SOME VERSIONS OF ANDERSON'S AND MAHER'S INEQUALITIES I

SALAH MECHERI

Received 11 January 2003

We prove the orthogonality (in the sense of Birkhoff) of the range and the kernel of an important class of elementary operators with respect to the Schatten p -class.

2000 Mathematics Subject Classification: 47A30, 47B47, 47B10, 47B20.

1. Introduction. Let H be a separable infinite-dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded operators on H into itself. Given $A, B \in B(H)$, we define the generalized derivation $\delta_{A,B} : B(H) \rightarrow B(H)$ by $\delta_{A,B}(X) = AX - XB$ and the elementary operator derivation $\Delta_{A,B} : B(H) \rightarrow B(H)$ by $\Delta_{A,B}(X) = AXB - X$. Denote $\delta_{A,A} = \delta_A$ and $\Delta_{A,A} = \Delta_A$.

In [1, Theorem 1.7], Anderson shows that if A is normal and commutes with T , then, for all $X \in B(H)$,

$$\|T + \delta_A(X)\| \geq \|T\|. \quad (1.1)$$

It is shown in [10] that if the pair (A, B) has the Fuglede-Putnam property (in particular, if A and B are normal operators) and $AT = TB$, then, for all $X \in B(H)$,

$$\|T + \delta_{A,B}(X)\| \geq \|T\|. \quad (1.2)$$

Duggal [4] showed that the above inequality (1.2) is also true when $\delta_{A,B}$ is replaced by $\Delta_{A,B}$. The related inequality (1.1) was obtained by the author [11] showing that if the pair (A, B) has the Fuglede-Putnam property $(FP)_{C_p}$, then

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (1.3)$$

for all $X \in B(H)$, where C_p is the von Neumann-Schatten class, $1 \leq p < \infty$, and $\|\cdot\|_p$ is its norm for all $X \in B(H)$ and for all $T \in C_p \cap \ker \delta_{A,B}$. In all of the above results, A was not arbitrary. In fact, certain normality-like assumptions have been imposed on A . A characterization of $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(\delta_A|_{C_p})$ (the range of $\delta_A|_{C_p}$) for a general operator A , has

been carried out by Kittaneh [7], showing that if T has the polar decomposition $T = U|T|$, then

$$\|T + \delta_A(X)\|_p \geq \|T\|_p \quad (1.4)$$

for all $X \in C_p$ ($1 < p < \infty$) if and only if $|T|^{p-1}U^* \in \ker \delta_A$. By a simple modification in the proof of the above inequality, we can prove that this inequality is also true in the general case, that is, if T has the polar decomposition $T = U|T|$, then $\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p$ for all $X \in C_p$ ($1 < p < \infty$) if and only if $|T|^{p-1}U^* \in \ker \delta_{B,A}$. In Sections 1, 2, 3, and 4, we prove these results in the case where we consider $E_{A,B}$ instead of $\delta_{A,B}$, which leads us to prove that if $T \in C_p$ and $\ker E_{A,B} \subseteq \ker E_{A,B}^*$, then

$$\|T + E_{A,B}(X)\|_p \geq \|T\|_p \quad (1.5)$$

for all $X \in C_p$ ($1 < p < \infty$) if and only if $T \in \ker E_{A,B}$. In Sections 5 and 6, we minimize the map $\|S + E_{A,B}(X)\|_p$ and we classify its critical points.

2. Preliminaries. Let $T \in B(H)$ be compact and let $s_1(X) \geq s_2(X) \geq \dots \geq 0$ denote the singular values of T , that is, the eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p -class C_p if

$$\|T\|_p = \left[\sum_{j=1}^{\infty} s_j(T)^p \right]^{1/p} = [\operatorname{tr}(T)^p]^{1/p}, \quad 1 \leq p < \infty, \quad (2.1)$$

where tr denotes the trace functional. Hence, C_1 is the trace class, C_2 is the Hilbert-Schmidt class, and C_∞ is the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\| \quad (2.2)$$

denoting the usual operator norm. For the general theory of the Schatten p -classes, the reader is referred to [8, 13].

Recall that the norm $\|\cdot\|$ of the B -space V is said to be Gateaux differentiable at nonzero elements $x \in V$ if

$$\lim_{t \rightarrow 0, t \in \mathbb{R}} \frac{\|x + t\mathcal{Y}\| - \|x\|}{t} = \Re D_x(\mathcal{Y}) \quad (2.3)$$

for all $\mathcal{Y} \in V$. Here $\mathbb{R}$ denotes the set of reals, $\Re$ denotes the real part, and D_x is the unique support functional (in the dual space V^*) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$. The Gateaux differentiability of the norm at x implies that x is a smooth point of the sphere of radius $\|x\|$.

It is well known (see [8] and the references therein) that, for $1 < p < \infty$, C_p is a uniformly convex Banach space. Therefore, every nonzero $T \in C_p$ is a smooth point and, in this case, the support functional of T is given by

$$D_T(X) = \operatorname{tr} \left[\frac{|T|^{p-1} U X^*}{\|T\|_p^{p-1}} \right] \quad (2.4)$$

for all $X \in C_p$, where $T = U|T|$ is the polar decomposition of T .

DEFINITION 2.1. Let E be a complex Banach space. We define the orthogonality in E . We say that $b \in E$ is orthogonal to $a \in E$ if, for all complex λ , there holds

$$\|a + \lambda b\| \geq \|a\|. \quad (2.5)$$

This definition has a natural geometric interpretation, namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, that is, if and only if this complex line is a tangent one. Note that if b is orthogonal to a , then a needs not be orthogonal to b . If E is a Hilbert space, then from (2.5), it follows that $\langle a, b \rangle = 0$, that is, orthogonality in the usual sense.

3. Main results. In this section, we characterize $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(\Delta_{A,B}|C_p)$ (the range of $\Delta_{A,B}|C_p$) for a general pair of operators A, B .

