

## HEEGAARD SPLITTINGS AND MORSE-SMALE FLOWS

RALF GAUTSCHI, JOEL W. ROBBIN, and DIETMAR A. SALAMON

Received 16 October 2002

We describe three theorems which summarize what survives in three dimensions of Smale's proof of the higher-dimensional Poincaré conjecture. The proofs require Smale's cancellation lemma and a lemma asserting the existence of a 2-gon. Such 2-gons are the analogues in dimension two of Whitney disks in higher dimensions. They are also embedded lunes; an (immersed) lune is an index-one connecting orbit in the Lagrangian Floer homology determined by two embedded loops in a 2-manifold.

2000 Mathematics Subject Classification: 57R58, 57N75.

**1. Introduction.** This is an expository paper. We wrote it to teach ourselves some low-dimensional topology. Our objective was to understand the speculation of Hsiang [9] concerning Floer homology and the Poincaré conjecture.

**INTERSECTION NUMBERS.** For transverse embedded closed curves  $\alpha$  and  $\beta$  in an orientable 2-manifold  $\Sigma$ , there are three ways to count the number of points in their intersection.

- (1) The *numerical intersection number*  $\text{num}(\alpha, \beta)$  is the actual number of intersection points.
- (2) The *geometric intersection number*  $\text{geo}(\alpha, \beta)$  is defined as the minimum of the numbers  $\text{num}(\alpha, \beta')$  over all embedded loops  $\beta'$  that are transverse to  $\alpha$  and isotopic to  $\beta$ .
- (3) The *algebraic intersection number*  $\text{alg}(\alpha, \beta)$  is the absolute value  $\text{alg}(\alpha, \beta) = |\alpha \cdot \beta|$  of the sum  $\alpha \cdot \beta = \sum_{x \in \alpha \cap \beta} \pm 1$ , where the plus sign is chosen if and only if the two orientations of  $T_x \Sigma = T_x \alpha \oplus T_x \beta$  match. This definition is independent of the choice of orientations of  $\alpha$ ,  $\beta$ , and  $\Sigma$ .

The inequalities

$$\text{alg}(\alpha, \beta) \leq \text{geo}(\alpha, \beta) \leq \text{num}(\alpha, \beta) \tag{1.1}$$

are immediate.

**REMARK 1.1.** Two embedded loops in  $\Sigma$  are homotopic if and only if they are isotopic (see [4]). Hence, if in the definition of geometric intersection number the word *isotopic* is replaced by the word *homotopic*, the value of  $\text{geo}(\alpha, \beta)$  remains unchanged.

**MORSE-SMALE/FLOER SYSTEMS.** Throughout this section  $M$  is a compact  $m$ -manifold, possibly with boundary. We assume throughout that  $\xi$  is a vector field on  $M$ , transverse to the boundary, and denote by  $\phi^t$  the flow of  $\xi$  and by  $P(\xi)$  the set of rest points. The stable and unstable manifolds of the rest point  $p$  are

$$\begin{aligned} W^s(p) &:= W^s(p; \xi) := \left\{ z \in M \mid \lim_{t \rightarrow \infty} \phi(t, z) = p \right\}, \\ W^u(p) &:= W^u(p; \xi) := \left\{ z \in M \mid \lim_{t \rightarrow -\infty} \phi(t, z) = p \right\}. \end{aligned} \tag{1.2}$$

The vector field  $\xi$  is called *gradient-like* if  $P(\xi)$  is a finite set and there exists a smooth *height function*  $h : M \rightarrow \mathbb{R}$  such that  $dh(z)\xi(z) \leq 0$  for all  $z \in M$  with equality if and only if  $z \in P(\xi)$ . It follows that

$$M = \bigcup_{p \in P(\xi)} W^s(p; \xi) = \bigcup_{p \in P(\xi)} W^u(p; \xi). \tag{1.3}$$

If  $\xi$  has only hyperbolic rest points, we write

$$P(\xi) = \bigcup_{k=0}^m P_k(\xi), \tag{1.4}$$

where  $P_k(\xi)$  denotes the set of rest points of Morse index  $k$ . A vector field  $\xi$  is called *Morse-Smale* (our terminology is nonstandard in that for us a Morse-Smale system has no periodic orbits) if and only if it is gradient-like and has only hyperbolic rest points (which implies that the stable and unstable manifolds are submanifolds of  $M$ ) such that  $W^u(p; \xi)$  and  $W^s(q; \xi)$  intersect transversally for all  $p, q \in P(\xi)$ . A gradient-like vector field  $\xi$  is called *Morse-Floer* if all its rest points are hyperbolic, if  $W^u(q; \xi)$  and  $W^s(p; \xi)$  intersect transversally for all  $p \in P_k(\xi)$  and  $q \in P_{k+1}(\xi)$ , and if there exists a  $z \in W^u(q; \xi) \cap W^s(p; \xi)$  with  $W^u(q; \xi) \pitchfork_z W^s(p; \xi)$  whenever  $W^u(q; \xi) \cap W^s(p; \xi) \neq \emptyset$  (cf. [19, Axiom B]). Note that if  $M$  has dimension three, then a Morse-Floer vector field is automatically Morse-Smale.

**REMARK 1.2.** Every Morse-Floer vector field  $\xi$  on  $M$  admits a *self-indexing* height function  $h : M \rightarrow \mathbb{R}$ , that is, one which satisfies  $h(p) = k$  for  $p \in P_k(\xi)$  and is constant on each boundary component (see [11]).

Define the *Smale order* on  $P(\xi)$  by  $p \preceq_\xi q$  if and only if there exists a sequence of rest points  $p = p_0, p_1, \dots, p_{n-1}, p_n = q$  such that  $W^u(p_i; \xi) \cap W^s(p_{i-1}; \xi) \neq \emptyset$  for  $i = 1, \dots, n$ . If  $\xi$  is gradient-like, this is a partial order. For a Morse-Floer vector field, it is equivalent to take  $n = 1$ :

$$p \preceq_\xi q \iff W^u(q; \xi) \cap W^s(p; \xi) \neq \emptyset. \tag{1.5}$$

(This is the “ $\lambda$ -Lemma” of Palis, see [12, 19].)

**HMS STRUCTURES.** Henceforth,  $Y$  is a *closed* (i.e., compact and without boundary) connected oriented smooth 3-manifold.

**DEFINITION 1.3.** An *HMS structure* (Heegaard-Morse-Smale structure) on  $Y$  is a triple  $(Y_0, Y_1, \xi)$  consisting of a Morse-Smale vector field  $\xi$  on  $Y$  and a decomposition  $Y = Y_0 \cup Y_1$  of  $Y$  into two 3-submanifolds intersecting in their common boundary:

$$Y = Y_0 \cup Y_1, \quad Y_0 \cap Y_1 = \partial Y_0 = \partial Y_1, \tag{1.6}$$

such that

- (i)  $\xi$  has one rest point  $p_0$  of index zero, one rest point  $q_0$  of index three,  $g$  rest points  $p_1, \dots, p_g$  of index one, and  $g$  rest points  $q_1, \dots, q_g$  of index two;
- (ii)  $p_0, p_1, \dots, p_g \in Y_0$  and  $q_0, q_1, \dots, q_g \in Y_1$ ;
- (iii)  $\xi$  is transverse to  $\Sigma$ .

A *Heegaard splitting* of  $Y$  is a decomposition  $Y = Y_0 \cup Y_1$  as in (1.6) which arises from some HMS structure.

**REMARK 1.4.** If a Morse-Smale vector field on  $Y$  has exactly one critical point of index zero and exactly one critical point of index three, then (by [Theorem 3.1](#)) the number of critical points of index one must equal the number of critical points of index two. In [Corollary 3.3](#), we show that this number is equal to the genus of  $\Sigma$ ; we call it the *genus* of the HMS structure.

**DEFINITION 1.5.** Let  $\alpha := \alpha_1 \cup \dots \cup \alpha_g$  and  $\beta := \beta_1 \cup \dots \cup \beta_g$  be the 1-submanifolds of  $\Sigma := Y_0 \cap Y_1$  defined by

$$\alpha_i := W^s(p_i) \cap \Sigma, \quad \beta_j := W^u(q_j) \cap \Sigma, \quad i, j = 1, \dots, g. \tag{1.7}$$

The pair  $(\alpha, \beta)$  is called *the trace* of the HMS structure  $(Y_0, Y_1, \xi)$  and *a trace* of the Heegaard splitting  $(Y, Y_0, Y_1)$ . Each connecting orbit from  $q_j$  to  $p_i$  intersects  $\Sigma$  in an intersection point of  $\alpha_i$  and  $\beta_j$ . It is said that an HMS structure is

$$\left\{ \begin{array}{l} \text{algebraically} \\ \text{geometrically} \\ \text{numerically} \end{array} \right\} \text{ reduced iff } \left\{ \begin{array}{l} \text{alg } (\alpha_i, \beta_j) = \delta_{ij} \\ \text{geo } (\alpha_i, \beta_j) = \delta_{ij} \\ \text{num } (\alpha_i, \beta_j) = \delta_{ij} \end{array} \right\} \tag{1.8}$$

for  $i, j = 1, \dots, g$ .

**REMARK 1.6.** Let  $\Sigma$  be a closed connected oriented 2-manifold of genus  $g$ . A *trace* in  $\Sigma$  is a closed 1-submanifold  $\alpha \subset \Sigma$  such that the complement  $\Sigma \setminus \alpha$  is connected. In [Appendix A](#), we show that a 1-submanifold  $\alpha \subset \Sigma$  is a trace if and only if it arises from an HMS structure as in [Definition 1.5](#). There, we also explain how to reconstruct the HMS structure  $(Y_0, Y_1, \xi)$  from a transverse pair of traces  $\alpha, \beta \subset \Sigma$ . Indeed, up to an appropriate notion of equivalence, a closed

connected oriented 3-manifold is the same as a 2-manifold equipped with a transverse pair of traces.

## 2. Main theorems

**THEOREM 2.1.** *Every closed connected oriented 3-manifold  $Y$  admits an HMS structure.*

**THEOREM 2.2.** *A closed connected oriented 3-manifold  $Y$  is an integral homology 3-sphere if and only if it admits an algebraically reduced HMS structure.*

**THEOREM 2.3.** *For every closed connected oriented 3-manifold  $Y$ , the following are equivalent:*

- (i)  $Y$  is diffeomorphic to the 3-sphere;
- (ii)  $Y$  admits an HMS structure of genus zero;
- (iii)  $Y$  admits a numerically reduced HMS structure;
- (iv)  $Y$  admits a geometrically reduced HMS structure.

When we began to work on this project, we hoped that the mere existence of an algebraically reduced HMS structure that is not geometrically reduced would imply that the homology 3-sphere  $Y$  has nontrivial Floer homology and is therefore not simply connected (and that the difficulty in establishing the Poincaré conjecture lies in proving nontriviality of Floer homology under this hypothesis). However, there is an algebraically reduced HMS structure on  $S^3$  which is not geometrically reduced, see [Example D.1](#).

**ROADMAP.** Except for the implication (iv) $\Rightarrow$ (iii) in [Theorem 2.3](#), the proofs of these theorems are the same as, or refinements of, the proofs used in the higher-dimensional Poincaré conjecture. (The standard exposition is [\[11\]](#).)

[Theorem 2.1](#) is explicitly stated in [\[18\]](#). Its proof uses the cancellation lemma (see [Theorem 4.1](#)) and the “Morse homology theory” described below. We give a proof of [Theorem 2.1](#) in [Section 4](#).

[Theorem 2.2](#) also uses this Morse homology theory and a “handle-sliding argument”; the proof is the same as in higher dimensions and is carried out in [Section 3](#).

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) in [Theorem 2.3](#) are obvious.

The implication (ii) $\Rightarrow$ (i) is essentially a smooth version of Reeb’s theorem [\[14\]](#). It follows easily from that fact that the group  $\text{Diff}_+(S^2)$  of orientation-preserving diffeomorphisms of the 2-sphere is connected. We give a proof of this well-known fact as well as the details of the proof of (ii) $\Rightarrow$ (i) in [Appendix B](#).

To prove (iii) $\Rightarrow$ (ii), we cancel critical points as in the higher-dimensional case. This only requires an alteration of the vector field in an arbitrarily small neighborhood of the connecting orbit. Hence, the cancellation of critical points can be carried out on a numerically reduced HMS structure so as to leave another numerically reduced HMS structure. The proof of the cancellation lemma is given in [Appendix C](#) and the proof of (iii) $\Rightarrow$ (ii) in [Section 4](#).

The implication (iv)⇒(iii) is proved in Section 5, the existence of a 2-gon is used here.

**FLOER HOMOLOGY.** The traces  $\alpha$  and  $\beta$  of an HMS structure  $(Y, Y_0, Y_1, \xi)$  can be interpreted as Lagrangian submanifolds of  $\Sigma := Y_0 \cap Y_1$  (with respect to any area form). The connecting orbits of the Morse complex (3.4) are intersections points of  $\alpha$  and  $\beta$ , and hence, can be interpreted as the critical points in Floer homology. The 2-gons appear as connecting orbits of index one in the Floer complex. In general, the Floer connecting orbits of index one need not be embedded, but are immersed half disks with boundary arcs in  $\alpha$  and  $\beta$ , respectively (see Section 6).

**3. Morse homology.** Let  $M$  be a compact  $m$ -manifold with boundary

$$\partial M = \Sigma_0 \cup \Sigma_1 \tag{3.1}$$

and let  $\xi$  be a Morse-Floer vector field on  $M$  that points in on  $\Sigma_1$  and points out on  $\Sigma_0$ . When the index difference of  $q$  and  $p$  is not one, let  $n(q, p) := n(q, p; \xi) := 0$ ; for  $p \in P_k(\xi)$  and  $q \in P_{k+1}(\xi)$ , we denote the number of connecting orbits by

$$n(q, p) := n(q, p; \xi) := \#(W^u(q; \xi) \cap W^s(p; \xi)) / \mathbb{R}. \tag{3.2}$$

Similarly, we define the algebraic number  $v(q, p) = v(q, p; \xi)$  of connecting orbits to be zero when the index difference of  $q$  and  $p$  is not one; for  $p \in P_k(\xi)$  and  $q \in P_{k+1}(\xi)$ , this number is defined as follows. Orient each  $W^u(p)$  arbitrarily. For every integral curve  $u : \mathbb{R} \rightarrow M$  of  $\xi$  running from  $q$  to  $p$ , choose an invariant complement  $E_t$  to  $\mathbb{R}\xi(u(t))$  in  $T_{u(t)}W^u(q)$ . This complement inherits an orientation from  $W^u(q)$  and, as  $t$  tends to infinity, converges to  $\pm T_p W^u(p)$  in the Grassmann bundle of oriented  $k$ -planes in  $TM$ . Denote the sign by  $\varepsilon(u)$  and define

$$v(q, p) := \sum_{[u]} \varepsilon(u), \tag{3.3}$$

where the sum runs over the equivalence classes  $[u]$  of integral curves of  $\xi$  from  $q$  to  $p$ ; the equivalence relation is given by time translation. If  $M$  is oriented, then  $W^s(p)$  can be oriented so that the product orientation of  $T_p M \cong T_p W^u(p) \oplus T_p W^s(p)$  is the orientation of  $T_p M$ . In this case,  $v(q, p)$  is the algebraic intersection number of  $W^u(q) \cap h^{-1}(k + 1/2)$  with  $W^s(p) \cap h^{-1}(k + 1/2)$  for  $q \in P_{k+1}$  and  $p \in P_k$ , where  $h$  is a self-indexing height function. Define  $\partial : C_{*+1} \rightarrow C_*$  by

$$C_k := \bigoplus_{p \in P_k} \mathbb{Z}p, \quad \partial q := \sum_{p \in P_k} v(q, p)p, \quad q \in P_{k+1}. \tag{3.4}$$

This chain complex is usually ascribed to Witten [20] and Floer [6], but the following theorem is older. (A proof may be found in [10] and other proofs can be found in [16, 17].)

