary 4.4.** Let $G$ be a finite group and let $M$ be $A_p$, $A_{p+1}$, $A_{p+2}$, $S_p$, or $S_{p+1}$. If $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, then $G \cong M$.

**Proof.** By assumption, we must have $OC(G) = OC(M)$. Thus the corollary follows by Theorems 3.3 and 3.9. 

**References**


[3] ———, A new characterization of $G_2(q)$, $[q \equiv 0(\text{mod} 3)]$, J. Southwest China Normal Univ. (1996), 47–51.


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