LEMMA 3.1 [7]. *Let u and v be two elements of a Banach space V with norm $\|\cdot\|$. If u is a smooth point, then $D_u(v) = 0$ if and only if*

$$\|u + zv\| \geq \|u\| \quad (3.1)$$

for all $z \in \mathbb{C}$ (the complex numbers).

THEOREM 3.2. *Let $A, B \in B(H)$ and $T \in C_p$ ($1 < p < \infty$). Then*

$$\|T + \Delta_{A,B}(X)\|_p \geq \|T\|_p \quad (3.2)$$

for all $X \in B(H)$ with $\Delta_{A,B}(X) \in C_p$ if and only if $\operatorname{tr}(|T|^{p-1} U^* \Delta_{A,B}(X)) = 0$ for all such X .

PROOF. The theorem is an immediate consequence of equality (2.4) and Lemma 3.1. $\square$

THEOREM 3.3. *Let $A, B \in B(H)$ and $T \in C_p$ ($1 < p < \infty$). Then*

$$\|T + \Delta_{A,B}(X)\|_p \geq \|T\|_p \quad (3.3)$$

for all $X \in C_p$ if and only if $\tilde{T} = |T|^{p-1} U^* \in \ker \Delta_{B,A}$.

PROOF. By virtue of [Theorem 3.2](#), it is sufficient to show that $\text{tr}(\tilde{T}\Delta_{A,B}(X)) = 0$ for all $X \in C_p$ if and only if $\tilde{T} \in \ker \Delta_{B,A}$.

Choose X to be the rank-one operator $f \otimes g$ for some arbitrary elements f and g in H ; then $\text{tr}(\tilde{T}(AXB - X)) = \text{tr}((B\tilde{T}A - \tilde{T})X) = 0$ implies that $\langle \Delta_{B,A}(\tilde{T})f, g \rangle = 0 \Leftrightarrow \tilde{T} \in \ker \Delta_{B,A}$. Conversely, assume that $\tilde{T} \in \ker \Delta_{B,A}$, that is, $B\tilde{T}A = \tilde{T}$.

Since $\tilde{T}X$ and $\tilde{T}\Delta_{B,A}$ are trace classes for all $X \in C_p$, we get

$$\begin{aligned} \text{tr}(\tilde{T}(AXB - X)) &= \text{tr}(\tilde{T}AXB - \tilde{T}X) \\ &= \text{tr}(XB\tilde{T}A - X\tilde{T}) \\ &= \text{tr}(X\Delta_{B,A}(\tilde{T})) = 0. \end{aligned} \quad (3.4) \quad \square$$

LEMMA 3.4. *Let $A, B \in B(H)$ and $S \in B(H)$ such that $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$. If $AU|S|^{p-1}B = U|S|^{p-1}$, where $p > 1$ and $S = U|S|$ is the polar decomposition of S , then $AU|S|B = U|S|$.*

PROOF. If $T = |S|^{p-1}$, then

$$AUTB = UT. \quad (3.5)$$

We prove that

$$AUT^nB = UT^n. \quad (3.6)$$

If $ATB = T = A^*TB^*$, then $BT^*T = BT^*ATB = T^*TB$, and thus $B|T| = |T|B$ and $BT^2 = T^2B$. Since B commutes with the positive operator T^2 , then B commutes with its square roots, that is,

$$BT = TB. \quad (3.7)$$

By (3.5) and (3.7) we obtain (3.6). Let $f(t)$ be the map defined on $\sigma(T) \subset \mathbb{R}^+$ by

$$f(t) = t^{1/(p-1)}, \quad 1 < p < \infty. \quad (3.8)$$

Since f is the uniform limit of a sequence (P_i) of polynomials without constant term (since $f(0) = 0$), it follows from (3.3) that $AUP_i(T)B = UP_i(T)$. Therefore, $AUT^{1/(p-1)}B = UT^{1/(p-1)}$. $\square$

THEOREM 3.5. *Let A and B be operators in $B(H)$ such that $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$. Then $T \in \ker \Delta_{A,B} \cap C_p$ if and only if*

$$\|S + \Delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.9)$$

for all $X \in C_p$.

PROOF. If $S \in \ker \Delta_{A,B}$, then, by applying [11, Theorem 3.4], it follows that

$$\|S + \Delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.10)$$

for all $X \in C_p$. Conversely, if

$$\|S + \Delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.11)$$

for all $X \in C_p$, then, from Theorem 3.3, $A|S|^{p-1}U^*B = |S|^{p-1}U^*$. Since $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$, $A^*|S|^{p-1}U^*B^* = |S|^{p-1}U^*$. By taking adjoints, we get $AU|S|^{p-1}B = U|S|^{p-1}$. From Lemma 3.4, it follows that $AU|S|B = U|S|$. That is, $S \in \ker \Delta_{A,B}$. $\square$

THEOREM 3.6. Let $A, B \in B(H)$. If

(1) $A, B \in \mathcal{L}(H)$ such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in \mathcal{H}$,

(2) A is invertible and B is such that $\|A^{-1}\| \|B\| \leq 1$,

(3) $A = B$ is a cyclic subnormal operator,

then, $T \in \ker \Delta_{A,B} \cap C_p$ if and only if

$$\|S + \Delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.12)$$

for all $X \in C_p$.

PROOF. The result of Tong [14, Lemma 1] guarantees that the above condition implies that for all $T \in \ker(\delta_{A,B}|\mathcal{H}(\mathcal{H}))$, $\overline{R(T)}$ reduces A , $\ker(T)^\perp$ reduces B , and $A|_{\overline{R(T)}}$ and $B|_{\ker(T)^\perp}$ are unitary operators. Take $\mathcal{H}_1 = \mathcal{H} = \overline{\text{ran } S} \oplus \overline{\text{ran } S}^\perp$ and $\mathcal{H}_2 = \mathcal{H} = \ker S \oplus \ker S^\perp$. According to the decomposition of $\mathcal{H}$ and for $A_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1$, $A_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_2$, and $S : \mathcal{H}_2 \rightarrow \mathcal{H}_1$, we can write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B^* = \begin{pmatrix} B_1^* & 0 \\ 0 & B_2^* \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.13)$$

From $ASB = S$, it follows that $A_1SB_1 = S$, and since A_1 and B_1 are unitary operators, we obtain $A_1^*SB_1^* = S$, and the result holds by the above theorem.