**THEOREM 3.1.** *The operator  $\partial$  defined in (3.4) satisfies  $\partial \circ \partial = 0$  and its (co)homology is isomorphic to the singular (co)homology of the pair  $(M, \Sigma_0)$ . Namely, for every abelian group,*

$$\frac{\text{Kernel}(\partial : C_k \otimes \Lambda \rightarrow C_{k-1} \otimes \Lambda)}{\text{Image}(\partial : C_{k+1} \otimes \Lambda \rightarrow C_k \otimes \Lambda)} \cong H_k(M, \Sigma_0; \Lambda),$$

$$\frac{\text{Kernel}(\partial^* : \text{Hom}(C_k, \Lambda) \rightarrow \text{Hom}(C_{k+1}, \Lambda))}{\text{Image}(\partial^* : \text{Hom}(C_{k-1}, \Lambda) \rightarrow \text{Hom}(C_k, \Lambda))} \cong H^k(M, \Sigma_0; \Lambda).$$
(3.5)

**COROLLARY 3.2** (Poincaré duality). *These groups satisfy*

$$H^k(M, \Sigma_0; \Lambda) \cong H_{m-k}(M, \Sigma_1; \Lambda).$$
(3.6)

Hence, if  $\Lambda$  is a field,

$$H_k(M, \Sigma_0; \Lambda) \cong H_{m-k}(M, \Sigma_1; \Lambda).$$
(3.7)

**PROOF.** Reverse the flow and use [Theorem 3.1](#). □

**COROLLARY 3.3.** *Let  $Y_0$  be a compact connected oriented smooth 3-manifold with boundary and let  $\xi$  be a Morse-Smale vector field on  $Y_0$  that points in on the boundary and has only rest points of index zero and one. Then the 2-manifold  $\Sigma = \partial Y_0$  is connected and has genus*

$$g := 1 - \#P_1(\xi) + \#P_0(\xi).$$
(3.8)

**PROOF.** Take  $\Lambda := \mathbb{Q}$ . By [Theorem 3.1](#), we have

$$H_2(Y_0) = \{0\}, \quad H_1(Y_0, \Sigma) = \{0\}.$$
(3.9)

(The latter is proved by reversing the flow.) Hence, since the Euler characteristic of the chain complex agrees with the Euler characteristic of its homology, we have

$$\dim H_1(Y_0) - \dim H_0(Y_0) = \#P_1(Y_0) - \#P_0(Y_0).$$
(3.10)

Since  $Y_0$  is connected, it follows that

$$\dim H_1(Y_0) = g, \quad \dim H_2(Y_0, \Sigma) = g.$$
(3.11)

(The latter is proved by reversing the flow.) Hence, the homology exact sequence of the pair  $(Y, \Sigma)$  has the form

$$0 \rightarrow H_2(Y, \Sigma) \rightarrow H_1(\Sigma) \rightarrow H_1(Y) \rightarrow 0.$$
(3.12)

So  $\dim H_2(\Sigma) = 2g$  as claimed. □

**PROOF OF THEOREM 2.2 (ASSUMING THEOREM 2.1).** Take  $M = Y$  and  $\xi$  the vector field of an HMS structure. Then (3.4) is

$$\partial q_0 = 0, \quad \partial q_j = \sum_{i=1}^g (\alpha_i \cdot \beta_j) p_i, \quad \partial p_i = 0. \tag{3.13}$$

Thus,  $Y$  is an integral homology sphere if and only if the intersection matrix with entries

$$v_{ij} := \alpha_i \cdot \beta_j \tag{3.14}$$

is unimodular. This is certainly the case if the HMS structure is algebraically reduced.

For the converse, assume that  $Y$  is an integral homology 3-sphere. By Theorem 2.1, there exists an HMS structure  $(Y_0, Y_1, \xi)$  on  $Y$ . Let  $(v_{ij})$  be the corresponding intersection matrix. By Theorem 3.1, the matrix  $(v_{ij})$  is unimodular. Any integer matrix may be diagonalized by elementary row and column operations: scale, swap, and shear. The scale operation reverses the sign of a row or column, the swap operation interchanges two rows or columns, and the shear operation adds a row or column to a different one. Each operation may be realized by a corresponding operation on the HMS structure. Reversing the sign of the  $j$ th column corresponds to reversing the orientation of  $W^u(q_j)$ , and hence, of  $\beta_j$ . Interchanging rows or columns corresponds to relabeling the components of  $\alpha$  or  $\beta$ . To perform the shear which adds column  $i$  to column  $j$ , we will replace  $\beta_i$  by the connected sum

$$\beta'_i \cong \beta_i \# \beta_j. \tag{3.15}$$

To construct  $\beta'_i$ , choose an embedding  $\gamma : [0, 1] \rightarrow \Sigma$  such that

$$\gamma(0) \in \beta_i, \quad \gamma(1) \in \beta_j, \quad \gamma((0, 1)) \cap \beta = \emptyset, \tag{3.16}$$

and  $\gamma$  intersects  $\beta_i$  and  $\beta_j$  with opposite signs. This is possible because  $\Sigma \setminus \beta$  is connected. Use this path as a guide to construct  $\beta'_i$  as an embedded path near the one that traces out  $\beta_i$ ,  $\gamma$ ,  $\beta_j$ , and  $\gamma^{-1}$ . We construct a Morse-Smale vector field  $\xi'$  with trace  $(\alpha, \beta')$ , where

$$\beta' := \beta_1 \cup \dots \cup \beta_{i-1} \cup \beta'_i \cup \beta_{i+1} \cup \dots \cup \beta_g, \tag{3.17}$$

as follows. Let  $h : Y \rightarrow \mathbb{R}$  be a height function for  $\xi$ , that is,  $dh \cdot \xi$  is negative on the complement of the rest points. We assume that

$$\max_v h(p_v) < h(\Sigma) < \min_{v \neq i, j} h(q_v) \leq \max_{v \neq i, j} h(q_v) < h(q_j) < h(q_i). \tag{3.18}$$

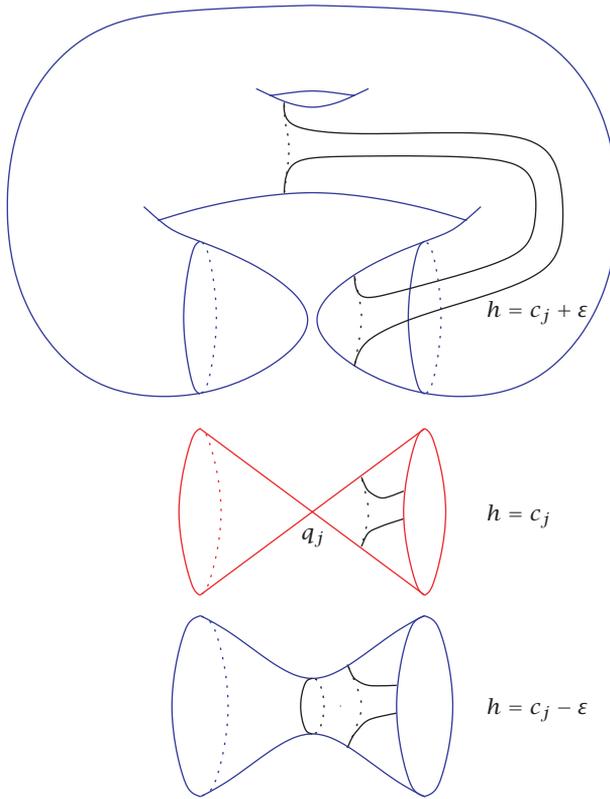


FIGURE 3.1. The backward orbit of  $\beta_i \# \beta_j$  near  $q_j$ .

Then the level set  $h^{-1}(c)$  is diffeomorphic to the 2-torus for  $h(q_j) < c < h(q_i)$ . Choose  $c$  and  $c'$  such that

$$h(q_j) < c' < c < h(q_i). \tag{3.19}$$

Let  $b_i$  be the intersection of the backwards orbit of  $\beta_i$  with  $h^{-1}(c)$  and let  $b'_i$  be the intersection of the backwards orbit of  $\beta'_i$  with  $h^{-1}(c')$ . Then  $b_i = W^u(q_i) \cap h^{-1}(c)$  and  $b'_i$  is isotopic to  $W^u(q_i) \cap h^{-1}(c')$  (see Figure 3.1). By familiar arguments,  $h^{-1}([c', c])$  is diffeomorphic to  $\mathbb{T}^2 \times [c', c]$  with orbits  $\{\text{pt}\} \times [c', c]$  (see [11]). Modify the flow in  $h^{-1}([c', c])$  so that it carries  $b_i$  to  $b'_i$ . □

**4. The cancellation lemma.** The following is an improved form of Smale’s cancellation lemma with essentially the same proof (see Appendix C).

**THEOREM 4.1** (cancellation lemma). *Suppose that  $\xi$  is a Morse-Floer vector field on  $M$  and let  $\bar{p}, \bar{q} \in P(\xi)$  be such that*

$$n(\bar{q}, \bar{p}; \xi) = 1. \tag{4.1}$$

*Let  $\Gamma$  denote the closure of the connecting orbit. Then, for every neighborhood  $U$  of  $\Gamma$ , there exists a Morse-Floer vector field  $\eta$  on  $M$  which agrees with  $\xi$  on the complement of  $U$  and satisfies*

$$P(\eta) = P(\xi) \setminus \{\bar{p}, \bar{q}\}, \tag{4.2}$$

$$p \preceq_{\eta} q \iff p \preceq_{\xi} q \text{ or } p \preceq_{\xi} \bar{q}, \bar{p} \preceq_{\xi} q, \tag{4.3}$$

$$n(q, p; \eta) = n(q, p; \xi) + n(q, \bar{p}; \xi)n(\bar{q}, p; \xi), \tag{4.4}$$

for  $p, q \in P(\eta)$ .

**REMARK 4.2.** If  $n(q, \bar{p}; \xi) = 0$ , then the closure of  $W^u(q; \xi)$  does not intersect the closure of the connecting orbit from  $\bar{q}$  to  $\bar{p}$ . Hence,  $W^u(q; \eta) = W^u(q; \xi)$  for every vector field  $\eta$  which agrees with  $\xi$  outside of a sufficiently small neighborhood of the connecting orbit from  $\bar{q}$  to  $\bar{p}$ . In this case, the formula (4.4) holds trivially. A similar argument deals with the case  $n(\bar{q}, p; \xi) = 0$ .

**REMARK 4.3.** If  $n(\bar{q}, \bar{p}; \xi) = \nu(\bar{q}, \bar{p}; \xi) = 1$ , then the algebraic numbers of connecting orbits of  $\eta$  are given by

$$\nu(q, p; \eta) = \nu(q, p; \xi) - \nu(q, \bar{p}; \xi)\nu(\bar{q}, p; \xi). \tag{4.5}$$

This follows from a refinement of the proof of [Theorem 4.1](#) which we will not discuss in this paper. Using (4.5), one can use standard arguments (see [5]) to construct a chain homotopy equivalence from the Morse complex of  $\xi$  to the Morse complex of  $\eta$ . This argument gives rise to an alternative proof of the fact that the Morse homology is independent of the Morse-Floer vector field  $\xi$  used to define it. Namely, in a generic one-parameter family of Morse-Floer vector fields, the boundary operator changes only through cancellation of critical points of index difference one.

**PROOF OF THEOREM 2.1.** By transversality,  $Y$  admits a Morse-Smale vector field  $\xi$ . For  $q \in P_1(\xi)$  and  $p \in P_0(\xi)$ , we have  $n(q, p) \in \{0, 1, 2\}$  and  $\nu(q, p) = 0$  if  $n(q, p) \in \{0, 2\}$ . Hence, by [Theorem 3.1](#), there must be a pair with  $n(q, p) = 1$  if  $P_0(\xi)$  has more than one element. Then, by [Theorem 4.1](#), we may find another Morse-Smale vector field  $\eta$  with  $P_0(\eta)$  of smaller size than  $P_0(\xi)$ . The same argument works to reduce  $P_3(\xi)$ . □

**PROOF OF (iii)⇒(ii) IN THEOREM 2.3.** The proof uses the cancellation lemma only under the hypothesis  $n(q, \bar{p}; \xi) = n(\bar{q}, p; \xi) = 0$  (see [Remark 4.2](#)). In this case, [Theorem 4.1](#) says that we can modify a numerically reduced HMS structure so as to produce another numerically reduced HMS structure of genus one less. The result now follows by induction. □

## 5. Isotopy

**LEMMA 5.1** (isotopy lemma). *Let  $(Y_0, Y_1, \xi)$  be an HMS structure on  $Y$  with  $\Sigma := Y_0 \cap Y_1$  and trace*

$$\alpha = \alpha_1 \cup \dots \cup \alpha_g, \quad \beta = \beta_1 \cup \dots \cup \beta_g. \quad (5.1)$$

*Suppose that  $f : \Sigma \rightarrow \Sigma$  is a diffeomorphism isotopic to the identity such that  $f(\beta)$  is transverse to  $\alpha$ . Then there is an HMS structure  $(Y_0, Y_1, \xi')$  on  $Y$  with trace*

$$\alpha = \alpha_1 \cup \dots \cup \alpha_g, \quad f(\beta) = f(\beta_1) \cup \dots \cup f(\beta_g). \quad (5.2)$$

**PROOF.** Use the graph of the isotopy to modify the flow.  $\square$

**Lemma 5.1** does not suffice to prove (iv) $\Rightarrow$ (iii) in **Theorem 2.3**. If the HMS structure is geometrically reduced but not numerically reduced, there is a pair of indices  $(i_0, j_0)$  and a diffeomorphism  $f$  isotopic to the identity with

$$\delta_{i_0, j_0} = \text{geo}(\alpha_{i_0}, \beta_{j_0}) = \text{num}(\alpha_{i_0}, f(\beta_{j_0})) < \text{num}(\alpha_{i_0}, \beta_{j_0}). \quad (5.3)$$

This does not prove (iv) $\Rightarrow$ (iii) because we do not know that

$$\text{num}(\alpha_i, f(\beta_j)) \leq \text{num}(\alpha_i, \beta_j) \quad (5.4)$$

for all  $i, j = 1, 2, \dots, g$ . We need to choose  $f$  more carefully. For this, we require the following lemma which is proved as in [8, Lemma 3.1, page 108]. The formulation here has additional hypotheses (which hold in our application) but our proof is the same as the proof in [8].