The above inequality holds in particular if $A = B$ is isometric; in other words, $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$.

(2) In this case, it suffices to take $A_1 = \|B\|^{-1}A$ and $B_1 = \|B\|^{-1}B$, then $\|A_1x\| \geq \|x\| \geq \|B_1x\|$, and the result holds by (1) for all $x \in \mathcal{H}$.

(3) Since T commutes with A , it follows that T is subnormal [15]. But any compact subnormal operator is normal; hence, T is normal. By applying Fuglede-Putnam theorem, we get that $ATA = T$ implies $A^*TA^* = T$. $\square$

4. The case where $n > 1$. Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ be n -tuples of operators in $B(H)$. In this section, we characterize $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(E_{A,B}|C_p)$ (the range of $E_{A,B}|C_p$) for a general pair of operators A and B .

By the same argument used in the proofs of Theorems 3.2 and 3.3, we prove the following theorems.

THEOREM 4.1. *Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ be n -tuples of operators in $B(H)$ and $T \in C_p$ ($1 < p < \infty$). Then*

$$\|T + E_{A,B}(X)\|_p \geq \|T\|_p \quad (4.1)$$

for all $X \in B(H)$ with $E_{A,B}(X) \in C_p$ if and only if $\text{tr}(|T|^{p-1}U^*E_{A,B}(X)) = 0$ for all such X .

THEOREM 4.2. *Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ be n -tuples of operators in $B(H)$ and $T \in C_p$ ($1 < p < \infty$). Then*

$$\|T + E_{A,B}(X)\|_p \geq \|T\|_p \quad (4.2)$$

for all $X \in C_p$ if and only if $\tilde{T} = |T|^{p-1}U^ \in \ker E_{A,B}$.*

LEMMA 4.3. *Let $C = (C_1, C_2, \dots, C_n)$ be n -tuple of operators in $B(H)$ such that $\sum_{i=1}^n C_i C_i^* \leq 1$, $\sum_{i=1}^n C_i^* C_i \leq 1$, and $\ker E_C \subseteq \ker E_{C^*}$. If*

$$\sum_{i=1}^n C_i U |S|^{p-1} C_i = U |S|^{p-1}, \quad (4.3)$$

where $p > 1$ and $S = U|S|$ is the polar decomposition of S , then

$$\sum_{i=1}^n C_i U |S| C_i = U |S|. \quad (4.4)$$

PROOF. If $T = |S|^{p-1}$, then

$$\sum_{i=1}^n C_i U T C_i = U T. \quad (4.5)$$

We prove that

$$\sum_{i=1}^n C_i U T^n C_i = U T^n. \quad (4.6)$$

It is known that if $\sum_{i=1}^n C_i C_i^* \leq 1$, $\sum_{i=1}^n C_i^* C_i \leq 1$, and $\ker E_C \subseteq \ker E_{C^*}$, then the eigenspaces corresponding to distinct nonzero eigenvalues of the compact positive operator $|S|^2$ reduce each C_i (see [3, Theorem 8], [14, Lemma 2.3]). In particular, $|S|$ commutes with C_i for all $1 \leq i \leq n$. This implies also that $|S|^{p-1} = T$ commutes with each C_i for all $1 \leq i \leq n$. Hence $C_i |T| = |T| C_i$ and $C_i T^2 = T^2 C_i$.

Since C_i commutes with the positive operator T^2 , then C_i commutes with its square roots, that is,

$$C_i T = T C_i. \quad (4.7)$$

By the same arguments used in the proof of [Lemma 3.4](#), the proof of this lemma can be completed. $\square$

THEOREM 4.4. *Let $C = (C_1, C_2, \dots, C_n)$ be n -tuple of operators in $B(H)$ such that $\sum_{i=1}^n C_i C_i^* \leq 1$, $\sum_{i=1}^n C_i^* C_i \leq 1$, and $\ker E_C \subseteq \ker E_{C^*}$. Then $S \in \ker E_C \cap C_p$ ($1 < p < \infty$) if and only if*

$$\|S + E_C(X)\|_p \geq \|S\|_p \quad (4.8)$$

for all $X \in C_p$.

PROOF. If $S \in \ker E_C$, then, from [14, Theorem 2.4], it follows that $\|S + E_C(X)\|_p \geq \|S\|_p$ for all $X \in C_p$. Conversely, if $\|S + E_C(X)\|_p \geq \|S\|_p$ for all $X \in C_p$, then, from [Theorem 4.2](#), $\sum_{i=1}^n C_i |S|^{p-1} U^* C_i = |S|^{p-1} U^*$. Since $\ker E_C \subseteq \ker E_{C^*}$, $\sum_{i=1}^n C_i^* |S|^{p-1} U^* C_i^* = |S|^{p-1} U^*$. Taking adjoints, we get $\sum_{i=1}^n C_i U \times |S|^{p-1} C_i = U |S|^{p-1}$, and from [Lemma 4.3](#), it follows that $\sum_{i=1}^n C_i U |S| C_i = U |S|$, that is, $S \in \ker E_C$. $\square$

THEOREM 4.5. *Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ be n -tuples of operators in $B(H)$ such that $\sum_{i=1}^n A_i A_i^* \leq 1$, $\sum_{i=1}^n A_i^* A_i \leq 1$, $\sum_{i=1}^n B_i B_i^* \leq 1$, $\sum_{i=1}^n B_i^* B_i \leq 1$, and $\ker E_{A,B} \subseteq \ker E_{A^*,B^*}$.*

Then $T \in \ker E_{A,B} \cap C_p$ if and only if

$$\|S + E_{A,B}(X)\|_p \geq \|S\|_p \quad (4.9)$$

for all $X \in C_p$.

PROOF. It suffices to take the Hilbert space $H \oplus H$ and the operators

$$C_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \quad S = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \quad (4.10)$$

and apply [Theorem 4.4](#). $\square$

5. Remarks. (1) It is known (see [8] and the references therein) that the smooth points of $K(H)$ are those compact operators that attain their norm at a unique (up to multiplication by a constant of modulus one) unit vector. It has been shown in [8] that a nonzero $T \in B(H)$ is a smooth point if and only if T attains its norm at a unique (up to multiplication by a constant of modulus one) unit vector $e \in H$ and $\|T\|_e \leq \|T\|$, where $\|T\|_e$ is the essential

norm of T , that is, the norm of $\pi(T)$, where π is the quotient map of $B(H)$ onto $B(H)/K(H)$. In this case,

$$D_T(X) = \operatorname{tr} \left[\frac{(e \otimes Te)}{\|T\|} X \right] = \left\langle Xe, \frac{Te}{\|T\|} \right\rangle \quad (5.1)$$

for all $X \in B(H)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on H and $e \otimes Te$ is the rank-one operators defined by $(e \otimes Te)f = \langle f, Te \rangle e$ for all $f \in H$.

Hence, for the usual operator norm, Theorems 3.2, 3.3, 4.1, and 4.2 can be combined in the following formulation. Let $A, B \in B(H)$ and $T \in B(H)$ be a smooth point. If $\tilde{T} = e \otimes Te$, then the following statements are equivalent:

- (i) $\|T + E_{A,B}(X)\| \geq \|T\|$ for all $X \in B(H)$,
- (ii) $\operatorname{tr}(T \tilde{T} E_{A,B}(X)) = 0$ for all $X \in B(H)$,
- (iii) $\tilde{T} \in \ker E_{A,B}$.

(2) It is still possible to give a characterization similar to this given in the usual operator norm for the norm $\|\cdot\|_\infty$. However, in this case, we have to assume that T is a smooth point, that is, the given norm is Gateaux differentiable at T and $\tilde{T} = e \otimes Te$, where e is the unique (up to multiplication by a constant of modulus one) unit vector at which T attains its norm.

(3) It is well known that the Hilbert-Schmidt class C_2 is a Hilbert space under the inner product $\langle Y, Z \rangle = \operatorname{tr} Z^* Y$.

We remark here that, for the Hilbert Schmidt norm $\|\cdot\|_2$, the orthogonality results in Theorems 3.3, 3.5, 4.1, and 4.2 are to be understood in the usual Hilbert-space sense. Note in the case $\tilde{T} = |T|U^* = T^*$ that

$$\|T + E_{A,B}(X)\|_2^2 = \|E_{A,B}(X)\|_2^2 + \|T\|_2^2 \quad (5.2)$$

for all $X \in C_2$ if and only if $T^* \in \ker E_{A,B}$.

(4) Theorem 4.4 does not hold in the case $0 < p \leq 1$ because the functional calculus argument involving the function $t \mapsto t^{1/(p-1)}$, where $0 \leq t < \infty$, is only valid for $1 < p < \infty$. We ask if there is another proof where this theorem still holds in the case $0 < p < 1$. For the case $p = 1$, this theorem still holds see [12, Theorem 2.3].

6. On minimizing $\|T - (AXB - X)\|_p^p$. Maher [9, Theorem 3.2] showed that, if A is normal, $AT = TA$, $1 \leq p < \infty$, and $S \in \ker \delta_{A,B} \cap C_p$; then the map F_p defined by $F_p(X) = \|S - (AX - XA)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AV - VA = 0$. In other words, we have

$$\|S - (AX - XA)\|_p^p \geq \|T\|_p^p \quad (6.1)$$

if, and for $1 < p < \infty$ only if, $AV - VA = 0$. In [10] we generalized Maher's result, showing that if the pair (A, B) has the property $(FP)_{C_p}$, that is, $(AT = TB, \text{ where } T \in C_p \text{ implies } A^*T = TB^*)$, $1 \leq p < \infty$ and $S \in \ker \delta_{A,B} \cap C_p$, then the map F_p

defined by $F_p(X) = \|S - (AX - XB)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In other words, we have

$$\|S - (AX - XB)\|_p^p \geq \|T\|_p^p \quad (6.2)$$

if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In this paper, we obtain an inequality similar to (6.1), where the operator $AX - XB$ is replaced by the operator $\Delta_{A,B}(X) = AXB - X$ (in the case $n = 1$). We prove that if $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$ and $T \in \ker \Delta_{A,B} \cap C_p$, then the map F_p defined by $F_p(X) = \|T - (AXB - X)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AVB - V = 0$. In other words, we have

$$\|T - (AXB - X)\|_p^p \geq \|T\|_p^p \quad (6.3)$$

if, and for $1 < p < \infty$ only if, $AVB - V = 0$. Additionally, we show that if $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$ and $T \in \ker \Delta_{A,B} \cap C_p$, $1 < p < \infty$, then the map F_p has a critical point at W if and only if $AWB - W = 0$, that is, if $\mathcal{D}_W F_p$ is the Frechet derivative at W of F_p , the set

$$\{W \in \mathcal{B}(H) : \mathcal{D}_W F_p = 0\} \quad (6.4)$$

coincides with $\ker \Delta_{A,B}$ (the kernel of $\Delta_{A,B}$).

THEOREM 6.1 [2]. *If $1 < p < \infty$, then the map*

$$F_p : C_p \mapsto \mathbb{R}^+ \quad (6.5)$$

defined by $X \mapsto \|X\|_p^p$ is differentiable at every $X \in C_p$ with derivative $\mathcal{D}_X F_p$ given by

$$\mathcal{D}_X F_p(T) = p \operatorname{Re} \operatorname{tr}(|X|^{p-1} U^* T), \quad (6.6)$$

where tr denotes trace, $\operatorname{Re} z$ is the real part of a complex number z , and $X = U|X|$ is the polar decomposition of X . If $\dim \mathcal{H} < \infty$, then the same result holds for $0 < p \leq 1$ at every invertible X .