**LEMMA 5.2.** *Let  $\Sigma$  be a closed oriented 2-manifold and let  $\alpha, \beta \subset \Sigma$  be two noncontractible transverse embedded loops. Assume that*

$$\text{geo}(\alpha, \beta) < \text{num}(\alpha, \beta). \quad (5.5)$$

*Then there exists a smooth orientation preserving embedding  $u : \mathbb{D} \rightarrow \Sigma$  of the half disk*

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1, \text{Im } z \geq 0\} \quad (5.6)$$

*such that*

$$u(\mathbb{D} \cap \mathbb{R}) \subset \alpha, \quad u(\mathbb{D} \cap S^1) \subset \beta. \quad (5.7)$$

A subset  $L$  of an oriented 2-manifold  $\Sigma$  is called a 2-gon if it is the image of an orientation preserving embedding  $u : \mathbb{D} \rightarrow \Sigma$ . The points  $u(-1)$  and  $u(1)$  are called the *corner points* of  $L$ , respectively, and the arcs  $u(\mathbb{D} \cap \mathbb{R})$  and  $u(\mathbb{D} \cap S^1)$  are called the *boundary arcs* of  $L$ , respectively.

**LEMMA 5.3.** *Let  $A, B \subset \mathbb{R}^2$  be embedded arcs intersecting only in their endpoints  $x$  and  $y$ . Let  $U$  denote the bounded component of  $\mathbb{R}^2 \setminus (A \cup B)$ . Then the following are equivalent.*

- (i) *The closure  $L$  of  $U$  is a 2-gon.*
- (ii) *The interior angles of  $U$  at the two corners are less than  $\pi$ .*

**PROOF.** That (i) implies (ii) is obvious. To prove the converse, construct the diffeomorphism  $u : \mathbb{D} \rightarrow L$  near the corners, extend it to a collar neighborhood of the boundary, and, by Morse theory, extend it to all of  $\mathbb{D}$ . □

**LEMMA 5.4.** *Let  $\Sigma$ ,  $\alpha$ , and  $\beta$  be as in Lemma 5.2. Let  $\pi : \tilde{\Sigma} \rightarrow \Sigma$  be a covering. Call two intersection points  $x, y \in \alpha \cap \beta$   $\pi$ -equivalent if there exist lifts  $\tilde{\alpha}$  and  $\tilde{\beta}$  of  $\alpha$  and  $\beta$ , respectively, and points  $\tilde{x}, \tilde{y} \in \tilde{\alpha} \cap \tilde{\beta}$  such that  $\pi(\tilde{x}) = x$  and  $\pi(\tilde{y}) = y$ . If  $\text{num}(\alpha, \beta) > \text{geo}(\alpha, \beta)$ , then there exists a pair of distinct, but equivalent, intersection points.*

**PROOF.** Let  $[0, 1] \times S^1 \rightarrow \Sigma : (t, \theta) \mapsto b(t, \theta) = b_t(\theta)$  be an isotopy such that  $b_0(S^1) = \beta$ ,  $b$  and  $b_1$  are transverse to  $\alpha$ , and  $\text{num}(\alpha, b_1(S^1)) = \text{geo}(\alpha, \beta)$ . Since  $\text{num}(\alpha, b_0(S^1)) > \text{num}(\alpha, b_1(S^1))$ , there must be a component of the 1-manifold  $b^{-1}(\alpha)$  with both endpoints in  $\{0\} \times S^1$ . The images of these endpoints under  $b_0$  are distinct intersection points of  $\alpha$  and  $\beta$ . By the covering space theory, they are equivalent. □

**PROOF OF LEMMA 5.2.** Let  $\pi : \mathbb{R}^2 = \tilde{\Sigma} \rightarrow \Sigma$  be the universal cover. A 2-gon  $\tilde{L} \subset \tilde{\Sigma}$  is called *admissible* if

$$\partial \tilde{L} \subset \pi^{-1}(\alpha) \cup \pi^{-1}(\beta). \tag{5.8}$$

It follows that one of the boundary arcs is contained in  $\pi^{-1}(\alpha)$  and the other in  $\pi^{-1}(\beta)$ . The set  $\mathcal{L}$  of admissible 2-gons is partially ordered by inclusion.

By Lemma 5.4, there exists a pair of distinct, but  $\pi$ -equivalent, intersection points of  $\alpha$  and  $\beta$ . Hence, there exist lifts  $\tilde{\alpha}$  and  $\tilde{\beta}$  of  $\alpha$  and  $\beta$ , respectively, and intersection points  $\tilde{x}, \tilde{y} \in \tilde{\alpha} \cap \tilde{\beta}$  such that  $\pi(\tilde{x}) \neq \pi(\tilde{y})$ . Changing  $\tilde{y}$ , if necessary, we may assume that the arc  $\tilde{B} \subset \tilde{\beta}$  from  $\tilde{x}$  to  $\tilde{y}$  lies on one side of  $\tilde{\alpha}$ . Let  $\tilde{A}$  be the arc in  $\tilde{\alpha}$  from  $\tilde{x}$  to  $\tilde{y}$ . Then, by Lemma 5.3,  $\tilde{A}$  and  $\tilde{B}$  bound an admissible 2-gon. Hence,  $\mathcal{L} \neq \emptyset$ , and hence,  $\mathcal{L}$  contains a minimal element  $\tilde{L}$ . Every such minimal 2-gon satisfies

$$\text{int}(\tilde{L}) \cap \pi^{-1}(\alpha) = \text{int}(\tilde{L}) \cap \pi^{-1}(\beta) = \emptyset. \tag{5.9}$$

This is because no component of  $\pi^{-1}(\alpha)$  or  $\pi^{-1}(\beta)$  can lie entirely inside a bounded open set; hence any such component which intersects the interior would have to exit and therefore cut off a smaller admissible 2-gon.

Let  $\tilde{L}$  be a minimal admissible 2-gon with corner points  $\tilde{x}, \tilde{y} \in \pi^{-1}(\alpha) \cap \pi^{-1}(\beta)$  and boundary arcs  $\tilde{A} \subset \pi^{-1}(\alpha)$  and  $\tilde{B} \subset \pi^{-1}(\beta)$ . It remains to show that  $\pi|_{\tilde{L}} : \tilde{L} \rightarrow \Sigma$  is injective. To see this, let  $\mathcal{g} : \tilde{\Sigma} \rightarrow \tilde{\Sigma}$  be a deck transformation

other than the identity. Then

$$g(\text{int}(\tilde{L})) \cap \text{int}(\tilde{L}) = \emptyset. \tag{5.10}$$

Otherwise,  $g(\text{int}(\tilde{L})) = \text{int}(\tilde{L})$ , so  $g(\tilde{L}) = \tilde{L}$ , and hence,  $g$  has a fixed point, a contradiction. Moreover,  $g(\tilde{x}) \neq \tilde{y}$  and  $g(\tilde{y}) \neq \tilde{x}$  because  $g$  is orientation preserving and the intersection numbers of  $\tilde{A}$  and  $\tilde{B}$  at  $\tilde{x}$  and  $\tilde{y}$  are opposite. It follows that  $g(\tilde{x}) \notin \tilde{A}$  and  $g(\tilde{y}) \notin \tilde{A}$ , and hence,

$$g(\tilde{A}) \cap \tilde{A} = \emptyset = g(\tilde{B}) \cap \tilde{B}. \tag{5.11}$$

Thus,  $g(\tilde{L}) \cap \tilde{L} = \emptyset$  for every nontrivial deck transformation  $g$ , and so  $\pi|_{\tilde{L}}$  is injective as claimed.  $\square$

**PROOF OF THEOREM 2.3 (IV) $\Rightarrow$ (III).** Let  $(Y_0, Y_1, \xi)$  be a geometrically reduced HMS structure on  $Y$  with  $\Sigma := Y_0 \cap Y_1$  and trace  $\alpha = \alpha_1 \cup \dots \cup \alpha_g$ ,  $\beta = \beta_1 \cup \dots \cup \beta_g$ . Assume that this HMS structure is not numerically reduced so that

$$\text{geo}(\alpha_{i_0}, \beta_{j_0}) < \text{num}(\alpha_{i_0}, \beta_{j_0}) \tag{5.12}$$

for some pair  $(i_0, j_0)$ . As in [Definition A.6](#), the homology classes of  $\alpha_1, \dots, \beta_g$  form an integral basis of  $H_1(\Sigma; \mathbb{Z})$ . In particular,  $\alpha_{i_0}$  and  $\beta_{j_0}$  are not contractible.

By [Lemma 5.2](#), there is a smooth embedding  $u : \mathbb{D} \rightarrow \Sigma$  with  $u(\mathbb{D} \cap \mathbb{R}) \subset \alpha_{i_0}$  and  $u(\mathbb{D} \cap S^1) \subset \beta_{j_0}$ . We will use this embedding to deform  $\beta_{j_0}$  by an ambient isotopy to remove the two intersections between  $\alpha_{i_0}$  and  $\beta_{j_0}$  at the corners of the 2-gon. Under this isotopy, none of the numbers  $\text{num}(\alpha_i, \beta_j)$  increases. More precisely, extend  $u$  to an embedding (still denoted by  $u$ ) of the open set

$$\mathbb{D}_\varepsilon := \{z \in \mathbb{C} \mid \text{Im} z > -\varepsilon, |z| < 1 + \varepsilon\} \tag{5.13}$$

for  $\varepsilon > 0$  sufficiently small such that

$$\begin{aligned} u(\mathbb{D}_\varepsilon) \cap \beta_{j_0} &= u(\mathbb{D}_\varepsilon \cap S^1), & u(\mathbb{D}_\varepsilon) \cap \alpha_{i_0} &= u(\mathbb{D}_\varepsilon \cap \mathbb{R}), \\ u(\{z \in \mathbb{D}_\varepsilon \mid |z| > 1\}) \cap \beta_j &= \emptyset, & u(\{z \in \mathbb{D}_\varepsilon \mid \text{Re} z < 0\}) \cap \alpha_i &= \emptyset, \end{aligned} \tag{5.14}$$

for all  $i$  and  $j$ . Choose an isotopy  $\psi_t : \Sigma \rightarrow \Sigma$  supported in  $u(\mathbb{D}_\varepsilon)$  such that  $\psi_0 = \text{id}$  and

$$\psi_1(\mathbb{D}) \subset \{z \in \mathbb{D}_\varepsilon \mid \text{Im} z < 0\} \tag{5.15}$$

(see [Figure 5.1](#)).

Now replace  $\beta_j$  by

$$\beta'_j := \psi_1(\beta_j). \tag{5.16}$$

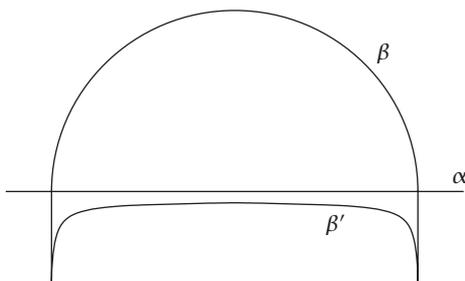


FIGURE 5.1. Removing a 2-gon.

Then

$$\text{num}(\alpha_{i_0}, \beta'_{j_0}) \leq \text{num}(\alpha_{i_0}, \beta_{j_0}) - 2 \tag{5.17}$$

and  $\text{num}(\alpha_i, \beta'_j) \leq \text{num}(\alpha_i, \beta_j)$  for all  $i$  and  $j$ . □

**6. Floer homology.** The Lagrangian Floer homology  $\text{HF}(\alpha, \beta)$  for pairs of loops  $\alpha$  and  $\beta$  on a Riemann surface  $\Sigma$  can be viewed as an infinite-dimensional analogue of the Morse homology described in Section 3: the manifold  $M$  is replaced by the space of paths in  $\Sigma$  from  $\alpha$  to  $\beta$  and the “critical points” are the constant paths, that is, the points of  $\alpha \cap \beta$ . To define an operator as in (3.4), we require a notion of “connecting orbit of index (difference) one” and a way of counting these connecting orbits. In the present (two-dimensional case), the connecting orbits can be defined combinatorially, following Vin de Silva [1], rather than analytically as in Floer’s original approach [5]. In this section, we describe this combinatorial definition; the proof of Theorem 6.2 is given in [2].

**DEFINITION 6.1.** Throughout,  $\alpha$  and  $\beta$  are transverse embedded loops in a closed orientable 2-manifold  $\Sigma$ . A *smooth  $(\alpha, \beta)$ -lune* is an equivalence class of orientation-preserving immersions  $u : \mathbb{D} \rightarrow \Sigma$  such that  $u(\mathbb{D} \cap \mathbb{R}) \subset \alpha$ ,  $u(\mathbb{D} \cap S^1) \subset \beta$ . The equivalence relation is defined by

$$[u] = [u'] \tag{6.1}$$

if and only if there is an orientation-preserving diffeomorphism  $\phi : \mathbb{D} \rightarrow \mathbb{D}$  such that

$$\phi(-1) = -1, \quad \phi(1) = 1, \quad u' = u \circ \phi. \tag{6.2}$$

That  $u$  is an immersion means that  $u$  is smooth and  $du$  is injective in all of  $\mathbb{D}$ , even at the corners  $\pm 1$ . The *endpoints* of the lune are intersection points

$$u(-1), u(1) \in \alpha \cap \beta \tag{6.3}$$

of  $\alpha$  and  $\beta$ . When  $x = u(-1)$  and  $y = u(1)$ , we say that the lune is *from  $x$  to  $y$* . The image of an embedded lune is a 2-gon as defined in Section 5. These notions are clearly independent of the choice of the immersion  $u$  representing the smooth lune.

In the remainder of this section,  $\Sigma$  is a closed connected oriented 2-manifold of positive genus. For each pair  $\alpha$  and  $\beta$  of transverse noncontractible embedded loops which are not isotopic to each other, we define

$$CF(\alpha, \beta) = \bigoplus_{x \in \alpha \cap \beta} \mathbb{Z}_2 x, \tag{6.4}$$

and a linear map  $\partial : CF(\alpha, \beta) \rightarrow CF(\alpha, \beta)$ , called the *Floer boundary operator*, by

$$\partial x = \sum_y (n(x, y) \bmod 2) y, \tag{6.5}$$

where  $n(x, y)$  denotes the number of smooth  $(\alpha, \beta)$ -lunes from  $x$  to  $y$ .

**THEOREM 6.2.** (a) For all  $x, y \in \alpha \cap \beta$ ,  $n(x, y) \in \{0, 1\}$ .

(b) The operator  $\partial : CF(\alpha, \beta) \rightarrow CF(\alpha, \beta)$  is a chain complex, that is,  $\partial \circ \partial = 0$ . Its homology will be denoted by

$$HF(\alpha, \beta) := \ker \partial / \text{im } \partial \tag{6.6}$$

and is called the *Floer homology of the pair  $(\alpha, \beta)$* .