THEOREM 6.2 [6]. *If $\mathcal{U}$ is a convex set of C_p with $1 < p < \infty$, then the map $X \mapsto \|X\|_p^p$, where $X \in \mathcal{U}$, has at most a global minimizer.*

DEFINITION 6.3. Let $\mathcal{U}(A,B) = \{X \in \mathcal{B}(H) : AXB - X \in C_p\}$ and let $F_p : \mathcal{U} \mapsto \mathbb{R}^+$ be the map defined by $F_p(X) = \|T - (AXB - X)\|_p^p$, where $T \in \ker \Delta_{A,B} \cap C_p$ ($1 \leq p < \infty$).

By a simple modification in the proof of Lemma 4.3, we can proof the following lemma.

LEMMA 6.4. *Let $A, B \in \mathcal{B}(H)$ and $S \in \mathcal{B}(H)$ such that $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$. If $A|S|^{p-1}U^*B = |S|^{p-1}U^*$, where $p > 1$ and $S = U|S|$ is the polar decomposition of S , then $A|S|U^*B = |S|U^*$.*

THEOREM 6.5. *Let $A, B \in B(H)$. If $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$ and $T \in \ker \Delta_{A,B} \cap C_p$, then, for $1 \leq p < \infty$, the map F_p has a global minimizer at W if, and for $1 < p < \infty$ only if, $AWB - W = 0$.*

PROOF. If $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$, then it follows from [Theorem 3.5](#) that $\|T - (AXB - X)\|_p^p \geq \|T\|_p^p$, that is, $F_p(X) \geq F_p(W)$. Conversely, if F_p has a minimum, then

$$\|T - (AWB - WB)\|_p^p = \|S\|_p^p. \quad (6.7)$$

Since $\mathcal{U}$ is convex, then the set $\mathcal{V} = \{T - (AXB - X); X \in \mathcal{U}\}$ is also convex. Thus [Theorem 6.2](#) implies that $S - (AWB - W) = S$. $\square$

THEOREM 6.6. *Let $A, B \in B(H)$. If $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$ and $S \in \ker \Delta_{A,B} \cap C_p$, then, for $1 < p < \infty$, the map F_p has a critical point at W if and only if $AWB - W = 0$.*

PROOF. Let $W, S \in \mathcal{U}$ and let ϕ and φ be two maps defined, respectively, by $\phi : X \mapsto S - (AXB - X)$ and $\varphi : X \mapsto \|X\|_p^p$.

Since the Frechet derivative of F_p is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \rightarrow 0} \frac{F_p(W + hT) - F_p(W)}{h}, \quad (6.8)$$

it follows that $\mathcal{D}_W F_p(T) = [\mathcal{D}_{S-(AWB-W)}](ATB - T)$. If W is a critical point of F_p , then $\mathcal{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying [Theorem 6.1](#), we get

$$\begin{aligned} \mathcal{D}_W F_p(T) &= p \operatorname{Retr} \left[|S - (AWB - W)|^{p-1} W^* (ATB - T) \right] \\ &= p \operatorname{Retr} [Y(ATB - T)] = 0, \end{aligned} \quad (6.9)$$

where $S - (AWB - W) = W|S - (AWB - W)|$ is the polar decomposition of the operator $S - (AWB - W)$, and $Y = |S - (AWB - W)|^{p-1} W^*$.

An easy calculation shows that $AYB - Y = 0$, that is,

$$A|S - (AWB - W)|^{p-1} W^* B = |S - (AWB - W)|^{p-1} W^*. \quad (6.10)$$

It follows from [Lemma 6.4](#) that

$$A|S - (AWB - W)|W^* B = |S - (AWB - W)|W^*. \quad (6.11)$$

By taking adjoints and since $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$, we get $A(T - (AWB - W))B = (T - (AWB - W))$. Then $A(AWB - W)B = (AWB - W)$.

Hence $AWB - W \in R(\Delta_{A,B}) \cap \ker \Delta_{A,B}$. It is easy to see that (arguing as in the proof of [Theorem 3.5](#)) if $A, B \in \mathcal{B}(H)$, $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$, and $T \in \ker \Delta_{A,B}$,

where $T \in \mathcal{B}(H)$, then

$$\|T - (AXB - X)\| \geq \|T\| \quad (6.12)$$

holds for all $X \in \mathcal{B}(H)$ and for all $T \in \ker \Delta_{A,B}$. Hence $AWB - W = 0$.

Conversely, if $AWB = W$, then W is a minimum, and since F_p is differentiable, then W is a critical point. $\square$

THEOREM 6.7. *Let $A, B \in \mathcal{B}(H)$ such that $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*, B^*}$, $S \in \ker \Delta_{A,B} \cap C_p$ ($0 < p \leq 1$), $\dim \mathcal{H} < \infty$, and $S - (AWB - W)$ is invertible. Then F_p has a critical point at W if $AWB - W = 0$.*

PROOF. Suppose that $\dim \mathcal{H} < \infty$. If $AWB - W = 0$, then S is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$. If we take

$$Y = |S|^{p-1} U^* \quad (6.13)$$

with $S = U|S|$ the polar decomposition and since $ASB = S$ implies $BS^*A = S^*$, then $AS^*S = AS^*BSA = S^*SA$, and this implies that $|S|^2A = A|S|^2$ and $|S|A = A|S|$.

Since $BS^*A = S^*$, that is, $A|S|U^*B = |S|U^*$, $|S|(AU^*B - U^*) = 0$, and since $A|S|^{p-1} = |S|^{p-1}A$, then

$$AYB - Y = A|S|^{p-1}U^*B - |S|^{p-1}U^* = |S|^{p-1}(AU^*B - U^*) \quad (6.14)$$

so that $AYB - Y = 0$ and $\text{tr}[(AYB - Y)T] = 0$ for all $T \in \mathcal{B}(H)$. Since $S = S - (AWB - W)$, then

$$\begin{aligned} 0 &= \text{tr}[YATB - YAT] = \text{tr}[Y(ATB - T)] \\ &= p \text{Retr}[Y(ATB - AT)] = p \text{Retr}[|S|^{p-1}U^*(ATB - T)] \\ &= (\mathcal{D}_T \phi)(ATB - T) = (\mathcal{D}_W F_p)(T). \end{aligned} \quad (6.15)$$

$\square$

REMARK 6.8. (1) In [Theorem 6.6](#), the implication “ W is a critical point $\Rightarrow AWB - WB = 0$ ” does not hold in the case $0 < p \leq 1$ because the functional calculus argument involving the function $t \mapsto t^{1/(p-1)}$, where $0 \leq t < \infty$, is only valid for $1 < p < \infty$.