(c) If  $\alpha', \beta' \subset \Sigma$  are transverse embedded loops such that  $\alpha$  is isotopic to  $\alpha'$  and  $\beta$  is isotopic to  $\beta'$ , then

$$HF(\alpha, \beta) \cong HF(\alpha', \beta'). \tag{6.7}$$

(d) If the Floer boundary operator  $\partial : CF(\alpha, \beta) \rightarrow CF(\alpha, \beta)$  is nonzero, then there exists an embedded  $(\alpha, \beta)$ -lune.

**COROLLARY 6.3.** It holds that

$$\dim CF(\alpha, \beta) = \text{num}(\alpha, \beta), \quad \dim HF(\alpha, \beta) = \text{geo}(\alpha, \beta). \tag{6.8}$$

**PROOF.** The first statement follows from the definition of  $CF(\alpha, \beta)$ . To prove the second statement, choose  $\beta'$  isotopic to  $\beta$  so that  $\beta'$  is transverse to  $\alpha$  and  $\text{num}(\alpha, \beta') = \text{geo}(\alpha, \beta)$ . Then the boundary operator of the pair  $(\alpha, \beta')$  is zero; if not, then, by (d), there is an embedded  $(\alpha, \beta')$ -lune and hence, as in the proof of (iv) $\Rightarrow$ (iii) in Theorem 2.3, there exists an embedded loop  $\beta''$  isotopic to  $\beta'$  with  $\text{num}(\alpha, \beta'') < \text{num}(\alpha, \beta')$ , a contradiction. Hence, by (c),

$$\dim HF(\alpha, \beta) = \dim HF(\alpha, \beta') = \text{num}(\alpha, \beta') = \text{geo}(\alpha, \beta), \tag{6.9}$$

as claimed. □

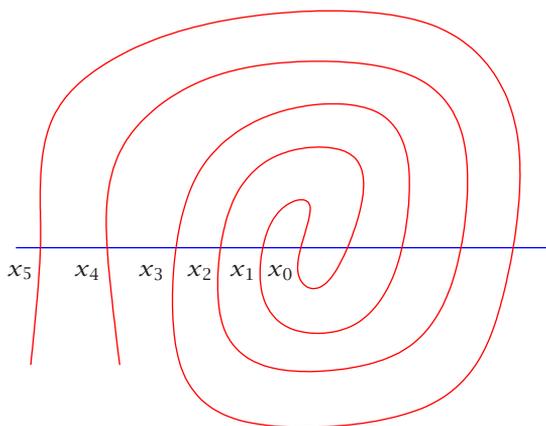


FIGURE 6.1. Lunes from  $x_i$  to  $x_{i-1}$ .

**REMARK 6.4.** It is easy to show that if there is a lune, then there is an embedded lune. Hence, Corollary 6.3 provides another proof of Lemma 5.2.

**REMARK 6.5.** The proof of [2, Theorem 5.2(a)] is based on a combinatorial characterization of smooth lunes which shows that a smooth lune is uniquely determined by its boundary arcs. In contrast, there exists an immersion of the circle into the plane with transverse self intersections which extends in nonequivalent ways to an immersion of the disk (see [13]).

**REMARK 6.6.** If  $x, y \in \alpha \cap \beta$  such that  $n(x, y) = 1$ , then  $\alpha$  and  $\beta$  have opposite intersection numbers at  $x$  and  $y$ . In particular,  $n(x, x) = 0$ . This shows that the Floer homology groups have a mod 2 grading. Namely, orient  $\alpha$  and  $\beta$  and write

$$CF(\alpha, \beta) = CF_0(\alpha, \beta) \oplus CF_1(\alpha, \beta), \tag{6.10}$$

where  $CF_i(\alpha, \beta)$  is generated by those intersection points where the intersection number is  $(-1)^i$ . Then the Floer boundary operator interchanges  $CF_0$  and  $CF_1$ .

**REMARK 6.7.** Define a relation  $x \preceq y$  on  $\alpha \cap \beta$  by  $x \preceq y$  if and only if there is a sequence  $x = x_0, \dots, x_k = y$  in  $\alpha \cap \beta$  with  $k \geq 0$  such that  $n(x_i, x_{i-1}) \neq 0$  for each  $i > 0$  (see Figure 6.1). Then  $x \preceq y$  is a partial order. To prove this, let  $\Omega_{\alpha, \beta}$  denote the space of all smooth curves  $z : [0, 1] \rightarrow \Sigma$  satisfying the boundary conditions  $z(0) \in \alpha$  and  $z(1) \in \beta$ . The intersection points of  $\alpha \cap \beta$  are the constant curves in  $\Omega_{\alpha, \beta}$ . Each component of the space  $\Omega_{\alpha, \beta}$  is simply connected, and hence, for every area form on  $\Sigma$ , the symplectic action is single valued. It is monotone with respect to the relation  $x \preceq y$ . This means that there is a function  $\mathcal{A} : \Omega_{\alpha, \beta} \rightarrow \mathbb{R}$  (the “action functional”) such that for any

curve  $\{z_s\}_{0 \leq s \leq 1}$  in  $\Omega_{\alpha, \beta}$ , the number  $\mathcal{A}(z_0) - \mathcal{A}(z_1)$  is the area of the region swept out. This function satisfies  $\mathcal{A}(x_{i-1}) < \mathcal{A}(x_i)$  for every  $i > 0$ , and hence, by induction,

$$x \preceq y \implies \mathcal{A}(x) \leq \mathcal{A}(y). \tag{6.11}$$

The relation  $x \preceq y$  is called the *Smale order* determined by  $(\alpha, \beta)$ .

**REMARK 6.8.** The proof of [2, Theorem 5.2(c)] establishes the following analog of the cancellation lemma ([Theorem 4.1](#)). Suppose that the isotopy is elementary in the sense that

$$\alpha' \cap \beta' = \alpha \cap \beta \setminus \{x, y\} \tag{6.12}$$

and the change in the number of intersection points occurs just at one parameter value and in the manner suggested by [Figure 5.1](#). Then, for  $x', y' \in \alpha' \cap \beta'$ , we have

$$\begin{aligned} x' \preceq' y' &\iff x' \preceq y' \text{ or } x' \preceq y, x \preceq y', \\ n'(x', y') &= n(x', y') + n(x', y)n(x, y'), \end{aligned} \tag{6.13}$$

where  $n(x', y')$  denotes the number of  $(\alpha, \beta)$ -lunes from  $x'$  to  $y'$ ,  $n'(x', y')$  denotes the number of  $(\alpha', \beta')$ -lunes from  $x'$  to  $y'$ , and  $x' \preceq' y'$  is the Smale order of  $(\alpha', \beta')$ .

**REMARK 6.9.** In Floer’s original theory, the number  $n(x, y)$  is defined as the (oriented) number of index-one holomorphic strips from  $x$  to  $y$ . To relate this definition to the above one must show the following.

- (i) The linearized Fredholm operator is surjective for every holomorphic strip. It follows that the number of index-one holomorphic strips from  $x$  to  $y$  (modulo time shift) is finite and is independent of the complex structure on  $\Sigma$ .
- (ii) The Fredholm index is one if and only if the holomorphic strip factors through an  $(\alpha, \beta)$ -lune.
- (iii) The correspondence between index-one holomorphic strips and the lunes in (ii) is bijective.

These assertions are specific to the two-dimensional case. The proof of (ii) follows from the asymptotic analysis established in [15] and an identity relating the Maslov index to the number of branch points. This approach leads to another proof of [Theorem 6.2](#). Details will appear elsewhere.

**REMARK 6.10.** Without the assumptions that  $\alpha$  and  $\beta$  are not contractible and not isotopic to each other, it can happen that  $\partial \circ \partial \neq 0$  (so there is no homology theory) or that  $\partial \circ \partial = 0$  but the resulting homology theory is not

invariant under isotopy. As an example of the former, take  $\alpha := S^1 \times \{\text{pt}\} \subset \mathbb{T}^2$  and take  $\beta$  to be a small circle intersecting  $\alpha$  transversely in two points. As an example of the latter, take  $\alpha := S^1 \times \{\text{pt}\} \subset \mathbb{T}^2$  and  $\beta$  to be the graph of a smooth map  $f : S^1 \rightarrow S^1$ . If  $\alpha$  and  $\beta$  do not intersect, then  $\text{HF}(\alpha, \beta) = 0$ , and if they do, then  $\text{HF}(\alpha, \beta) \cong H_*(S^1)$ . Floer's original theory is invariant only under Hamiltonian isotopy and only applies to the case where  $\alpha$  and  $\beta$  are not contractible and are Hamiltonian isotopic to each other. In their recent work [7], Fukaya et al. developed an obstruction theory for Floer homology of Lagrangian intersections which allows the construction of Floer homology groups in some cases where  $\partial \circ \partial \neq 0$ .

## Appendices

### A. Handlebodies

**DEFINITION A.1.** Let  $Y_0$  be a compact connected oriented 3-manifold with boundary  $\partial Y_0$ . A *handlebody structure* on  $Y_0$  is a Morse-Smale vector field  $\xi$  that points in on the boundary and has a single rest point  $p_0$  of index zero, rest points  $p_1, \dots, p_g$  of index one, and no other rest point. The *trace* of the handlebody structure is the 1-submanifold

$$\alpha = \alpha_1 \cup \dots \cup \alpha_g \tag{A.1}$$

of  $\partial Y_0$  defined by

$$\alpha_i = W^s(p_i) \cap \partial Y_0; \tag{A.2}$$

we say that  $\alpha$  is the *trace* of  $(Y_0, \xi)$  and a *trace* of  $Y_0$ . It follows that  $\partial Y_0$  is a closed connected oriented 2-manifold of genus  $g$  (see [Corollary 3.3](#)). A *handlebody* is a compact connected oriented 3-manifold  $Y_0$  which admits a handlebody structure.

**REMARK A.2.** A compact connected oriented 3-manifold  $Y_0$  is a handlebody if and only if it admits a Morse-Smale vector field  $\xi$  which points in on the boundary and has only rest points of index zero and one, that is, excess rest points of index zero can be cancelled. Namely, if  $\#P_0(\xi) > 1$ , then, as  $H_0(Y_0; \mathbb{Q}) = \mathbb{Q}$ , there must exist a pair of rest points  $p \in P_0(\xi)$  and  $q \in P_1(\xi)$  with  $n(q, p) = 1$ . Use the cancellation lemma repeatedly to reduce  $\#P_0(\xi)$ .

**THEOREM A.3.** *Two handlebodies whose boundaries have the same genus are diffeomorphic. More precisely, let  $Y_0$  and  $\tilde{Y}_0$  be handlebodies with traces  $\alpha$  and  $\tilde{\alpha}$ , respectively. Suppose that  $\partial Y_0$  and  $\partial \tilde{Y}_0$  have the same genus  $g$ . Then there exists a diffeomorphism  $\phi : \partial Y_0 \rightarrow \partial \tilde{Y}_0$  such that  $\phi(\alpha) = \tilde{\alpha}$  and any such  $\phi$  extends to a diffeomorphism  $\psi_0 : Y_0 \rightarrow \tilde{Y}_0$ .*

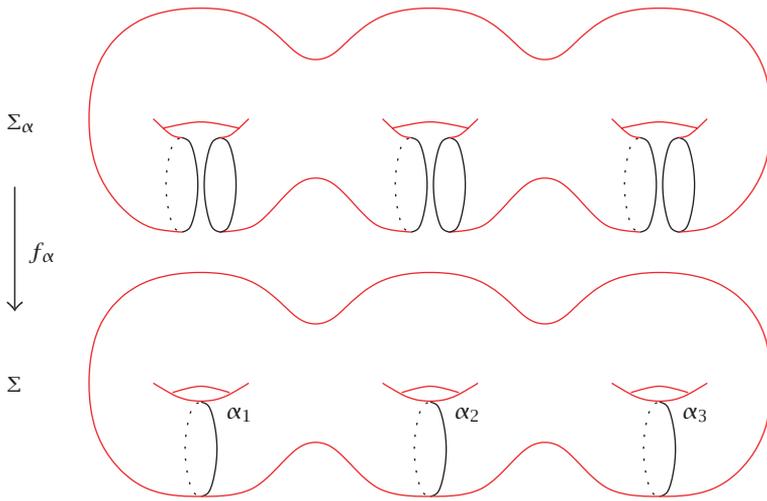


FIGURE A.1. Cutting  $\Sigma$  along  $\alpha$ .

**DEFINITION A.4.** Let  $\Sigma$  be a closed connected oriented 2-manifold and let  $\alpha \subset \Sigma$  be a compact 1-submanifold, that is,

$$\alpha = \alpha_1 \cup \dots \cup \alpha_n, \tag{A.3}$$

where  $\alpha_1, \dots, \alpha_n$  are disjoint embedded loops. (We do not assume here that  $n$  is the genus of  $\Sigma$ .) There are a compact oriented 2-manifold  $\Sigma_\alpha$  (with boundary) and a smooth map  $f_\alpha : \Sigma_\alpha \rightarrow \Sigma$  such that  $f_\alpha$  has an invertible derivative at every point, restricts to a diffeomorphism from the interior of  $\Sigma_\alpha$  to  $\Sigma \setminus \alpha$ , and restricts to a trivial orientation preserving double covering  $\partial\Sigma_\alpha \rightarrow \alpha$ . The manifold  $\Sigma_\alpha$  is unique in the sense that if  $f'_\alpha : \Sigma'_\alpha \rightarrow \Sigma$  is another such map, then there is a unique diffeomorphism  $\phi : \Sigma'_\alpha \rightarrow \Sigma_\alpha$  with  $f_\alpha \circ \phi = f'_\alpha$ . It is said that  $\Sigma_\alpha$  results by *cutting  $\Sigma$  along  $\alpha$*  (see Figure A.1).

**DEFINITION A.5.** Let  $(Y_0, \xi)$  be a handlebody structure with rest points  $p_0, \dots, p_g$  and let

$$A := \bigcup_{i=1}^g A_i, \quad A_i := W^s(p_i). \tag{A.4}$$

There is compact oriented 3-manifold  $Y_A$  with corners and a smooth map

$$F_A : Y_A \rightarrow Y_0 \tag{A.5}$$

such that  $F_A$  has an invertible derivative at every point, restricts to a diffeomorphism from  $Y_A \setminus F_A^{-1}(A)$  to  $Y \setminus A$ , and restricts to a trivial orientation preserving double covering from  $F_A^{-1}(A)$  to  $A$ . The manifold  $Y_A$  is unique in

the sense that if  $F'_A : Y'_A \rightarrow Y_0$  is another such map, then there is a unique diffeomorphism  $\Phi : Y'_A \rightarrow Y_A$  such that  $F_A \circ \Phi = F'_A$ . It is said that  $Y_A$  is the 3-manifold with corners that results by *cutting*  $Y_0$  *along*  $A$ . As a topological manifold,  $Y_A$  is homeomorphic to the 3-ball. As a smooth manifold,  $Y_A$  is diffeomorphic to a 3-ball with  $2g$  spherical caps sliced off. To prove this, cut out tubular neighborhood of the disks  $A_i$  to obtain a submanifold with corners  $Y' \subset Y_0 \setminus A$  that is diffeomorphic to  $Y_A$ . Choose a smooth submanifold with boundary  $Y'' \subset Y_0 \setminus A$  that contains  $Y'$ . The orbits of  $\xi$  define a diffeomorphism from the 3-ball centered at  $p_0$  to  $Y''$ . The preimage of  $Y'$  under this diffeomorphism is the required 3-ball with the caps sliced off. The vector field  $\xi$  on  $Y_0$  pulls back under  $F_A$  to a vector field  $\xi_A$  on  $Y_A$  which is tangent to the  $2g$  disks that form the preimage of  $A$  and otherwise points in on the boundary. It has a critical point of index one on each of these disks and a unique critical point of index zero in the interior.