(2) Theorems [3.5](#), [6.5](#), [6.6](#), and [6.7](#) hold in particular if A and B are contractions. Indeed, it is known [\[4\]](#) that if A and B are contractions and $\Delta_{A,B}(S) = 0$, where $S \in C_p$, then $\Delta_{A^*, B^*}(S) = \delta_{A^*, B}(S) = \delta_{A, B^*}(S) = 0$.

(3) The set

$$\mathcal{S} = \{X : AXB - X \in C_p\} \quad (6.16)$$

contains C_p for if $X \in C_p$, then $X \in \mathcal{S}$ and, for example, $I \in \mathcal{S}$ but $I \notin C_p$. If $A \in C_p$, the conclusions of Theorems [6.5](#), [6.6](#), and [6.7](#) hold for all $X \in \mathcal{B}(H)$.

7. On minimizing $\|T - (\sum_{i=1}^n A_i X B_i - X)\|_p^p$. Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ be n -tuples of operators in $B(H)$. We define the elementary operator $E_{A,B} : B(H) \rightarrow B(H)$ by $E_{A,B}(X) = \sum_{i=1}^n A_i X B_i - X$.

Denote $E_{A,A} = E_A$. In this section, we prove that if $\sum_{i=1}^n A_i A_i^* \leq 1$, $\sum_{i=1}^n A_i^* A_i \leq 1$, $\sum_{i=1}^n B_i B_i^* \leq 1$, $\sum_{i=1}^n B_i^* B_i \leq 1$, $\ker E_{A,B} \subseteq \ker E_{A^*,B^*}$, and $T \in \ker \Delta_{A,B} \cap C_p$, then the map F_p defined by $F_p(X) = \|T - E_{A,B}(X)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $\sum_{i=1}^n A_i V B_i - V = 0$. In other words, we have

$$\|T - E_{A,B}(X)\|_p^p \geq \|T\|_p^p \quad (7.1)$$

if, and for $1 < p < \infty$ only if, $\sum_{i=1}^n A_i V B_i - V = 0$. Additionally, we show that if $\ker E_{A,B} \subseteq E_{A^*,B^*}$ and $T \in \ker E_{A,B} \cap C_p$ ($1 < p < \infty$), then the map F_p has a critical point at W if and only if $\sum_{i=1}^n A_i W B_i - W = 0$, that is, if $D_W F_p$ is the Frechet derivative of F_p at W , the set

$$\{W \in L(H) : D_W F_p = 0\} \quad (7.2)$$

coincides with $\ker E_{A,B}$ (the kernel of $E_{A,B}$).

DEFINITION 7.1. Let $\mathcal{U}(A,B) = \{X \in B(H) : (\sum_{i=1}^n C_i X C_i - X) \in C_p\}$ and let $F_p : \mathcal{U} \rightarrow \mathbb{R}^+$ be the map defined by $F_p(X) = \|T - (\sum_{i=1}^n C_i X C_i - X)\|_p^p$, where $T \in \ker E_C \cap C_p$ ($1 \leq p < \infty$).

LEMMA 7.2. Let $C = (C_1, C_2, \dots, C_n)$ be n -tuple of operators in $B(H)$ such that $\sum_{i=1}^n C_i C_i^* \leq 1$, $\sum_{i=1}^n C_i^* C_i \leq 1$, and $\ker E_C \subseteq \ker E_C^*$. If $\sum_{i=1}^n C_i |S|^{p-1} U^* C_i = |S|^{p-1} U^*$, where $p > 1$ and $S = U|S|$ is the polar decomposition of S , then $\sum_{i=1}^n C_i |S| U^* C_i = |S| U^*$.

PROOF. By the same arguments as in the proof of [Lemma 4.3](#), the proof can be completed. $\square$

THEOREM 7.3. Let $C = (C_1, C_2, \dots, C_n)$ be n -tuple of operators in $B(H)$. If $\sum_{i=1}^n C_i C_i^* \leq 1$, $\sum_{i=1}^n C_i^* C_i \leq 1$, $\ker E_C \subseteq \ker E_C^*$, and $T \in \ker \Delta_{A,B} \cap C_p$, then, for $1 \leq p < \infty$, the map F_p has a global minimizer at W if, and for $1 < p < \infty$ only if, $\sum_{i=1}^n C_i W C_i - W = 0$.

PROOF. If $\sum_{i=1}^n C_i C_i^* \leq 1$, $\sum_{i=1}^n C_i^* C_i \leq 1$, and $\ker E_C \subseteq \ker E_C^*$, it follows from [Theorem 4.4](#) that

$$\left\| T - \left(\sum_{i=1}^n C_i X C_i - X \right) \right\|_p^p \geq \|T\|_p^p, \quad (7.3)$$

that is, $F_p(X) \geq F_p(W)$. Conversely, if F_p has a minimum, then

$$\left\| T - \left(\sum_{i=1}^n C_i W C_i - W \right) \right\|_p^p = \|T\|_p^p. \quad (7.4)$$

Since $\mathcal{U}$ is convex, then the set

$$\mathcal{V} = \left\{ T - \left(\sum_{i=1}^n C_i X C_i - X \right); X \in \mathcal{U} \right\} \quad (7.5)$$

is also convex. Thus [Theorem 6.2](#) implies that $T - (\sum_{i=1}^n C_i W C_i - W) = T$. $\square$