**DEFINITION A.6.** Let  $(\Sigma, \alpha)$  be as in [Definition A.4](#) and assume that  $n = g$ , that is, the number of components of  $\alpha$  is the genus of  $\Sigma$ . Another embedded 1-submanifold  $\beta$  is said to be *dual* to  $\alpha$  if it also has  $g$  components, say

$$\beta = \beta_1 \cup \dots \cup \beta_g, \tag{A.6}$$

where  $\beta_1, \dots, \beta_g$  are disjoint embedded loops, and (for a suitable choice of orientations)

$$\alpha_i \cdot \beta_j = \delta_{ij} \tag{A.7}$$

for all  $i$  and  $j$ . It follows that the homology classes of  $\alpha_1, \dots, \beta_g$  form an integral basis of  $H_1(\Sigma; \mathbb{Z})$ . To see this, express  $\alpha_1, \dots, \beta_g$  in terms of a symplectic integral basis of  $H_1(\Sigma; \mathbb{Z})$ . Since

$$\alpha_i \cdot \beta_j = \delta_{ij}, \quad \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0 \tag{A.8}$$

for all  $i$  and  $j$ , the matrix of coefficients is symplectic, and hence, unimodular.

**THEOREM A.7.** *Let  $(\Sigma, \alpha)$  be as in [Definition A.4](#) and assume  $n = g$ . Then the following are equivalent:*

- (i) *there exist a handlebody  $Y_0$  and a diffeomorphism  $\iota : \Sigma \rightarrow \partial Y_0$  such that  $\iota(\alpha)$  is a trace of  $Y_0$ ;*
- (ii) *the manifold  $\Sigma_\alpha$  has genus zero;*
- (iii) *the open set  $\Sigma \setminus \alpha$  is connected;*
- (iv) *the homology classes of  $\alpha_1, \dots, \alpha_g$  are linearly independent in  $H_1(\Sigma; \mathbb{Q})$ ;*
- (v) *the homology classes of  $\alpha_1, \dots, \alpha_g$  extend to a free basis of  $H_1(\Sigma; \mathbb{Z})$ ;*
- (vi) *there exists a 1-manifold  $\beta$  dual to  $\alpha$ .*

*If these equivalent conditions are satisfied, then  $\alpha$  is called a trace in  $\Sigma$ .*

**PROOF.** The pattern of proof is (ii) $\Rightarrow$ (vi) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). Let  $f_\alpha : \Sigma_\alpha \rightarrow \Sigma$  be as in [Definition A.4](#) and write

$$\partial\Sigma_\alpha = \alpha'_1 \cup \cdots \cup \alpha'_g, \quad f_\alpha(\alpha'_i) = f_\alpha(\alpha''_i) = \alpha_i. \tag{A.9}$$

We prove that (ii) implies (vi). Since  $\Sigma_\alpha$  has genus zero, it embeds in a 2-sphere, that is,

$$\Sigma_\alpha = S^2 \setminus \bigcup_{i=1}^g (C'_i \cup C''_i), \quad \alpha'_i = \partial\bar{C}'_i, \quad \alpha''_i = \partial\bar{C}''_i, \tag{A.10}$$

where  $\bar{C}'_i$  and  $\bar{C}''_i$  are embedded closed disks with interiors  $C'_i$  and  $C''_i$ , respectively. Connect  $\alpha'_j$  to  $\alpha''_j$  with an arc  $b_j \subset \Sigma_\alpha$ ; do this in such a way that the  $b_j$  are disjoint,  $b_j$  intersects  $\partial\Sigma_\alpha$  only in the endpoints,  $f_\alpha$  maps the two endpoints of  $b_j$  to the same point in  $\Sigma$ , and, for  $j = 1, \dots, g$ , the image  $\beta_j := f_\alpha(b_j)$  is a smooth submanifold of  $\Sigma$  transverse to  $\alpha_j$ . Then  $\beta = \beta_1 \cup \cdots \cup \beta_g$  is dual to  $\alpha$  as required.

We prove that (vi) implies (v) implies (iv). Let  $\beta = \beta_1 \cup \cdots \cup \beta_g$  be dual to  $\alpha$ . As in [Definition A.6](#), the homology classes of  $\alpha_1, \dots, \beta_g$  form an integral basis of  $H_1(\Sigma; \mathbb{Z})$ . This proves (v). That (v) implies (iv) is trivial.

We prove that (iv) implies (iii). Assume that (iii) fails. Let  $C$  be the closure of a connected component of  $\Sigma \setminus \alpha$ . Then  $C \neq \Sigma$ . Hence, the boundary of  $C$  is homologous to zero and gives rise to a nontrivial relation among the homology classes of the  $\alpha_i$ . Hence (iv) fails.

We prove that (iii) implies (ii). Assume that  $\Sigma \setminus \alpha$  is connected. Then  $\Sigma_\alpha$  is also connected. Each identification  $f(\alpha'_i) = f(\alpha''_i)$  contributes one to the genus, so  $\Sigma_\alpha$  must have genus zero. Also note that the fact that (ii) implies (iii) is obvious.

We prove that (ii) implies (i) implies (iii). To prove that (ii) implies (i), reverse the construction of [Definition A.5](#). Now assume (i) and let  $\xi$  be a handlebody structure on  $Y_0$  with trace  $\iota(\alpha)$ . Choose points  $x, y \in \Sigma \setminus \alpha$ . The forward orbits of  $\iota(x)$  and  $\iota(y)$  get close to  $p_0$ , and hence, may be connected by an arc in  $Y_0$  which, by transversality, misses  $\bigcup_{i=1}^g W^u(p_i)$ . Now let this arc flow backwards out of  $Y_0$ . The exit points trace out an arc in  $\partial Y_0 \setminus \iota(\alpha)$  connecting  $\iota(x)$  to  $\iota(y)$ . □

**PROOF OF THEOREM A.3.** The existence of  $\phi$  follows from item (ii) in [Theorem A.7](#). Namely, let  $\Sigma := \partial Y_0$  and  $\tilde{\Sigma} := \partial \tilde{Y}_0$ , and choose a diffeomorphism  $\Sigma_\alpha \rightarrow \tilde{\Sigma}_\alpha$  which maps pairs of equivalent boundary circles to pairs of equivalent boundary circles. Then isotope so that the diffeomorphism descends to the quotient. Given  $\phi$ , extend it to a diffeomorphism  $U \rightarrow \tilde{U}$ , where  $U$  is a neighborhood of  $\partial Y_0 \cup A$ ,  $\tilde{U}$  is a neighborhood of  $\partial \tilde{Y}_0 \cup \tilde{A}$ ,  $A = \bigcup_{i=1}^g W^s(p_i) \subset Y_0$ , and  $\tilde{A} = \bigcup_{i=1}^g W^s(\tilde{p}_i) \subset \tilde{Y}_0$ . The argument in [Definition A.5](#) shows that these neighborhoods can be chosen such that the complements  $Y_0 \setminus U$  and  $\tilde{Y}_0 \setminus \tilde{U}$  are smooth submanifolds with boundary, each diffeomorphic to the 3-ball. Since

the group of orientation-preserving diffeomorphisms of the 2-sphere is connected,  $\phi$  extends to a diffeomorphism  $\psi_0 : Y_0 \rightarrow \tilde{Y}_0$  as required.  $\square$

**DEFINITION A.8.** Let  $\Sigma$  be a closed oriented 2-manifold. Two traces  $\alpha, \beta \subset \Sigma$  are called *equivalent* if there exist a handlebody  $Y_0$  and a diffeomorphism  $\iota : \Sigma \rightarrow \partial Y_0$  such that both  $\iota(\alpha)$  and  $\iota(\beta)$  are traces of  $Y_0$ . By [Theorem A.3](#), two traces  $\alpha, \beta \subset \Sigma$  are equivalent if and only if, for every handlebody  $Y_0$  and every diffeomorphism  $\iota : \Sigma \rightarrow \partial Y_0$ , it is true that  $\iota(\alpha)$  is a trace of  $Y_0$  if and only if  $\iota(\beta)$  is a trace of  $Y_0$ . Hence, equivalence of traces is an equivalence relation.

**REMARK A.9.** Equivalent traces generate the same Lagrangian subspace of  $H_1(\Sigma; \mathbb{Z})$ , namely, the kernel of the map  $\iota_* : H_1(\Sigma; \mathbb{Z}) \rightarrow H_1(Y_0; \mathbb{Z})$ .

**REMARK A.10.** Isotopic traces are equivalent ([Lemma 5.1](#)). To prove this, use the isotopy to modify the flow on a collar neighborhood of the boundary.

**REMARK A.11.** Let  $\Sigma$  be a closed connected oriented 2-manifold, let  $Y_0$  be a handlebody, and let  $\iota : \Sigma \rightarrow Y_0$  be a diffeomorphism. Let  $\text{Diff}(\Sigma, \iota) \subset \text{Diff}(\Sigma)$  denote the subgroup of all diffeomorphisms  $\phi : \Sigma \rightarrow \Sigma$  that extend to  $Y_0$  in the sense that there exists a diffeomorphism  $\psi_0 : Y_0 \rightarrow Y_0$  such that

$$\psi_0 \circ \iota = \iota \circ \phi. \tag{A.11}$$

Let  $\alpha \subset \Sigma$  be a trace such that  $\iota(\alpha)$  is a trace of  $Y_0$ . Then, by [Theorem A.3](#), a trace  $\beta \subset \Sigma$  is equivalent to  $\alpha$  if and only if there exists a diffeomorphism  $\phi \in \text{Diff}(\Sigma, \iota)$  such that  $\phi(\alpha) = \beta$ .

**EXAMPLE A.12.** A trace on a surface of genus one is a noncontractible embedded loop. Two such loops are equivalent as traces if and only if they are isotopic. For an example of two nonisotopic, but equivalent, traces on a surface of genus two, see [Example D.1](#).

An HMS structure  $(Y_0, Y_1, \xi)$  on a closed connected oriented 3-manifold  $Y$  determines two handlebody structures  $(Y_0, \xi|_{Y_0})$  and  $(Y_1, -\xi|_{Y_1})$ . Recall that the trace of  $(Y_0, Y_1, \xi)$  is the pair of 1-submanifolds  $\alpha, \beta \subset Y_0 \cap Y_1$  where  $\alpha$  is the trace of  $(Y_0, \xi|_{Y_0})$  and  $\beta$  is the trace of  $(Y_1, -\xi|_{Y_1})$ . The operation

$$(Y, Y_0, Y_1, \xi) \mapsto (Y_0 \cap Y_1, \alpha, \beta) \tag{A.12}$$

is bijective in the sense of the following two propositions.

**PROPOSITION A.13.** *Let  $(\alpha, \beta)$  be a transverse pair of traces in a closed connected oriented 2-manifold  $\Sigma$ . Then there is an HMS structure  $(Y_0, Y_1, \xi)$  on a closed connected oriented 3-manifold  $Y$  and a diffeomorphism  $\iota : \Sigma \rightarrow Y_0 \cap Y_1$  such that  $\iota(\alpha)$  is the trace of  $(Y_0, \xi|_{Y_0})$  and  $\iota(\beta)$  is the trace of  $(Y_1, -\xi|_{Y_1})$ .*

**PROOF.** By definition of trace, there exist handlebody structures  $(Y_0, \xi_0)$  and  $(Y_1, \xi_1)$  and diffeomorphisms  $\iota_0 : \Sigma \rightarrow \partial Y_0$  and  $\iota_1 : \Sigma \rightarrow \partial Y_1$  such that  $\iota_0(\alpha)$

is the trace of  $(Y_0, \xi_0)$  and  $\iota_1(\beta)$  is the trace of  $(Y_1, \xi_1)$ . Suppose, without loss of generality, that  $\iota_0$  is orientation preserving and  $\iota_1$  is orientation reversing. Then the flow of  $\xi_0$  and the embedding  $\iota_0$  define an orientation-preserving diffeomorphism from  $(-\varepsilon, 0] \times \Sigma$  to a tubular neighborhood  $U_0 \subset Y_0$  of the boundary. Likewise, the flow of  $\xi_1$  and the embedding  $\iota_1$  define an orientation-preserving diffeomorphism from  $[0, \varepsilon) \times \Sigma$  to a tubular neighborhood  $U_1 \subset Y_1$  of the boundary. There is a unique manifold structure on the union

$$Y := Y_0 \cup_{\Sigma} Y_1 \tag{A.13}$$

such that the map  $(-\varepsilon, \varepsilon) \times \Sigma \rightarrow U_0 \cup_{\Sigma} U_1$  is a diffeomorphism and the inclusions of  $Y_0$  and of  $Y_1$  are embeddings.  $\square$

**PROPOSITION A.14.** *Let  $\Sigma$  be a closed connected oriented 2-manifold and suppose that  $(Y, Y_0, Y_1, \xi, \iota, \alpha, \beta)$  and  $(\tilde{Y}, \tilde{Y}_0, \tilde{Y}_1, \tilde{\xi}, \tilde{\iota}, \tilde{\alpha}, \tilde{\beta})$  are as in the statement of Proposition A.13. Then the following are equivalent:*

- (i)  $\alpha$  is equivalent to  $\tilde{\alpha}$ , and  $\beta$  is equivalent to  $\tilde{\beta}$ ;
- (ii) there exists a diffeomorphism  $\psi : Y \rightarrow \tilde{Y}$  such that  $\psi \circ \iota = \tilde{\iota}$ .

**PROOF.** If (ii) holds, then the pullback  $\psi^*(\tilde{Y}_0, \tilde{Y}_1, \tilde{\xi})$  is an HMS structure on  $Y$  with traces  $\iota(\tilde{\alpha})$  and  $\iota(\tilde{\beta})$ . Hence (i) holds. Conversely, assume (i). Since  $\alpha$  is equivalent to  $\tilde{\alpha}$ , we have that  $\iota(\tilde{\alpha})$  is a trace of  $Y_0$ . Hence, the diffeomorphism  $\phi := \tilde{\iota} \circ \iota^{-1} : \partial Y_0 \rightarrow \partial \tilde{Y}_0$  maps a trace of  $Y_0$  to a trace of  $\tilde{Y}_0$ . Hence, by Theorem A.3, there exists a diffeomorphism  $\psi_0 : Y_0 \rightarrow \tilde{Y}_0$  such that  $\psi_0 \circ \iota = \tilde{\iota}$ . The same applies for  $Y_1$ , and this proves the proposition.  $\square$

**B. Diffeomorphisms of the two sphere.** Let  $\text{Diff}_+(S^2)$  denote the group of orientation-preserving diffeomorphisms of the 2-sphere and let  $\text{PSL}_2(\mathbb{C})$  denote the subgroup of fractional linear transformations.

**THEOREM B.1** (Smale). *The subgroup  $\text{PSL}_2(\mathbb{C})$  is a deformation retract of  $\text{Diff}_+(S^2)$ .*

**PROOF.** Our proof is inspired by [3] but uses a different PDE. Let  $\omega \in \Omega^2(S^2)$  be the standard volume form and denote by  $\mathcal{F}_+(S^2)$  the space of complex structures on  $S^2$  that are compatible with  $\omega$ . We prove that there is a fibration

$$\begin{array}{ccc} \text{PSL}_2(\mathbb{C}) & \longrightarrow & \text{Diff}_+(S^2) \\ & & \downarrow \\ & & \mathcal{F}_+(S^2). \end{array} \tag{B.1}$$

The projection  $\text{Diff}_+(S^2) \rightarrow \mathcal{F}_+(S^2)$  is given by  $\psi \mapsto \psi^* J_0$ , where  $J_0 \in \mathcal{F}_+(S^2)$  denotes the standard complex structure. We prove that this projection is in fact a fibration, that is, it has the path-lifting property. Let  $[0, 1] \rightarrow \mathcal{F}_+(S^2) : t \mapsto J_t$

be a smooth path in  $\mathcal{F}_+(S^2)$ . We must prove that there is an isotopy  $t \mapsto \psi_t$  of  $S^2$  such that

$$\psi_t^* J_t = J_0. \tag{B.2}$$

Suppose that the isotopy  $\psi_t$  is generated by a smooth family of vector fields  $X_t \in \text{Vect}(S^2)$  via

$$\frac{d}{dt} \psi_t = X_t \circ \psi_t, \quad \psi_0 = \text{id}. \tag{B.3}$$

Then (B.2) is equivalent to

$$\mathcal{L}_{X_t} J_t + \dot{J}_t = 0, \tag{B.4}$$

where  $\dot{J}_t := (d/dt)J_t \in C^\infty(\text{End}(TS^2))$ . Since  $J_t^2 = -\text{id}$ , we have

$$\dot{J}_t J_t + J_t \dot{J}_t = 0. \tag{B.5}$$

This means that  $\dot{J}_t : TS^2 \rightarrow TS^2$  is complex antilinear with respect to  $J_t$ . Hence, we can think of  $\dot{J}_t$  as a  $(0, 1)$ -form on  $S^2$  with values in the complex line bundle

$$E_t := (TS^2, J_t). \tag{B.6}$$

The vector field  $X_t$  is a section of this line bundle. Let

$$\bar{\partial}_{J_t} : C^\infty(E_t) \rightarrow \Omega^{0,1}(E_t) \tag{B.7}$$

denote the Cauchy-Riemann operator associated to the metric  $\omega(\cdot, J_t \cdot)$  on  $S^2$  and the Levi-Civita connection of this metric on  $E_t$ . Thus

$$\bar{\partial}_{J_t} X = \frac{1}{2} (\nabla X + J_t \circ \nabla X \circ J_t). \tag{B.8}$$

Now, for every vector field  $Y \in \text{Vect}(S^2)$ , we have

$$\begin{aligned} (\mathcal{L}_{X_t} J_t) Y &= \mathcal{L}_{X_t} (J_t Y) - J_t \mathcal{L}_{X_t} Y \\ &= [J_t Y, X_t] - J_t [Y, X_t] \\ &= \nabla_{X_t} (J_t Y) - \nabla_{J_t Y} X_t - J_t \nabla_{X_t} Y + J_t \nabla_Y X_t \\ &= J_t \nabla_Y X_t - \nabla_{J_t Y} X_t \\ &= 2J_t (\bar{\partial}_{J_t} X_t)(Y). \end{aligned} \tag{B.9}$$

The penultimate equality uses the fact that  $J_t$  is integrable and so  $\nabla J_t = 0$ . Hence, (B.4) can be expressed in the form

$$\bar{\partial}_{J_t} X_t = -\frac{1}{2} J_t \dot{J}_t. \tag{B.10}$$

Now the line bundle  $E_t$  has Chern number  $c_1(E_t) = 2$ , and hence, by the Riemann-Roch theorem, the Cauchy-Riemann operator  $\bar{\partial}_{J_t}$  has real Fredholm index six and is surjective for every  $t$ . Denote by

$$\bar{\partial}_{J_t}^* : \Omega^{0,1}(E_t) \longrightarrow C^\infty(E_t) \tag{B.11}$$

the formal  $L^2$ -adjoint operator of  $\bar{\partial}_{J_t}$ . By elliptic regularity, the formula

$$X_t := -\frac{1}{2} \bar{\partial}_{J_t}^* (\bar{\partial}_{J_t} \bar{\partial}_{J_t}^*)^{-1} (J_t \dot{J}_t) \tag{B.12}$$

defines a smooth family of vector fields on  $S^2$ , and this family obviously satisfies (B.10). Hence, the isotopy  $\psi_t$  generated by  $X_t$  satisfies (B.2).

Thus we have proven that the projection  $\text{Diff}_+(S^2) \rightarrow \mathcal{F}_+(S^2)$  is a fibration and, in particular, is surjective. Since the space  $\mathcal{F}^+(S^2)$  is contractible (it is the space of sections of a bundle over  $S^2$  with contractible fibres), it follows from the homotopy exact sequence for a fibration that  $\text{Diff}_+(S^2)$  is homotopy equivalent to  $\text{PSL}(2, \mathbb{C})$ .  $\square$

**COROLLARY B.2.** *The group  $\text{Diff}_+(S^2)$  is connected.*

We emphasize that our proof of [Theorem B.1](#) uses the integrability of almost complex structures in dimension two, the Riemann-Roch theorem, and elliptic regularity.

**PROOF OF THEOREM 2.3 (II)  $\Rightarrow$  (I).** Choose an HMS structure  $(Y, Y_0, Y_1, \xi)$  so that  $\Sigma := Y_0 \cap Y_1$  is a 2-sphere. Then  $\xi$  is a Morse-Smale vector field on  $Y$  with exactly two critical points,  $p_0$  of index zero and  $q_0$  of index three, in particular,

$$W^s(p_0, \xi) = Y \setminus \{q_0\}, \quad W^u(q_0, \xi) = Y \setminus \{p_0\}. \tag{B.13}$$

Let  $\phi$  denote the flow of  $\xi$ . After modifying  $\xi$  near  $p_0$  and  $q_0$ , we may assume that there are diffeomorphisms  $u : \mathbb{R}^3 \rightarrow Y \setminus \{q_0\}$  and  $v : \mathbb{R}^3 \rightarrow Y \setminus \{p_0\}$  so that

$$u(e^{-t}x) = \phi^t(u(x)), \quad v(e^t y) = \phi^t(v(y)). \tag{B.14}$$

After a further modification of  $\xi$  away from  $p_0$  and  $q_0$ , we may assume that  $u(S^2) = v(S^2)$ . It follows that

$$|u^{-1}(v(x))| = |x|^{-1} \tag{B.15}$$

for  $x \in \mathbb{R}^3 \setminus \{0\}$ . We may assume that  $u^{-1} \circ v|_{S^2}$  is orientation preserving. (If not, replace  $v$  by  $v$  composed with a reflection.) As  $\text{Diff}_+(S^2)$  is connected (see [Corollary B.2](#)), there is a diffeomorphism  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that

$$|f(x)| = |x|, \tag{B.16}$$

$$f(x) = \begin{cases} x, & \text{for } |x| \leq 1, \\ |x|^2 u^{-1}(v(x)), & \text{for } |x| \geq 2. \end{cases}$$

Define  $g : \mathbb{R}^3 \rightarrow Y$  by

$$g(x) := u(f(x)). \tag{B.17}$$

Let  $y \in \mathbb{R}^3$  with  $|y| \leq 1/2$  and denote  $T := -\ln|y|^2$  so that  $e^T = |y|^{-2}$ . Then

$$g(|y|^{-2}y) = u(e^T u^{-1}(v(e^T y))) = \phi^{-T}(u(u^{-1}(v(e^T y)))) = v(y). \tag{B.18}$$

This shows that  $g \circ \sigma$  extends to a diffeomorphism  $S^3 \rightarrow Y$ , where  $\sigma : S^2 \setminus \{(0,0,0,1)\} \rightarrow \mathbb{R}^3$  is the stereographic projection.  $\square$

**C. Proof of the cancellation lemma.** Before giving the proof, we give some preliminary definitions and lemmas. Let  $(P, \leq)$  be a finite poset. An ordered pair  $(p, q) \in P \times P$  is called *adjacent* if  $p \leq q$ ,  $p \neq q$ , and

$$p \leq r \leq q \implies r \in \{p, q\}. \tag{C.1}$$

Fix an adjacent pair  $(\bar{p}, \bar{q}) \in P \times P$  and consider the relation  $\leq'$  on  $P' = P \setminus \{\bar{p}, \bar{q}\}$  defined by

$$p \leq' q \iff p \leq q, \text{ or } \bar{p} \leq q, p \leq \bar{q}. \tag{C.2}$$

**LEMMA C.1.**  $(P', \leq')$  is a poset.

**PROOF.** We prove that the relation  $\leq'$  is transitive. Let  $p, q, r \in P'$  such that  $p \leq' q$  and  $q \leq' r$ . There are four cases. If  $p \leq q$  and  $q \leq r$ , then  $p \leq r$ , and hence,  $p \leq' r$ . The second case is  $p \not\leq q$  and  $q \leq r$ . In this case,  $\bar{p} \leq q \leq r$  and  $p \leq \bar{q}$ , and hence,  $p \leq' r$ . The third case is  $p \leq q$  and  $q \not\leq r$ , and the argument is as in the second case. The fourth case is  $p \not\leq q$  and  $q \not\leq r$ . In this case, it follows that  $p \leq \bar{q}$  and  $\bar{p} \leq r$ , and hence,  $p \leq' r$ .

Next we prove that the relation  $\leq'$  is antisymmetric. Hence, assume that  $p, q \in P'$  such that  $p \leq' q$  and  $q \leq' p$ . We claim that  $p \leq q$  and  $q \leq p$ . Assume otherwise that  $p \not\leq q$ . Then  $\bar{p} \leq q$  and  $p \leq \bar{q}$ . Since  $q \leq' p$ , it follows that  $\bar{p} \leq p \leq \bar{q}$  and  $\bar{p} \leq q \leq \bar{q}$ , and hence,  $\{p, q\} \subset \{\bar{p}, \bar{q}\}$ , a contradiction. Thus we have shown that  $p \leq q$ . Similarly,  $q \leq p$ , and hence,  $p = q$ .  $\square$

Two Morse-Floer vector fields are called *MF-equivalent* if there exists a diffeomorphism  $\psi : M \rightarrow M$  such that

$$P_k(\xi') = P_k(\psi^* \xi) \tag{C.3}$$

for  $k = 0, \dots, m$  and

$$\psi(p) \leq_{\xi'} \psi(q) \iff p \leq_{\xi} q, \quad n(\psi(q), \psi(p); \xi') = n(q, p; \xi) \tag{C.4}$$

for all  $p, q \in P(\xi)$ . Morse-Floer vector fields are stable in the sense that equivalence classes are open in the  $C^1$ -topology. Moreover, Morse-Floer vector fields

are stable under certain  $C^0$ -perturbations as we explain next. Let  $\xi$  be a Morse-Floer vector field on  $M$  and  $r \in P(\xi)$ . A compact set  $U \subset M$  is called  $\xi$ -unvisited if no  $\xi$ -orbit exits  $U$  and then returns to  $U$ . A neighborhood  $U_r$  of  $r \in P(\xi)$  is called  $\xi$ -admissible if and only if it is  $\xi$ -unvisited and satisfies the following conditions:

- (i) if  $r \not\leq_\xi q$ , then  $\overline{W^u}(q) \cap U_r = \emptyset$ ;
- (ii) if  $p \not\leq_\xi r$ , then  $\overline{W^s}(p) \cap U_r = \emptyset$ ;
- (iii) if  $p, q \in P(\xi) \setminus \{r\}$  such that  $p \leq_\xi q$ , then there is a transverse connecting orbit from  $q$  to  $p$  that misses  $U_r$ .

Call a vector field  $\xi'$  on  $M$  an *admissible perturbation of  $\xi$*  (supported near  $r \in P(\xi)$ ) if and only if it satisfies the following conditions:

- (iv)  $\xi = \xi'$  outside of some  $\xi$ -admissible neighborhood  $U_r$  of  $r$ ;
- (v)  $U_r \cap P(\xi) = U_r \cap P(\xi') = \{r\}$ ,  $r$  is a hyperbolic rest point of  $\xi'$ , and

$$\begin{aligned} W^u(r; \xi') \cap U_r &= W^u(r; \xi) \cap U_r, \\ W^s(r; \xi') \cap U_r &= W^s(r; \xi) \cap U_r; \end{aligned} \tag{C.5}$$

- (vi) every  $\xi'$ -orbit that stays in  $U_r$  in positive time lies in  $W^s(r; \xi')$ , and every  $\xi'$ -orbit that stays in  $U_r$  in negative time lies in  $W^u(r; \xi')$ .

**LEMMA C.2.** *Let  $\xi$  be a Morse-Floer vector field. Then every admissible perturbation of  $\xi$  is a Morse-Floer vector field and is MF-equivalent to  $\xi$ .*

**PROOF.** Let  $\xi'$  be a vector field on  $M$  that satisfies (iv), (v), and (vi). From (vi) and the unvisitedness of  $U_r$ , we conclude that

$$M = \bigcup_{p \in P(\xi)} W^s(p; \xi') = \bigcup_{p \in P(\xi)} W^u(p; \xi'). \tag{C.6}$$

We prove the assertion in three Steps.

**STEP 1.** For all  $p, q \in P(\xi)$ ,

$$W^u(q; \xi) \cap W^s(p; \xi) = \emptyset \implies W^u(q; \xi') \cap W^s(p; \xi') = \emptyset. \tag{C.7}$$

To see this, note that if  $W^u(q; \xi) \cap W^s(p; \xi) = \emptyset$ , then  $p \not\leq_\xi q$ , and hence, either  $r \not\leq_\xi q$  or  $p \not\leq_\xi r$ . Assume, without loss of generality, that  $r \not\leq_\xi q$ . Write  $P(\xi)$  as a disjoint union of a lower set  $Q$  containing  $q$  and an upper set  $R$  containing  $p$ :

$$Q := \{q' \in P(\xi) \mid q' \leq_\xi q\}, \quad R := P(\xi) \setminus Q. \tag{C.8}$$

Then the set

$$A = \bigcup_{q' \in Q} W^u(q'; \xi) \tag{C.9}$$

is an attractor for  $\xi$  and, in particular, is a compact subset of  $M$ . By the assumption that  $r \not\leq_\xi q$ , we have that  $r \in R$ . Hence,  $r \not\leq_\xi q'$  for every  $q' \in Q$ , and

hence, by (i),

$$U_r \cap A = \emptyset. \tag{C.10}$$

Now  $A$  is a  $\xi$ -attractor, and  $\xi$  and  $\xi'$  agree near  $A$ , so  $A$  is a  $\xi'$ -attractor. Since  $p \notin A$  and  $q \in A$ , it follows that there is no  $\xi'$ -orbit connecting  $q$  to  $p$ . Hence,  $W^u(q; \xi') \cap W^s(p; \xi') = \emptyset$  as claimed. This proves **Step 1**. It follows from **Step 1** and (C.6) that  $\xi'$  is a Morse-Floer vector field.