THEOREM 7.4. *Let $C = (C_1, C_2, \dots, C_n)$ be n -tuple of operators in $B(H)$. If $\sum_{i=1}^n C_i C_i^* \leq 1$, $\sum_{i=1}^n C_i^* C_i \leq 1$, $\ker E_C \subseteq \ker E_C^*$, and $T \in \ker E_C \cap C_p$, then, for $1 \leq p < \infty$, the map F_p has a critical point at W if, and for $1 < p < \infty$ only if,*

$$\sum_{i=1}^n C_i W C_i - W = 0. \quad (7.6)$$

PROOF. Let $W, S \in U$ and let ϕ and φ be two maps defined, respectively, by

$$\phi : X \mapsto S - \left(\sum_{i=1}^n C_i X C_i - X \right), \quad \varphi : X \mapsto \|X\|_p^p. \quad (7.7)$$

Since the Frechet derivative of F_p is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \rightarrow 0} \frac{F_p(W + hT) - F_p(W)}{h}, \quad (7.8)$$

it follows that

$$\mathcal{D}_W F_p(T) = \left[\mathcal{D}_{S - (\sum_{i=1}^n C_i W C_i - W)} \right] \left(\sum_{i=1}^n C_i T C_i - T \right). \quad (7.9)$$

If W is a critical point of F_p , then $\mathcal{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying [Theorem 6.1](#), we get

$$\begin{aligned} \mathcal{D}_W F_p(T) &= p \operatorname{Re} \operatorname{tr} \left[\left| S - \left(\sum_{i=1}^n C_i W C_i - W \right) \right|^{p-1} W^* \left(\sum_{i=1}^n C_i T C_i - T \right) \right] \\ &= p \operatorname{Re} \operatorname{tr} \left[Y \left(\sum_{i=1}^n C_i T C_i - T \right) \right] = 0, \end{aligned} \quad (7.10)$$

where

$$S - \left(\sum_{i=1}^n C_i W C_i - W \right) = W \left| S - \left(\sum_{i=1}^n C_i W C_i - W \right) \right| \quad (7.11)$$

is the polar decomposition of the operator $S - (\sum_{i=1}^n C_i W C_i - W)$, and

$$Y = \left| S - \left(\sum_{i=1}^n C_i W C_i - W \right) \right|^{p-1} W^*. \quad (7.12)$$

An easy calculation shows that

$$\left(\sum_{i=1}^n C_i Y C_i - Y \right) = 0, \quad (7.13)$$

that is,

$$\sum_{i=1}^n C_i \left| S - \left(\sum_{i=1}^n C_i W C_i - W \right) \right|^{p-1} W^* C_i = |S - (AWB - W)|^{p-1} W^*. \quad (7.14)$$

It follows from [Lemma 7.2](#) that

$$\sum_{i=1}^n C_i \left| S - \left(\sum_{i=1}^n C_i W C_i - W \right) \right| W^* C_i = \left| S - \left(\sum_{i=1}^n C_i W C_i - W \right) \right| W^*. \quad (7.15)$$

By taking adjoints, and since $\ker E_C \subseteq \ker E_C^*$, we get

$$\sum_{i=1}^n C_i \left(T - \left(\sum_{i=1}^n C_i W C_i - W \right) \right) C_i = \left(T - \left(\sum_{i=1}^n C_i W C_i - W \right) \right), \quad (7.16)$$

and then

$$\sum_{i=1}^n C_i \left[\left(\sum_{i=1}^n C_i W C_i - W \right) \right] C_i = \left(\sum_{i=1}^n C_i W C_i - W \right). \quad (7.17)$$

Hence

$$\sum_{i=1}^n C_i W C_i - W \in R(E_C) \cap \ker E_C^*. \quad (7.18)$$

It is easy to see that (arguing as in the proof of [\[14, Proposition 4.3\]](#)) if $C = (C_1, C_2, \dots, C_n)$ is n -tuple of operator in $B(H)$ such that

$$\sum_{i=1}^n C_i C_i^* \leq 1, \quad \sum_{i=1}^n C_i^* C_i \leq 1, \quad (7.19)$$

$\ker E_C \subseteq \ker E_C^*$, and $T \in \ker \Delta_C$, where $T \in B(H)$, then

$$\|T - \Delta_C X\| \geq \|T\| \quad (7.20)$$

holds for all $X \in B(H)$ and for all $T \in \ker E_C$. Hence $\sum_{i=1}^n C_i W C_i - W = 0$.

Conversely, if $\sum_{i=1}^n C_i W C_i = W$, then W is minimum, and since F_p is differentiable, W is a critical point. $\square$

THEOREM 7.5. *Let $C = (C_1, C_2, \dots, C_n)$ be n -tuple of operators in $B(H)$. If*

$$\sum_{i=1}^n C_i C_i^* \leq 1, \quad \sum_{i=1}^n C_i^* C_i \leq 1, \quad (7.21)$$

such that $\ker E_c \subseteq \ker E_c^*$, $S \in \ker E_c \cap C_p$ ($0 < p \leq 1$), $\dim H < \infty$, and $S - (\sum_{i=1}^n C_i W C_i - W)$ is invertible, then F_p has a critical point at W if $\sum_{i=1}^n C_i W C_i - W = 0$.

PROOF. Suppose that $\dim H < \infty$. If $\sum_{i=1}^n C_i W C_i - W = 0$, then S is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$ taking $Y = |S|^{p-1} U^*$, where $S = U|S|$ is the polar decomposition of S .

It is known that if

$$\sum_{i=1}^n C_i C_i^* \leq 1, \quad \sum_{i=1}^n C_i^* C_i \leq 1, \quad \ker E_c \subseteq \ker E_c^*, \quad (7.22)$$

the eigenspaces corresponding to distinct nonzero eigenvalues of the compact positive operator $|S|^2$ reduce each C_i (see [5, Theorem 8] and [14, Lemma 2.3]). In particular, $|S|$ commutes with C_i for all $1 \leq i \leq n$. Hence

$$C_i |S| = |S| C_i. \quad (7.23)$$

Since $\sum_{i=1}^n C_i S^* C_i = S^*$, that is,

$$\sum_{i=1}^n C_i |S| U^* C_i = |S| U^*, \quad (7.24)$$

then

$$|S| \left(\sum_{i=1}^n C_i U^* C_i - U^* \right) = 0, \quad (7.25)$$

and since

$$A |S|^{p-1} = |S|^{p-1} A, \quad (7.26)$$

then

$$\sum_{i=1}^n C_i Y C_i - Y = \sum_{i=1}^n C_i |S|^{p-1} U^* C_i - |S|^{p-1} U^* = |S|^{p-1} \left(\sum_{i=1}^n C_i U^* C_i - U^* \right) \quad (7.27)$$

so that $\sum_{i=1}^n C_i Y C_i - Y = 0$ and $\operatorname{tr}[(\sum_{i=1}^n C_i Y C_i - Y)T] = 0$ for all $T \in B(H)$. Since

$$S = S - \left(\sum_{i=1}^n C_i W C_i - W \right), \quad (7.28)$$

then

$$\begin{aligned}
 0 &= \operatorname{tr} \left[Y \sum_{i=1}^n C_i T C_i - Y T \right] = \operatorname{tr} \left[Y \left(\sum_{i=1}^n C_i T C_i - T \right) \right] \\
 &= p \operatorname{Re} \operatorname{tr} \left[Y \left(\sum_{i=1}^n C_i T C_i - T \right) \right] = p \operatorname{Re} \operatorname{tr} \left[|S|^{p-1} U^* \left(\sum_{i=1}^n C_i T C_i - T \right) \right] \quad (7.29) \\
 &= (\mathcal{D}_T \phi) \left(\sum_{i=1}^n C_i T C_i - T \right) = (\mathcal{D}_W F_p)(T).
 \end{aligned}$$

□

THEOREM 7.6. Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ be n -tuples of operators in $B(H)$ such that

$$\sum_{i=1}^n A_i A_i^* \leq 1, \quad \sum_{i=1}^n A_i^* A_i \leq 1, \quad \sum_{i=1}^n B_i B_i^* \leq 1, \quad \sum_{i=1}^n B_i^* B_i \leq 1. \quad (7.30)$$

If

$$\ker E_{A,B} \subseteq \ker E_{A^*,B^*} \quad (7.31)$$

and $T \in \ker E_{A,B} \cap C_p$, then for $1 \leq p < \infty$,

(i) the map F_p has a global minimizer at W if, and for $1 < p < \infty$ only if,

$$\sum_{i=1}^n A_i W B_i - W = 0; \quad (7.32)$$

(ii) the map F_p has a critical point at W if, and for $1 < p < \infty$ only if,

$$\sum_{i=1}^n A_i W B_i - W = 0; \quad (7.33)$$

(iii) for $0 < p \leq 1$, $\dim H < \infty$, and $S - (\sum_{i=1}^n C_i W C_i - W)$ invertible, F_p has a critical point at W if

$$\sum_{i=1}^n A_i W B_i - W = 0. \quad (7.34)$$

PROOF. It suffices to take the Hilbert space $H \oplus H$ and operators (4.10) and apply Theorems 7.3, 7.4, and 7.5. □

REMARK 7.7. (1) In Theorem 7.4, the implication “ W is a critical point $\Rightarrow \sum_{i=1}^n A_i W B_i - W = 0$ ” does not hold in the case $0 < p \leq 1$ because the functional calculus argument involving the function $t \mapsto t^{1/(p-1)}$, where $0 \leq t < \infty$, is only valid for $1 < p < \infty$.

(2) The set $S = \{X : AXB - X \in C_p\}$ contains C_p . If $X \in C_p$, then $X \in S$ and, for example, $I \in S$ but $I \notin C_p$. If $A \in C_p$, the conclusion of Theorem 7.6 holds for all $X \in B(H)$.

ACKNOWLEDGMENT. This work was supported by the Research Center Project No. Math/1422/10.

REFERENCES

- [1] J. H. Anderson, *On normal derivations*, Proc. Amer. Math. Soc. **38** (1973), 135-140.
- [2] S. Bouali and S. Cherki, *Approximation by generalized commutators*, Acta Sci. Math. (Szeged) **63** (1997), no. 1-2, 273-278.
- [3] R. G. Douglas, *On the operator equation $S^*XT = X$ and related topics*, Acta Sci. Math. (Szeged) **30** (1969), 19-32.
- [4] B. P. Duggal, *A remark on normal derivations*, Proc. Amer. Math. Soc. **126** (1998), no. 7, 2047-2052.
- [5] I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs, vol. 18, American Mathematical Society, Rhode Island, 1969.
- [6] D. Kečkić, *Orthogonality in C_1 and C -spaces and normal derivations*, to appear in J. Operator Theory.
- [7] F. Kittaneh, *Operators that are orthogonal to the range of a derivation*, J. Math. Anal. Appl. **203** (1996), no. 3, 868-873.
- [8] F. Kittaneh and R. Younis, *Smooth points of certain operator spaces*, Integral Equations Operator Theory **13** (1990), no. 6, 849-855.
- [9] P. J. Maher, *Commutator approximants*, Proc. Amer. Math. Soc. **115** (1992), no. 4, 995-1000.
- [10] S. Mecheri, *On minimizing $\|S - (AX - XB)\|_p^p$* , Serdica Math. J. **26** (2000), no. 2, 119-126.
- [11] ———, *On the orthogonality in von Neumann-Schatten class*, Int. J. Appl. Math. **8** (2002), no. 4, 441-447.
- [12] S. Mecheri and M. Bounkhel, *Some variants of Anderson's inequality in C_1 -classes*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Article 24, 1-6.
- [13] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Ergebnisse der Mathematik und ihrer Grenzgebiete. N. F., vol. 27, Springer-Verlag, Berlin, 1960.
- [14] Y. S. Tong, *Kernels of generalized derivations*, Acta Sci. Math. (Szeged) **54** (1990), no. 1-2, 159-169.
- [15] T. Yoshino, *Remark on the generalized Putnam-Fuglede theorem*, Proc. Amer. Math. Soc. **95** (1985), no. 4, 571-572.

Salah Mecheri: Department of Mathematics, College of Science, King Saud University,
P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: mecheri@ksu.edu.sa