**STEP 2.** For all  $p, q \in P(\xi)$ ,  $p \preceq_\xi q$  if and only if  $p \preceq_{\xi'} q$ .

It follows from **Step 1** that  $p \preceq_{\xi'} q$  implies that  $p \preceq_\xi q$ . The converse follows immediately from condition (iii) on  $U_r$ .

**STEP 3.** For all  $p, q \in P(\xi)$ ,  $n(q, p; \xi') = n(q, p; \xi)$ .

Suppose that  $q$  and  $p$  have index difference one (otherwise, the assertion is obvious). Assume first that  $q, p \in P(\xi) \setminus \{r\}$ . Then either  $p \not\prec_\xi r$  or  $r \not\prec_\xi q$ , and, by (i) and (ii),

$$W^s(p; \xi) \cap U_r = \emptyset \quad \text{or} \quad W^u(q; \xi) \cap U_r = \emptyset. \tag{C.11}$$

Hence, no  $\xi$ -orbit from  $q$  to  $p$  passes through  $U_r$ ; hence, the  $\xi$ -orbits from  $q$  to  $p$  survive as  $\xi'$ -orbits; and hence  $n(q, p; \xi) \leq n(q, p; \xi')$ . Suppose, by contradiction, that  $n(q, p; \xi) < n(q, p; \xi')$ . Then there exists a  $\xi'$ -orbit from  $q$  to  $p$  that passes through  $U_r$ . Hence,

$$W^u(q; \xi) \cap U_r \neq \emptyset, \quad W^s(p; \xi) \cap U_r \neq \emptyset, \tag{C.12}$$

contradicting (C.11). This proves **Step 3** in the case  $p, q \in P(\xi) \setminus \{r\}$ . Now it follows from (iv) and (v) that  $W^s(r; \xi') = W^s(r; \xi)$  and  $W^u(r; \xi') = W^u(r; \xi)$ . Hence,  $n(q, r; \xi') = n(q, r; \xi)$  and  $n(r, p; \xi') = n(r, p; \xi)$  for all  $p, q \in P(\xi)$ . This proves the lemma.  $\square$

**PROPOSITION C.3** (normal form). *Let  $\xi$  be a Morse-Floer vector field,  $\bar{p} \in P_k(\xi)$ , and  $\bar{q} \in P_{k+1}(\xi)$ . Let  $\Gamma$  denote the closure of a connecting orbit from  $\bar{q}$  to  $\bar{p}$ . Then, for every neighborhood  $U$  of  $\Gamma$ , there exist a compact neighborhood  $N \subset U$  of  $\Gamma$ , a diffeomorphism*

$$f : D^k \times D^{m-k-1} \times [-1, 2] \rightarrow N, \tag{C.13}$$

*a Morse-Floer vector field  $\tilde{\xi}$  on  $M$ , and a smooth function  $w : [-1, 2] \rightarrow \mathbb{R}$  such that  $f^*\tilde{\xi}$  has the form*

$$f^*\tilde{\xi}(x, y, z) = (x, -y, w(z)), \tag{C.14}$$

*$\tilde{\xi}$  agrees with  $\xi$  outside of  $U$ ,  $\tilde{\xi}$  is MF-equivalent to  $\xi$ , and*

$$w^{-1}(0) = \{0, 1\}, \quad w'(0) = -1, \quad w'(1) = 1. \tag{C.15}$$

**PROOF.** The proof consists of five steps.

**STEP 1.** There are an admissible perturbation  $\xi'$  of  $\xi$  supported near  $\bar{p}$ , and coordinates  $x_1 \in \mathbb{R}^k$ ,  $y_1 \in \mathbb{R}^{m-k-1}$ , and  $z_1 \in \mathbb{R}$  near  $\bar{p}$  such that  $\xi'$  is given by the equations  $\dot{x}_1 = x_1$ ,  $\dot{y}_1 = -y_1$ , and  $\dot{z}_1 = -z_1$ . Moreover, the connecting orbit  $\Gamma'$  is defined by  $x_1 = 0$ ,  $y_1 = 0$ , and  $z_1 \geq 0$ , and the unstable manifold  $W^u(\bar{q}; \xi')$  is defined by  $y_1 = 0$  and  $z_1 > 0$ .

Let  $B^u$  be a small ball in the unstable subspace  $T_{\bar{p}}W^u(\bar{p})$  and let  $B^s$  be a small ball in the stable subspace  $T_{\bar{p}}W^s(\bar{p})$ . Use the exponential map to identify the product  $B^u \times B^s$  with a neighborhood of  $\bar{p}$ . We may assume that the balls  $B^u$  and  $B^s$  and the exponential map have been chosen such that  $B^u \times \{0\}$  is a subset of  $W^u(\bar{p}; \xi)$ ,  $\{0\} \times B^s$  is a subset of  $W^s(\bar{p}; \xi)$ ,  $\xi$  points in on  $B^u \times \partial B^s$ , and  $\xi$  points out on  $\partial B^u \times B^s$ . Let  $\zeta$  be a product vector field on  $B^u \times B^s$  which is the radial vector field on the first factor and the negative of the radial vector field on the second. Consider the vector field  $\xi' := \beta\xi + (1 - \beta)\zeta$ , where  $\beta : B^u \times B^s \rightarrow [0, 1]$  is a cutoff function which is zero near  $\bar{p}$  and identically one near the boundary of  $B^u \times B^s$ . If  $\beta^{-1}((0, 1])$  is contained in a sufficiently small neighborhood of the boundary of  $B^u \times B^s$ , then  $\xi'$  satisfies the requirements of [Lemma C.2](#). In any linear coordinates  $x$  in  $B^u$  and  $(y, z)$  in  $B^s$ , the vector field  $\xi'$  has the required form. Choose these coordinates such that  $\Gamma$  has the required form. By transversality and invariance under the flow, the unstable manifold  $W^u(\bar{q}; \xi')$  has an equation of the form

$$y = zg(zx). \tag{C.16}$$

Make the further change of variables

$$(x_1, y_1, z_1) = (x, y - zg(zx), z) \tag{C.17}$$

to achieve the required equation for  $W^u(\bar{q}; \xi')$ .

**STEP 2.** There are an admissible perturbation  $\xi''$  of  $\xi'$  supported near  $\bar{q}$ , and coordinates  $x_2 \in \mathbb{R}^k$ ,  $y_2 \in \mathbb{R}^{m-k-1}$ , and  $z_2 \in \mathbb{R}$  near  $\bar{q}$  such that  $\xi''$  is given by the equations  $\dot{x}_2 = x_2$ ,  $\dot{y}_2 = -y_2$ , and  $\dot{z}_2 = z_2$ . Moreover, the connecting orbit  $\Gamma''$  is defined by  $x_2 = 0$ ,  $y_2 = 0$ , and  $z_2 \leq 0$ , and the stable manifold  $W^s(\bar{p}; \xi)$  is defined by  $x_2 = 0$  and  $z_2 < 0$ .

The proof is the same as for [Step 1](#). Henceforth, we drop the primes and assume that  $\xi$  satisfies the conclusions of [Steps 1](#) and [2](#).

Let  $L \subset M$  be the smooth (noncompact) one-dimensional submanifold determined by the conditions that it contains  $\Gamma$  in its interior and  $L \setminus \{p, q\}$  consists of three orbits of  $\phi$ . Thus  $L$  intersects each of the coordinate systems of [Steps 1](#) and [2](#) in the  $z$ -axis. Choose a diffeomorphism  $\ell : \mathbb{R} \rightarrow L$  such that  $\ell(0) = \bar{p}$ ,  $\ell(1) = \bar{q}$ , and the pullback vector field

$$w(z) := \ell^*\xi(z) \tag{C.18}$$

satisfies the following strengthened form of (C.15):

$$w(z) = -z \quad \text{for } z \approx 0, \quad w(z) = z - 1 \quad \text{for } z \approx 1. \tag{C.19}$$

**STEP 3.** The restriction  $T_L M$  of the tangent bundle  $TM$  of  $M$  to the curve  $L$  admits a smooth direct sum decomposition

$$T_L M = E^u \oplus E^s \oplus TL \tag{C.20}$$

which is invariant in the sense that

$$d\phi^t(z)E_z^u = E_{\phi^t(z)}^u, \quad d\phi^t(z)E_z^s = E_{\phi^t(z)}^s, \quad d\phi^t(z)T_z L = T_{\phi^t(z)} L \tag{C.21}$$

for  $z \in L$ , and satisfies

$$\begin{aligned} T_z W^u(q) &= E_z^u \oplus T_z L \quad \text{for } z \in W^u(q) \cap L, \\ T_z W^s(q) &= E_z^s \oplus T_z L \quad \text{for } z \in W^s(q) \cap L, \\ T_p W^u(p) &= E_p^u, \quad T_q W^s(q) = E_q^s. \end{aligned} \tag{C.22}$$

To construct  $E^u$ , choose  $E_z^u$  to agree with the  $x_2$ -subspace for  $z$  near  $\bar{q}$  in the coordinates of Step 2. Extend by invariance. Then, by transversality,  $E_z^u$  has the form

$$E_z^u = \text{graph}(\Lambda(z)) \times \{0\}, \quad \Lambda(z) = z_1^2 \Lambda_0 \tag{C.23}$$

for  $z = (0, 0, z_1) \in \Gamma$  near  $\bar{p}$  where  $\Lambda_0 : \mathbb{R}^k \rightarrow \mathbb{R}^{m-k-1}$  is linear. Extend to  $L \setminus \Gamma$  using the same formula (and invariance). The construction of  $E^s$  is similar.

**STEP 4.** There exists a diffeomorphism  $f : B^k \times B^{m-k-1} \times [-1, 2] \rightarrow N$ , where  $B^n$  denotes the closed unit ball in  $\mathbb{R}^n$  and  $N \subset U$  is a neighborhood of  $\Gamma$  in  $M$ , such that  $f^* \xi'$  has the form

$$\xi(x, y, z) = (\hat{a}(z)x, \hat{b}(z)y, w(z)) + O(\|x\|^2 + \|y\|^2), \tag{C.24}$$

where  $\hat{a}(z) \in \mathbb{R}^{k \times k}$  and  $\hat{b}(z) \in \mathbb{R}^{(m-k-1) \times (m-k-1)}$  satisfy

$$\hat{a}(z) = \text{id}, \quad \hat{b}(z) = -\text{id}, \tag{C.25}$$

for  $z$  near 0 and 1.

Choose a Riemannian metric on  $T_L M$  which agrees with the standard metric in the  $(x_1, y_1, z_1)$  coordinates near  $\bar{p}$  and agrees with the standard metric in the  $(x_2, y_2, z_2)$  coordinates near  $\bar{q}$ . The coordinate systems of Steps 1 and 2 determine trivializations of  $E^u \oplus E^s$  near  $\bar{p}$  and  $\bar{q}$ ; extend to a vector bundle trivialization

$$\mathbb{R}^k \times \mathbb{R}^{m-k-1} \times \mathbb{R} \rightarrow E^u \oplus E^s \tag{C.26}$$

that covers the diffeomorphism  $\ell : \mathbb{R} \rightarrow L$ . (It may be necessary to reverse the sign of one component of  $x_1$  and/or of one component of  $y_1$  to match orientations.) Compose with the exponential map to obtain a tubular neighborhood

$$\mathbb{R}^k \times \mathbb{R}^{m-k-1} \times \mathbb{R} \rightarrow M \tag{C.27}$$

of  $L$ . This gives coordinates  $(x, y, z)$  on a neighborhood of  $\Gamma$ . We use the same letters  $\phi$  and  $\xi$  to represent the flow and vector field in these coordinates. Thus  $\bar{p} = (0, 0, 0)$ ,  $\bar{q} = (0, 0, 1)$ , and  $\Gamma$  is the set of points  $(0, 0, z)$  where  $0 \leq z \leq 1$ . Since  $L = \{0\} \times \{0\} \times \mathbb{R}$  is invariant by  $\phi$ , the restriction has the form

$$\phi^t(0, 0, z) = (0, 0, \psi^t(z)). \tag{C.28}$$

By invariance of the splitting,  $d\phi^t(0, 0, z)$  has the form

$$d\phi^t(0, 0, z) = a^t(z) \oplus b^t(z) \oplus c^t(z), \tag{C.29}$$

where  $a^t(z) \in \text{GL}_k(\mathbb{R})$ ,  $b^t(z) \in \text{GL}_{m-k-1}(\mathbb{R})$ , and  $c^t(z) > 0$ . Differentiate (C.28) and (C.29) to deduce that the vector field  $\xi$  satisfies

$$\xi(0, 0, z) = (0, 0, w(z)), \quad d\xi(0, 0, z) = \hat{a}(z) \oplus \hat{b}(z) \oplus w'(z), \tag{C.30}$$

where

$$\hat{a}(z) = \left. \frac{\partial}{\partial t} a^t(z) \right|_{t=0}, \quad \hat{b}(z) = \left. \frac{\partial}{\partial t} b^t(z) \right|_{t=0}. \tag{C.31}$$

The construction of  $E^u$  and  $E^s$  shows that  $\hat{a}$  and  $\hat{b}$  satisfy (C.25). Use Taylor’s formula in  $(x, y)$  to obtain (C.24). Rescale  $(x, y)$  so that the coordinates are defined for  $\|x\|, \|y\| \leq 1$  and  $-1 \leq z \leq 2$ .

**STEP 5.** We prove Proposition C.3.

Construct a  $C^1$ -perturbation  $\tilde{\xi}$  of  $\xi$  near  $\Gamma$  using a cutoff function to eliminate the higher-order terms in (C.24). Then  $\tilde{\xi}$  is a Morse-Floer vector field with  $P(\tilde{\xi}) = P(\xi)$ ,  $\tilde{\xi}$  is MF-equivalent to  $\xi$ , and  $f^*\tilde{\xi}$  has the form

$$f^*\tilde{\xi}(x, y, z) = (\hat{a}(z)x, \hat{b}(z)y, w(z)) \tag{C.32}$$

in some neighborhood of  $\Gamma$ . Consider the coordinate change

$$(x, y, z) = g(\tilde{x}, \tilde{y}, z) := (\Phi(z)\tilde{x}, \Psi(z)\tilde{y}, z), \tag{C.33}$$

where

$$\begin{aligned} w(z)\partial_z\Phi(z) &= \Phi(z)(\text{id} - \hat{a}(z)), & \Phi(0) &= \text{id}, \\ w(z)\partial_z\Psi(z) &= \Psi(z)(\text{id} + \hat{b}(z)), & \Psi(0) &= \text{id}. \end{aligned} \tag{C.34}$$

By (C.25), we have  $\partial_z\Phi(z) = 0$  and  $\partial_z\Psi(z) = 0$  for  $z$  near 0 and 1. It follows that

$$g^*f^*\tilde{\xi}(x, y, z) = (x, -y, w(z)). \tag{C.35}$$

Now read  $f \circ g$  for  $f$ , rescale in  $(x, y)$ , and restrict the domain as required. This proves Proposition C.3.  $\square$

**PROOF OF THE CANCELLATION LEMMA (THEOREM 4.1).** Choose a finite set  $S$  of  $\xi$ -orbits which contains all the orbits between pairs of index difference one and also at least one orbit of transverse intersection of  $W^u(p, \xi) \cap E^s(q, \xi)$  for any pair of rest points  $p, q \in P(\xi) \setminus \{\tilde{p}, \tilde{q}\}$  with  $p \preceq_\xi q$ . Let  $U_{\tilde{p}}$  be a  $\xi$ -admissible neighborhood of  $\tilde{p}$  and let  $U_{\tilde{q}}$  be a  $\xi$ -admissible neighborhood of  $\tilde{q}$ . Suppose, without loss of generality, that the neighborhood  $U$  of  $\Gamma$  is so small that

$$U \cap S = \emptyset, \tag{C.36}$$

every  $\xi$ -orbit that enters  $U$  must first pass through  $U_{\tilde{p}} \cup U_{\tilde{q}}$ , and every  $\xi$ -orbit that leaves  $U$  passes afterwards through  $U_{\tilde{p}} \cup U_{\tilde{q}}$ . Thus, for every  $\xi$ -orbit  $\gamma : \mathbb{R}^\pm \rightarrow M$ ,

$$\gamma(\mathbb{R}^\pm) \cap U = \{\gamma(0)\} \implies \gamma(\mathbb{R}^\pm) \cap (U_{\tilde{p}} \cup U_{\tilde{q}}) \neq \emptyset, \tag{C.37}$$

where  $\mathbb{R}^+ := [0, \infty)$  and  $\mathbb{R}^- := (-\infty, 0]$ . By Proposition C.3, we may assume, without loss of generality, that  $\xi$  is in normal form near  $\Gamma$ , that is, there exist  $N \subset U$ ,  $f$ , and  $w$  such that the conclusions of Proposition C.3 hold with  $\tilde{\xi}$  replaced by  $\xi$ . Define the vector field  $\eta$  by  $\eta = \xi$  on  $M \setminus N$  and

$$f^*\eta(x, y, z) = (x, -y, v(x, y, z)), \tag{C.38}$$

where

$$v(x, y, z) = \beta(r)w(z) + (1 - \beta(r))(\rho(z)w(z) + (1 - \rho(z))\varepsilon). \tag{C.39}$$

Here  $r = \sqrt{x^2 + y^2}$ ,  $\beta : [0, 1] \rightarrow [0, 1]$  satisfies

$$\beta(r) = \begin{cases} 0, & \text{if } r \leq \frac{1}{3}, \\ 1, & \text{if } r \geq \frac{2}{3}, \end{cases} \tag{C.40}$$

and  $\rho : [-1, 2] \rightarrow \mathbb{R}$  is chosen such that

$$\rho(z) = \begin{cases} 1, & \text{if } z \leq -2\varepsilon, \\ 0, & \text{if } -\varepsilon \leq z \leq 1 + \varepsilon, \\ 1, & \text{if } z \geq 1 + 2\varepsilon. \end{cases} \tag{C.41}$$

By construction, the vector field  $\eta$  has only hyperbolic rest points,

$$P(\eta) = P(\xi) \setminus \{\bar{p}, \bar{q}\}, \quad (\text{C.42})$$

it agrees with  $\xi$  outside  $N$  (and hence, with  $\xi$  outside  $U$ ), and

$$M = \bigcup_{p \in P(\eta)} W^s(p; \eta) = \bigcup_{p \in P(\eta)} W^u(p; \eta). \quad (\text{C.43})$$

We must show that, for  $p, q \in P(\eta)$ , we have

- (a)  $p \preceq_{\xi} q \Rightarrow W^s(p; \eta) \cap W^u(q; \eta) \neq \emptyset$ ;
- (b)  $p \preceq_{\xi} \bar{q}$  and  $\bar{p} \preceq_{\xi} q \Rightarrow W^s(p; \eta) \cap W^u(q; \eta) \neq \emptyset$ ;
- (c)  $W^s(p; \eta) \cap W^u(q; \eta) \neq \emptyset$ ,  $p \not\preceq_{\xi} q \Rightarrow p \preceq_{\xi} \bar{q}$ , and  $\bar{p} \preceq_{\xi} q$ .

By [Lemma C.1](#), the right-hand side of formula (4.3) defines a partial order on  $P(\eta)$  whenever  $\bar{p}$  and  $\bar{q}$  are an adjacent pair in  $P(\xi)$ . Hence, it follows from (c) and (C.43) that  $\eta$  is gradient-like and that the Smale order  $\preceq_{\eta}$  is given by (4.3).

Assume that  $p \preceq_{\xi} q$ . Then there is a  $\xi$ -orbit from the set  $S$  which runs from  $q$  to  $p$ . By (C.36), the set  $U$  misses this orbit and  $\eta - \xi$  is supported in  $U$ ; hence, this orbit is an  $\eta$ -orbit. Hence,  $W^s(p; \eta) \cap W^u(q; \eta) \neq \emptyset$ . This proves (a). Next assume that  $W^s(p; \eta) \cap W^u(q; \eta) \neq \emptyset$  and  $p \not\preceq_{\xi} q$ . Then there exists an  $\eta$ -orbit from  $q$  to  $p$  that passes through  $U$ . By (C.37), this orbit must pass through  $U_{\bar{p}} \cup U_{\bar{q}}$  before entering  $U$  and must pass again through  $U_{\bar{p}} \cup U_{\bar{q}}$  after leaving  $U$ . Since  $U_{\bar{p}}$  and  $U_{\bar{q}}$  are  $\xi$ -admissible, it follows that there is a  $\xi$ -orbit from  $q$  to either  $\bar{p}$  or  $\bar{q}$  and another  $\xi$ -orbit from either  $\bar{p}$  or  $\bar{q}$  to  $p$ . Since  $p \not\preceq_{\xi} q$ , it follows that  $p \preceq_{\xi} \bar{q}$  and  $\bar{p} \preceq_{\xi} q$ , as claimed. This proves (c). Assertion (b) follows from a gluing argument. Namely, if there exists a  $\xi$ -orbit from  $q$  to  $\bar{p}$ , then  $W^u(q; \xi)$  intersects  $N$  in a slice along the  $x$ -plane near  $z = 1$ , provided that  $\varepsilon > 0$  is chosen sufficiently small. Likewise, if there exists a  $\xi$ -orbit from  $\bar{q}$  to  $p$ , then  $W^s(p; \xi)$  intersects  $N$  in a slice along the  $y$ -plane near  $z = 0$ . The orbits of  $\eta$  connect these two transverse slices. Moreover, the resulting  $\eta$ -orbit from  $q$  to  $p$  is transverse. The same argument shows, in the case where  $\bar{q}$  and  $q$  have the same index and  $\bar{p}$  and  $p$  have the same index, that every pair of connecting orbits from  $q$  to  $\bar{p}$  and from  $\bar{q}$  to  $p$  gives rise to a transverse  $\eta$ -orbit from  $q$  to  $p$ . Hence,  $\eta$  is a Morse-Floer vector field that satisfies (4.4).  $\square$

#### D. An example

**EXAMPLE D.1.** François Laudenbach and Denis Auroux showed us the following example of an algebraically reduced HMS structure on  $S^3$  which is not geometrically reduced. Let  $\Sigma = \partial Y_0 = \partial Y_1$  have genus two and let the embedded loops  $\alpha_1, \alpha_2, \beta_1$ , and  $\beta_2$  form a standard basis of  $H_1(\Sigma)$ . The embedded loop  $\gamma \subset \Sigma$  is homologous to zero in  $\Sigma$  and contractible in both handlebodies  $Y_0$  and  $Y_1$  (see [Figure C.1](#)). Hence, the Dehn twist  $\phi : \Sigma \rightarrow \Sigma$  along  $\gamma$  extends to a diffeomorphism of  $Y_1$ , and hence, by [Remark A.11](#), the trace  $\beta' := \phi(\beta)$  is equivalent to  $\beta$ . Hence, by [Proposition A.14](#), the pair  $(\alpha, \beta')$  is a trace of the

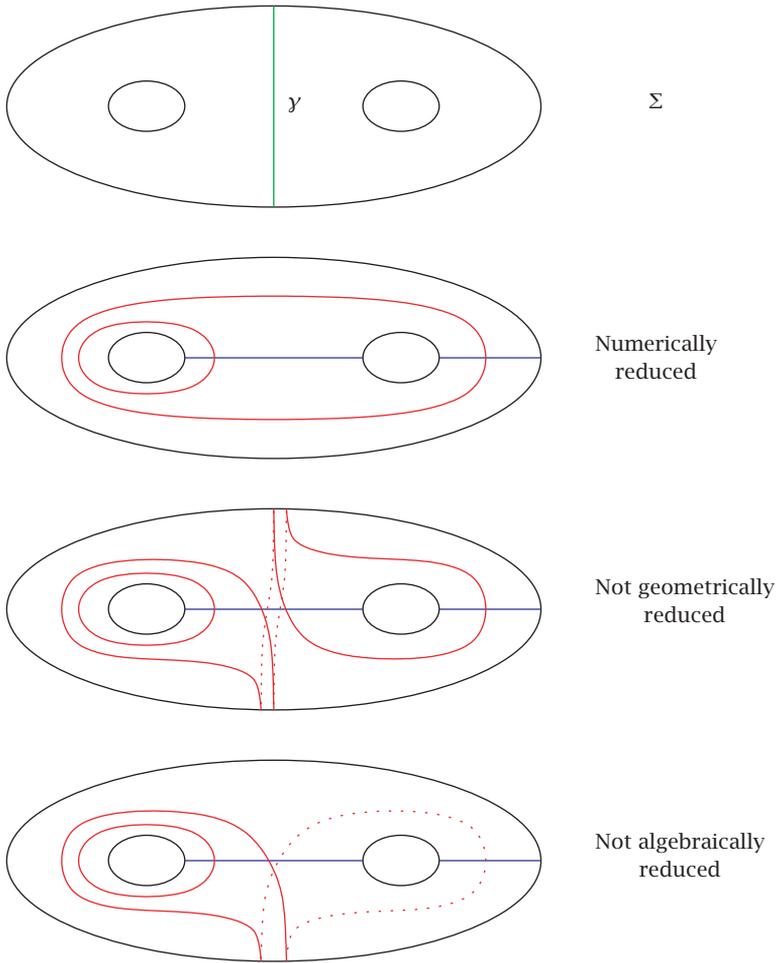


FIGURE C.1. Three HMS structures.

same Heegaard splitting of  $S^3$ . It is algebraically reduced, but not geometrically reduced. Replacing  $\phi$  by a diffeomorphism which rotates  $\Sigma$  by a half turn on one side of  $\gamma$  (i.e., a square root of  $\phi$ ), we obtain a trace  $(\alpha, \beta'')$  of the same Heegaard splitting which is not algebraically reduced.

**ACKNOWLEDGMENT.** Ralf Gautschi was partially supported by the SNF project number 21-64937.01.

**REFERENCES**

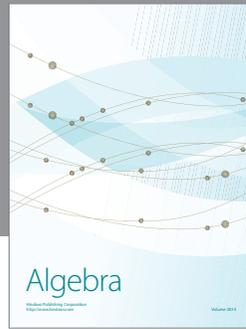
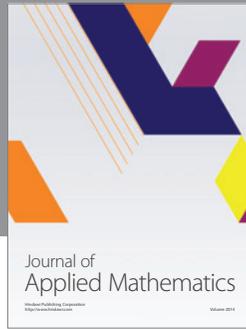
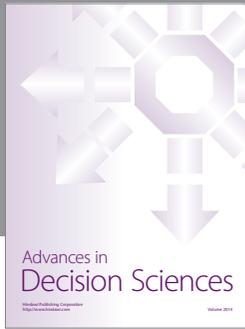
[1] Vin de Silva, *Products in the symplectic Floer homology of Lagrangian intersections*, Ph.D. thesis, Oxford University, 1998.

- [2] Vin de Silva, R. Gautschi, J. Robbin, and D. A. Salamon, *Combinatorial Floer homology*, ETH-Zürich, preprint, 2001.
- [3] C. J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geometry **3** (1969), 19–43.
- [4] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83–107.
- [5] A. Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28** (1988), no. 3, 513–547.
- [6] ———, *Witten's complex and infinite-dimensional Morse theory*, J. Differential Geom. **30** (1989), no. 1, 207–221.
- [7] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory— anomaly and obstruction*, preprint, 2000.
- [8] J. Hass and P. Scott, *Intersections of curves on surfaces*, Israel J. Math. **51** (1985), no. 1-2, 90–120.
- [9] W.-C. Hsiang, *A speculation on Floer theory and Nielsen theory*, Handwritten Notes, Princeton University, New Jersey, 1999.
- [10] J. Milnor, *Morse Theory*, Annals of Mathematics Studies, no. 51, Princeton University Press, New Jersey, 1963.
- [11] ———, *Lectures on the h-Cobordism Theorem*, Notes by L. Siebenmann and J. Sondow, Princeton University Press, New Jersey, 1965.
- [12] J. Palis Jr. and W. de Melo, *Geometric Theory of Dynamical Systems, An Introduction*, Springer-Verlag, New York, 1982.
- [13] V. Póenaru, *Extensions des immersions en codimension 1 (d'après Samuel Blank)*, Sémin. Bourbaki, Exposé 342, 1967/1968 (French).
- [14] G. Reeb, *Sur Certaines Propriétés Topologiques des Variétés Feuilletées*, Actualités Sci. Ind., no. 1183, Hermann & Cie., Paris, 1952 (French), 91–154.
- [15] J. W. Robbin and D. A. Salamon, *Asymptotic behaviour of holomorphic strips*, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), no. 5, 573–612.
- [16] D. A. Salamon, *Morse theory, the Conley index and Floer homology*, Bull. London Math. Soc. **22** (1990), no. 2, 113–140.
- [17] M. Schwarz, *Morse Homology*, Progress in Mathematics, vol. 111, Birkhäuser Verlag, Basel, 1993.
- [18] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961), 391–406.
- [19] ———, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817, reprinted in *The Mathematics of Time*, Springer Verlag, 1980.
- [20] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), no. 4, 661–692.

Ralf Gautschi: Navigation Support Office, European Space Operations Center, Robert-Bosch Straße 5, D-64293 Darmstadt, Germany  
*E-mail address:* [ralf.gautschi@esa.int](mailto:ralf.gautschi@esa.int)

Joel W. Robbin: Mathematics Department, University of Wisconsin, Madison, WI 53706, USA  
*E-mail address:* [robbin@math.wisc.edu](mailto:robbin@math.wisc.edu)

Dietmar A. Salamon: Department of Mathematics, ETH Zürich, 8092 Zürich, Switzerland  
*E-mail address:* [salamon@math.ethz.ch](mailto:salamon@math.ethz.ch)



# Hindawi

Submit your manuscripts at  
<http://www.hindawi.com>

